

## A GENERALIZATION OF THE EPSTEIN-PENNER CONSTRUCTION TO PROJECTIVE MANIFOLDS

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ABSTRACT. We extend the canonical cell decomposition due to Epstein and Penner of a hyperbolic manifold with cusps to the strictly convex setting.

### 1. INTRODUCTION

One of the powerful constructions in theory of cusped hyperbolic  $n$ -manifolds is a cellulation constructed by Epstein and Penner in [4], which in the particular case that the manifold has one cusp, gives rise to a canonical cell decomposition. In this note we extend their results to the case of strictly convex real projective manifolds.

The proof of [4] employs Minkowski space  $\mathbb{R}^{n,1}$  and shows that if  $p \in \mathbb{R}^{n,1}$  is a point on the lightcone that corresponds to a parabolic fixed point  $\mathbb{P}(p) \in \mathbb{R}P^n$ , then  $p$  has a *discrete* orbit. The convex hull of this orbit is an infinite sided polytope in Minkowski space that is preserved by the group. The boundary of the quotient of this polytope by the group gives the cell decomposition. This approach uses in an essential way the quadratic form  $\beta = x_1^2 + \cdots + x_n^2 - x_{n+1}^2$  that defines  $O(n, 1)$  to identify Minkowski space with its dual. This gives a bijection between points on the lightcone  $t \cdot p$  with  $t > 0$  in the direction of  $\mathbb{P}(p)$  and horoballs  $\mathcal{B}(t)$  centered on  $\mathbb{P}(p)$ . For  $t$  sufficiently large this horoball covers an *embedded* cusp, so the orbit consists of *disjoint* horoballs, which implies the orbit of  $p$  is discrete.

In this paper we use a *Vinberg hypersurface* to associate to  $p$  a horoball in the universal cover of the *dual* projective manifold that covers the dual cusp.

In the hyperbolic case in dimension 2 one obtains a cell decomposition of moduli space from the result of Epstein and Penner, [4]. For finite volume hyperbolic structures, Mostow-Prasad rigidity implies that in dimension at least 3 the moduli space is a point. No similar result holds in the strictly convex setting: there are examples of one cusped 3-manifolds with families of finite volume strictly convex projective structure. It can be shown that the results of this paper lead to a decomposition of the moduli space of such structures, but we do not know if the components of this decomposition are cells.

Background for theory of cusped projective manifolds can be found in [2]. We summarize the most important points here.

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A subset  $\Omega \subset \mathbb{R}P^n$  is *properly convex* if it is the interior of a compact convex set  $K$  that is disjoint from some codimension-1 projective hyperplane and *strictly convex* if in addition  $K$  contains no line segment of positive length in its boundary.

A *strictly convex real projective  $n$ -manifold* is  $M = \Omega/\Gamma$  where  $\Omega \subset \mathbb{R}P^n$  is strictly convex and  $\Gamma \cong \pi_1 M$  is a discrete group of projective transformations that preserves  $\Omega$  and acts freely on it. Since  $\Omega$  is properly convex, it has two connected preimages in the universal covering of  $\mathbb{R}P^n$  and we may (and will) lift  $\Gamma$  to a subgroup of  $SL(\Omega)$  which is the group of matrices of determinant  $\pm 1$  that preserve one of these components. The Hilbert metric on  $\Omega$  is invariant for the action of  $\Gamma$  and defines a Finsler metric, and thus a notion of volume, on  $M$ . All projective manifolds in this note will be assumed to have finite volume.

An element of  $SL(\Omega)$  is *parabolic* if all its eigenvalues have modulus 1 and it is not semisimple.

A *maximal cusp* in  $M = \Omega/\Gamma$  is a connected submanifold,  $B$ , such that  $\partial B = \overline{M \setminus B} \cap B$  is compact and

- C1 Every component  $\tilde{B}$  of the preimage of  $B$  in  $\Omega$  has strictly convex interior.
- C2  $p = cl(\tilde{B}) \cap \partial\Omega$  is a single point called a *parabolic fixed point*.
- C3 The stabilizer  $\Gamma_{\tilde{B}} \subset \Gamma$  of  $\tilde{B}$  fixes  $p$ .
- C4 There is a unique projective hyperplane  $H \subset \mathbb{R}P^n$  with  $p = H \cap \overline{\Omega}$ .
- C5 Every non-trivial element of  $\Gamma_{\tilde{B}}$  is parabolic and preserves  $H$ .
- C6  $\Gamma_{\tilde{B}}$  is conjugate into  $PO(n, 1)$  so contains  $\mathbb{Z}^{n-1}$  as a subgroup of finite index.

It is proved in [2] that a strictly convex finite volume real projective manifold has finitely many ends and each is a maximal cusp. Such cusps have the maximal virtual cohomological dimension.

We can identify a properly convex domain  $\Omega \subset \mathbb{R}P^n$  with a subset  $\Omega$  of some affine hyperplane in  $\mathbb{R}^{n+1}$ . Then  $\mathcal{C}\Omega = (\mathbb{R}_{>0}) \cdot \Omega \subset \mathbb{R}^{n+1}$  is an open cone based at 0 and  $\mathbb{P}(\mathcal{C}\Omega) = \Omega \subset \mathbb{R}P^n$  where  $\mathbb{P} : \mathbb{R}^{n+1} \setminus 0 \rightarrow \mathbb{R}P^n$  denotes projectivization. The (*positive*) *lightcone* of  $\Omega$  is the cone  $\mathcal{L} = \mathcal{C}(\partial\Omega)$ . It is the subset of the frontier of  $\mathcal{C}\Omega$  obtained by deleting 0 and  $\mathbb{P}(\mathcal{L}) = \partial\Omega$ .

**Theorem 1.1.** *Suppose  $M = \Omega/\Gamma$  is a strictly convex real projective  $n$ -manifold that contains a maximal cusp  $B$ , and that  $p \in \mathcal{L}$  is a point in the lightcone of  $\Omega$  such that  $\mathbb{P}(p) \in \partial\Omega$  is the parabolic fixed point of  $\pi_1 B$ . Then the  $\Gamma$ -orbit of  $p$  is a discrete subset of  $\mathbb{R}^{n+1}$ .*

This has as an immediate consequence the existence of ideal cell decompositions:

**Corollary 1.2.** *Suppose  $M = \Omega/\Gamma$  is a strictly convex real projective  $n$ -manifold of finite volume with at least one (maximal) cusp and  $\mathcal{Q} \subset \partial\Omega$  is the set of fixed points of parabolics in  $\Gamma$ .*

*Then there is a  $\Gamma$ -invariant tessellation of  $\Omega \cup \mathcal{Q}$  by convex polytopes, each of which is the convex hull of a finite subset  $V \subset \mathcal{Q}$ . The interiors of the cells of dimension greater than zero project to a cell decomposition of  $M$ .*

To prove this, one chooses a  $\Gamma$ -invariant collection  $\mathcal{P} \subset \mathcal{L}$  of points in the lightcone, one in the direction of each parabolic fixed point. If there are  $k$  cusps this amounts to choosing  $k$  positive reals. By Theorem 1.1,  $\mathcal{P}$  is a discrete set. The closed convex hull of  $\mathcal{P}$  is a  $\Gamma$ -invariant set  $C = C(\Gamma, \mathcal{P})$  in  $\mathbb{R}^{n+1}$ . We show that

$C$  has polyhedral boundary. The image under projectivization decomposes  $\Omega \cup \mathcal{Q}$  into convex cells.

Clearly this decomposition is unchanged by uniformly scaling  $\mathcal{P}$ , and so the  $k$  cusps result in a family of cell decompositions parameterized by a point in the interior of a simplex in  $\mathbb{R}P^{k-1}$ . In the case that the manifold has one cusp, this decomposition is canonical:

**Corollary 1.3.** *Suppose  $M = \Omega/\Gamma$  is a strictly convex real projective  $n$ -manifold with finite volume and that contains a unique (maximal) cusp  $B$ .*

*Then the cell decomposition obtained from  $C(\Gamma, \mathcal{P})$  is independent of the choice of  $\mathcal{P}$  and thus canonical.*

It follows immediately that the isometry group of a one-cusped strictly convex real projective  $n$ -manifold with finite volume preserves this cell decomposition, and is finite.

*Remark.* We refer the reader to [1] §1.7 for the (technical, but standard) details concerning the continuous variation of geometric structures. One can think of this as variations of the holonomy representation or of developing maps with an appropriate topology. Then with appropriate hypotheses, one can show that  $C(\Gamma)$  can be chosen to vary continuously with  $\Gamma$ . In particular, if all the cells are ideal simplices, then the combinatorics of the cell decomposition is locally constant.

Let  $V$  be a real vector space of dimension  $(n + 1)$  with dual  $V^*$  and  $\mathbb{P} : V \setminus 0 \rightarrow \mathbb{P}V$  the projectivization map. In the following discussion  $\Omega$  is a properly convex set in  $\mathbb{P}V$ ; we do not require the extra hypothesis of being strictly convex.

The relation between a vector space and its dual gives rise to projective duality. Given a properly convex  $\Omega \subset \mathbb{P}(V)$  the *dual cone*  $\mathcal{C}\Omega^* \subset V^*$ , is the interior set of linear functionals which take strictly positive values on  $\mathcal{C}\Omega$ . The *dual domain*  $\Omega^* = \mathbb{P}(\mathcal{C}\Omega^*) \subset \mathbb{P}(V^*)$  is also properly convex. If  $\Omega$  is strictly convex, then so is  $\Omega^*$ . The *dual lightcone*  $\mathcal{L}^*$  of  $\Omega$  is the lightcone of  $\Omega^*$ .

The dual action of an element  $\gamma \in PGL(V)$  on  $V^*$  is given by  $\gamma^*(\phi) = \phi \circ \gamma^{-1}$ . A choice of basis for  $V$  gives isomorphisms  $V \cong \mathbb{R}^{n+1} \cong V^*$  and  $PGL(V) \cong PGL(n + 1, \mathbb{R}) \cong PGL(V^*)$ . Using these identifications the dual action of  $PGL(V)$  on  $V^*$  then corresponds to the Cartan involution  $\theta(A) = (A^{-1})^t$  on  $PGL(n + 1, \mathbb{R})$ . If  $\Gamma \subset PGL(n + 1, \mathbb{R})$ , the *dual group* is  $\Gamma^* = \theta(\Gamma)$ .

In this way, (see [2]) we define the *dual manifold* of  $M = \Omega/\Gamma$  to be  $M^* = \Omega^*/\Gamma^*$ . If  $p \in \partial\Omega$  is the parabolic fixed point of a maximal cusp  $B \subset M$  by (C4) there is a unique supporting hyperplane  $H$  to  $\Omega$  at  $p$ . The *dual parabolic fixed point*  $[\phi] \in \partial\Omega^*$  is defined by  $\mathbb{P}(\ker \phi) = H$ . The dual action of  $\pi_1 B$  fixes  $[\phi]$  and there is a *dual cusp*  $B^*$ , well defined up to the equivalence relation generated by inclusion. Thus  $\phi$  is a point on the dual lightcone. Below we show that level sets of  $\phi$  determine a type of horosphere in  $\Omega$  centered at  $p$ .

The *hyperboloid model* of hyperbolic space is a certain level set of the quadratic form  $\beta$ . In general the holonomy of a strictly convex manifold does not preserve any non-degenerate quadratic form but it does preserve a certain convex function which has levels sets called *Vinberg hypersurfaces* [6] that provide a generalization of the hyperboloid. We briefly recall the construction here. Let  $d\psi$  be a volume form on  $V^*$ . Then the *characteristic function*  $f : \mathcal{C}\Omega \rightarrow \mathbb{R}$  is defined by

$$f(x) = \int_{\mathcal{C}\Omega^*} e^{-\psi(x)} d\psi.$$

This is real analytic, convex, and satisfies  $f(tx) = t^{-n}f(x)$  for  $t > 0$ . For each  $t > 0$  the level set  $S_t = f^{-1}(t)$  is called a *Vinberg hypersurface* and is convex. These sets foliate  $\mathcal{C}\Omega$  and are permuted by homotheties fixing the origin. For example, the hyperboloids  $z^2 = x^2 + y^2 + t$  are Vinberg hypersurfaces in the cone  $z^2 > x^2 + y^2$ . The surfaces  $S_t$  are all preserved by  $SL(\Omega)$ , (we recall that this is the group of matrices of determinant  $\pm 1$  that preserve  $\Omega$ ) and in particular by  $\Gamma$ . Henceforth, we fix some choice  $S := S_1$  which we refer to as *the* Vinberg surface for  $\Omega$ . It is a substitute for the hyperboloid model of hyperbolic space.

Let  $\pi : S \rightarrow \Omega$  be the restriction of the projectivization map. A point,  $\phi \in \mathcal{L}^*$ , in the dual lightcone of  $\Omega$  determines a *horofunction*

$$h_\phi = \phi \circ \pi^{-1} : \Omega \rightarrow \mathbb{R}.$$

Since  $\phi \in \mathcal{L}^*$  it follows that  $\ker \phi$  contains a ray  $(0, \infty) \cdot v \subset \mathcal{L}$  in the lightcone.

We show below that the sublevel set  $\mathcal{B}(\phi, t) = h_\phi^{-1}(0, t]$  is convex. This is called a *horoball* associated to  $\phi$ . It is in general different from the *algebraic horoballs* defined in [2]. The boundary

$$\mathcal{S}(\phi, t) = \partial\mathcal{B}(\phi, t) = h_\phi^{-1}(t)$$

of a horoball is called a *horosphere*, and is analytic.

**Lemma 1.4.** *Suppose  $\Omega \subset \mathbb{R}P^n$  is properly convex and  $\phi$  is a point in the dual lightcone. Then the horofunction  $h_\phi : \Omega \rightarrow (0, \infty)$  is a smooth surjective submersion. Hence for all  $t > 0$  the horoball  $\mathcal{B}(\phi, t)$  is non-empty and convex.*

*Proof.* Here is an overview of the proof: The implicit function theorem and smoothness of  $f$  imply  $h_\phi$  is smooth. Referring to the figure, notice the subset of  $\mathbb{R}^{n+1}$  above  $S$ , which we denote by  $W = f^{-1}(0, 1]$ , is convex. It follows that  $X_t = W \cap \phi^{-1}(0, t]$  is convex, since it is the intersection of convex sets. Hence  $\mathcal{B}(\phi, t) = \pi(X_t) \subset \Omega$  is convex.

We claim that for  $t > 0$  it is not empty. This follows from the fact that if  $v$  is a point in the lightcone of  $\Omega$  that is also in  $\ker \phi$ , then the vertical distance  $\delta(t \cdot v)$  between  $S$  and  $t \cdot v$  goes to zero as  $t \rightarrow \infty$ . This is shown by direct computation in the particular case that  $\Omega$  is an open simplex, and the general result follows by a comparison argument.

Here are the details:

To begin with, consider the special case that  $\mathcal{C}\Omega$  is the positive orthant in  $\mathbb{R}^{n+1}$ , which is the cone on an  $n$ -simplex  $\Omega = \sigma$ . In an appropriate basis, the group  $SL(\sigma)$  contains positive diagonal matrices of determinant 1. The orbits are the Vinberg hypersurfaces  $x_0 \cdot x_1 \cdots x_n = c$  for each  $c > 0$ . Each hypersurface is asymptotic to the ray  $(0, \infty) \cdot v$  which proves the result in this special case.

In the general case there is a simplex  $\sigma$  with interior in  $\Omega$  and  $v$  as a vertex. Then  $\mathcal{C}(\sigma) \subset \mathcal{C}\Omega$ , and it follows from the definition that  $f_{\mathcal{C}(\sigma)} \geq f_{\mathcal{C}\Omega}|_{\mathcal{C}(\sigma)}$ , which implies that the Vinberg surface for  $\Omega$  lies below that for  $\sigma$  along the direction given by  $v$ . The claim now follows from the special case.

It follows from this, and the convexity of  $S$ , that  $S$  is never tangent to a level set of  $\ker \phi$ , hence  $h_\phi$  is a submersion. □

It is immediate from the definition of horoball that if  $\gamma \in SL(\Omega)$  and  $\phi$  is in the dual lightcone, then

$$\gamma(\mathcal{B}(\phi, t)) = \mathcal{B}(\gamma^* \phi, t).$$

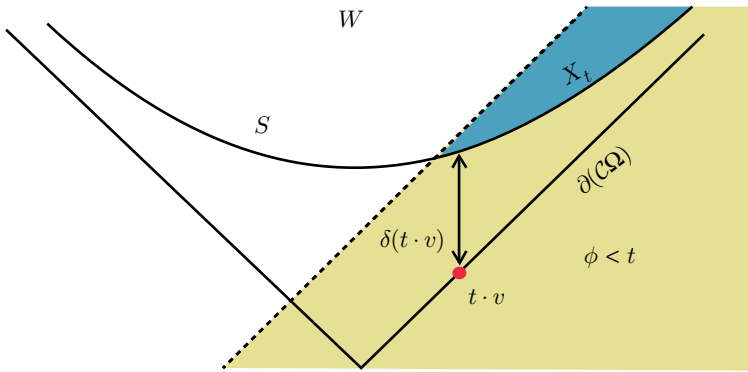


FIGURE 1. Vinberg hypersurface inside the lightcone

Thus if  $\gamma^*\phi = \phi$ , then the horoball  $\mathcal{B}(\phi, t)$  is preserved by  $\gamma$ . If  $B$  is a maximal cusp of  $M$ , then  $\pi_1 B$  preserves horoballs corresponding to the unique supporting hyperplane at the parabolic fixed point. The next result is that a sufficiently small such horoball projects to an (embedded) cusp in  $M$  called a *horocusp*. Recall that a subset  $U \subset \Omega$  is *precisely invariant* under a subgroup  $G \subset \pi_1 M$  if every element of  $G$  preserves  $U$  and every element of  $\pi_1 M \setminus G$  sends  $U$  to a subset disjoint from  $U$ .

**Corollary 1.5.** *Suppose  $M = \Omega/\Gamma$  is a properly convex manifold and  $B \subset M$  is a maximal cusp with holonomy  $\Gamma_B \subset \Gamma$  and with dual  $\Gamma_B^*$  that fixes the dual parabolic fixed point  $[\phi] \in \partial\Omega^*$ .*

*Then there is  $t > 0$  such that the horoball  $\mathcal{B}(\phi, t)$  is precisely invariant under  $\Gamma_B \subset \Gamma$ . For sufficiently small  $t > 0$  the manifold  $B' = \mathcal{B}(\phi, t)/\Gamma_B$  projects injectively onto a cusp  $B' \subset B \subset M$  and  $B \setminus B'$  is bounded. The horofunction  $h_\phi$  covers a proper smooth submersion  $h : B' \rightarrow (0, t]$ . The sublevel sets are convex and the level sets, called horomanifolds, are compact and give a product foliation.*

*Proof.* Let  $H$  be the supporting hyperplane to  $\Omega$  at the parabolic fixed point for  $\Gamma_B$ . By (C3)  $H$  is preserved by  $\Gamma_B$ . There is a codimension-1 subspace  $V \subset \mathbb{R}^{n+1}$  with  $\mathbb{P}(V) = H$ . Then  $V = \ker \phi$  iff  $\phi$  is a point on the dual lightcone  $\mathcal{L}^*$  such that  $[\phi] \in \partial\Omega^*$  is the parabolic fixed point for  $\Gamma_B^*$ .

The dual of a parabolic matrix is also parabolic, thus  $\phi \in V^*$  is an eigenvector with eigenvalue 1 for every element of  $\Gamma_B^*$ . This means  $\phi$  is a  $\Gamma_B$ -invariant function on  $V$ , thus  $h_\phi$  covers a well-defined function  $h : \Omega/\Gamma_B \rightarrow (0, \infty)$ . This is a submersion because  $h_\phi$  is, hence the level sets  $\mathcal{H}_t = h^{-1}(t)$  are a foliation of  $B'$ .

Clearly  $\pi_1 \mathcal{H}_t \cong \pi_1 B$  and, since  $\pi_1 B$  is a maximal cusp, by (C6)  $\pi_1 B$  has (virtual) cohomological dimension  $(n - 1)$ , and it follows that each horomanifold is compact. There is a transverse foliation by lines going out into the cusp, and these foliations give a product structure on  $B'$ . Convexity of sublevel sets follows from convexity of horoballs.  $\square$

*Proof of Theorem 1.1.* We will prove the theorem for the dual manifold  $M^* = \Omega^*/\Gamma^*$ . By (0.15) and (6.7) of [2]  $M^*$  is a strictly convex projective manifold with

finite volume. The result follows using the canonical isomorphism between a finite dimensional vector space and its double dual. As before we write  $V = \mathbb{R}^{n+1}$ .

Recall that  $B$  is a maximal cusp, and let  $\phi \in \mathcal{L}^*$  be a point on the dual lightcone fixed by the dual action of  $\pi_1 B$ . Let  $\gamma_n^* \in \Gamma^*$  be a sequence such that  $\gamma_n^* \phi = \phi \circ \gamma_n^{-1}$  converges to some  $\psi \in V^*$ . We must show this sequence is eventually constant. If not, we may assume  $\gamma_n^* \phi$  are all distinct. Clearly  $\psi \in \mathcal{L}^* \cup 0$ .

By Corollary 1.5 there is  $t > 0$  such that  $B$  contains the horocusp  $B' = \mathcal{B}(\phi, t)/\pi_1 B$ . Choose  $w \in \mathcal{C}\Omega$  with  $\psi(w) < t$ . Since  $\gamma_n^* \phi \rightarrow \psi$ , then for all  $n$  sufficiently large  $\gamma_n^* \phi(w) < t$ . Since  $\psi \in \mathcal{L}^*$  there is  $v \in \mathcal{L}$  with  $\psi(v) = 0$  thus  $\psi(w + s \cdot v) = \psi(w)$ . From the formula for the characteristic function  $f$  for  $\Omega$  we see that  $f(w + s \cdot v) \rightarrow 0$  as  $s \rightarrow \infty$  (also see Figure 1), so for  $s$  sufficiently large  $w + s \cdot v$  is above the Vinberg hypersurface, which implies  $w + s \cdot v \in \gamma_n \mathcal{B}(\phi, t) \cap \gamma_{n+1} \mathcal{B}(\phi, t)$ . Since  $B'$  is precisely invariant for  $\pi_1 B \subset \pi_1 M$  it follows that  $\gamma_{n+1}^{-1} \gamma_n \in \pi_1 B$ . But  $\phi$  is preserved by every element of this group thus  $\gamma_n^* \phi = \gamma_{n+1}^* \phi$  which is a contradiction. □

The next result is well known, but we include it for the convenience of the reader.

**Lemma 1.6** (Orbits are dense). *Suppose  $M = \Omega/\Gamma$  is a strictly convex projective manifold with finite volume. Then every  $\Gamma$  orbit is dense in  $\partial\Omega$ .*

*Proof.* Given a point  $b \in \partial\Omega$  define  $\Omega^-$  to be the intersection with  $\Omega$  of the closed convex hull of  $\Gamma b$ . Since  $\Omega$  is strictly convex,  $\Gamma b$  is dense in  $\partial\Omega$  iff  $\Omega = \Omega^-$ . The set  $\Omega^-$  is convex and  $\Gamma$  invariant. The projection,  $N = \Omega^-/\Gamma$ , of  $\Omega^-$  is a submanifold of  $M$ . Since  $M$  is finite volume and strictly convex, it is the union of a compact set and finitely many cusps, [2]. We replace  $\Omega^-$  by a  $K$ -neighborhood with  $K$  so large that the complement of  $N$  is now a subset of the cusps of  $M$ .

Suppose  $R$  is a component of  $\Omega \setminus \Omega^-$ , with stabilizer  $\Gamma_R \subset \Gamma$ . Then  $R/\Gamma_R$  is mapped injectively into a cusp  $B \subset M$  by the projection. Let  $\tilde{B}$  be the component of the preimage of  $B$  that contains  $R$ . Observe that  $\overline{R} \cap \partial\Omega$  contains more than one point, but  $cl(\tilde{B}) \cap \partial\Omega$  is one point by (C2), a contradiction. □

The hypothesis of strictly convex in the above cannot be weakened to properly convex because there is a properly convex projective torus that is the quotient of the interior of a triangle by a discrete group and each vertex of the triangle is an orbit.

**Lemma 1.7.** *Suppose  $M = \Omega/\Gamma$  is strictly convex and has finite volume. If  $x \in \mathcal{L}$  is a point in the lightcone, then 0 is an accumulation point of  $\Gamma x$  iff  $\mathbb{P}(x) \in \partial\Omega$  is not a parabolic fixed point.*

*Proof.* If  $\mathbb{P}(x)$  is a parabolic fixed point the result follows from Theorem 1.1. We follow the proof of (3.2) in [4] with some adaptations for our situation to show the corresponding result for the projective dual  $M^* = \Omega^*/\Gamma^*$ . The result then follows by duality. The proof is broken down into several steps.

Let  $H = \mathbb{P}(V)$  be the unique supporting hyperplane to  $\Omega$  at  $\mathbb{P}(x)$ . Let  $\phi \in \mathcal{L}^*$  be dual to  $H$  thus  $V = \ker \phi$ . This only defines  $\phi$  up to scaling. By Theorem 0.2 of [2], the manifold  $M$  is the union of a compact thick part  $K$  and finitely many cusp neighborhoods.

Choose a compact set  $\tilde{K}$  in the Vinberg hypersurface  $S$  for  $\Omega$  so that the projection of  $\tilde{K}$  contains  $K$ . Given  $t > 0$  there is a horoball  $\mathcal{B} = \mathcal{B}(\phi, t) \subset \Omega$ . If the  $\Gamma^*$  orbit of  $\phi$  does not accumulate on 0 we may choose  $t$  so small ( $\Rightarrow \mathcal{B}$  small) that

the  $\Gamma$  orbit of  $\mathcal{B}$  is disjoint from  $\tilde{K}$ . This implies the orbit of  $\mathcal{B}$  projects into a cusp of  $M$ . It follows that  $\phi$  is a parabolic fixed point of  $\Gamma^*$ .  $\square$

*Proof of Corollary 1.2.* We refer to the notation established in the paragraph after Corollary 1.2. Regard  $\mathbb{R}^{n+1}$  as an affine patch in  $\mathbb{R}P^{n+1} = \mathbb{R}^{n+1} \sqcup \mathbb{R}P^n_\infty$ . The image of  $\Omega$  in  $\mathbb{R}P^n_\infty$  under radial projection from  $0 \in \mathbb{R}^{n+1}$  is a set  $\Omega_\infty \subset \mathbb{R}P^n_\infty$  projectively equivalent to  $\Omega$ . Define  $K \subset \mathbb{R}P^{n+1}$  to be the cone that is the closure in  $\mathbb{R}P^{n+1}$  of  $\mathcal{C}\Omega$ ; then

$$K = \mathcal{C}\Omega \sqcup \mathcal{L} \sqcup \overline{\Omega}_\infty \sqcup 0.$$

It is disjoint from a codimension-1 projective hyperplane  $H$  that is a small perturbation of  $\mathbb{R}P^n_\infty$  and therefore  $K \subset \mathbb{A}^{n+1} = \mathbb{R}P^{n+1} \setminus H$  is a properly convex set. Recalling that  $\mathcal{P}$  was some  $\Gamma$ -invariant choice of points, one in the direction of each parabolic fixed point, observe that  $\mathcal{P} \subset \partial K$  and, since  $K$  is convex,  $\overline{C} \subset K$  where  $\overline{C}$  is the closure of  $C$  in  $\mathbb{R}P^{n+1}$ .

We claim that it follows from Lemma 1.6 that the set of accumulation points of  $\mathcal{P}$  is  $\partial\Omega_\infty$ . The argument is the following: Every open set  $U \subset \partial\Omega_\infty$  contains infinitely many parabolic fixed points. These correspond to an infinite subset of  $\mathcal{P}$ . By Theorem 1.1 this set is discrete in the lightcone so that all but finitely many of these points are very high in the lightcone and, in particular, very close to  $U$ .

This in turn implies  $\overline{C}$  contains  $\partial\Omega_\infty$  and therefore also contains  $\overline{\Omega}_\infty$  thus

$$\overline{C} = C \sqcup \overline{\Omega}_\infty.$$

It follows that  $\overline{C}$  is the closed convex hull in  $\mathbb{A}^{n+1}$  of  $\mathcal{P}$ .

The holonomy,  $\Gamma$ , of  $M$  lies in  $SL(n+1, \mathbb{R})$  and can be identified with a subgroup  $\Gamma^+ \subset \text{PGL}(n+2, \mathbb{R})$  that preserves  $K$ . In suitable coordinates  $\Gamma^+$  is block diagonal with a trivial block of size 1 and the other block is  $\Gamma$ .

Closely following §3 of [4] we establish the following claims:

- The dimension of  $C$  is  $n + 1$  because  $\overline{C}$  contains the  $n$ -dimensional set  $\Omega_\infty \subset \mathbb{R}P^n_\infty$  and also contains the points  $\mathcal{P}$  which are not in  $\mathbb{R}P^n_\infty$ .

- $w \in C \cap \mathcal{L}$  iff  $w = \alpha z$  for some  $z \in \mathcal{P}$  and  $\alpha \geq 1$ .

If  $w$  is not of this form, then the segment  $[0, w]$  is disjoint from  $\mathcal{P}$ . Since  $\mathcal{P}$  is discrete there is a small neighborhood  $U \subset \mathbb{R}^{n+1}$  of this segment that contains no point of  $\mathcal{P}$ . Hence there is a hyperplane that intersects  $\mathcal{C}\overline{\Omega}$  in a small, convex, codimension-1 set in  $U$  and separates  $[0, w]$  from  $\mathcal{P}$ , and hence from  $C$ . This means  $w \notin C$ .

For the converse, given  $z \in \mathcal{P}$  the image  $w \in \partial\Omega_\infty$  of  $z$  is in  $\overline{C}$ , hence  $[z, w] \subset \overline{C}$ . This contains all the points  $\alpha z$  with  $\alpha \geq 1$ .

- Each ray  $\lambda \subset \mathcal{C}\Omega$  that starts at 0 meets  $\partial C$  exactly once.

Since  $\mathcal{P}$  is discrete in  $\mathbb{R}^{n+1}$  it follows that  $0 \notin \overline{C}$  so  $\lambda$  starts outside  $\overline{C}$  and limits on  $q \in \Omega_\infty \subset \overline{C}$ . Thus  $\lambda$  contains points in the interior of  $\overline{C}$ . Since  $\overline{C}$  is convex  $\lambda$  meets  $\partial\overline{C}$  in a single point  $z$ . Since  $\lambda \subset \mathcal{C}\Omega$  it follows that  $z \in \mathcal{C}\Omega \cap \partial\overline{C} = \partial C$ .

- If  $W \subset \mathbb{R}^{n+1}$  is a supporting affine hyperplane for  $C$  at a point  $z \in \partial C \cap \mathcal{C}\Omega$ , then  $W \cap \mathcal{C}(\overline{\Omega})$  is compact and convex.

The closure  $\overline{W}$  of  $W$  in  $\mathbb{R}P^{n+1}$  is a projective hyperplane that is a supporting hyperplane for  $\overline{C}$  in  $\mathbb{R}P^{n+1}$ . Clearly  $\overline{W}$  is disjoint from  $\Omega_\infty$  and by the previous claim  $0 \notin \overline{W}$ . The ray from 0 through  $z$  limits on  $\Omega_\infty$  and crosses  $\partial C$  at  $z$  therefore  $\overline{W}$  separates 0 from  $\Omega_\infty$  in  $\mathbb{A}^{n+1}$ .

Let  $V$  be the vector subspace parallel to  $W$ . Then  $V = \ker \phi$  for some linear map  $\phi$ . We claim that  $V$  is disjoint from  $\mathcal{C}\overline{\Omega} = \mathcal{C}\Omega \sqcup \mathcal{L}$ . Observe that  $\overline{V}$  and  $\overline{W}$

have the same intersection with  $\mathbb{R}P_\infty^n$  and are disjoint from  $\Omega_\infty$ . Since  $V$  contains 0 it follows that  $V$  is disjoint from  $\mathcal{C}\Omega$ . It remains to show  $V$  is disjoint from  $\mathcal{L}$ .

Define an affine function  $\psi$  on  $\mathbb{R}^{n+1}$  by  $\psi(v) = \phi(v - z)$ . Then  $W = \psi^{-1}(0)$ , and since  $W$  is a supporting hyperplane for  $C$ , this means that  $\psi$  has constant sign on  $C$ . By replacing  $\phi$  by  $-\phi$  if needed we may assume  $\psi(v) \geq 0$  for all  $v \in C$ . Hence  $\phi(v) \geq \phi(z)$  for all  $v \in C$ . Since  $V$  is disjoint from  $\mathcal{C}\Omega$  it follows that  $\phi$  has constant sign on  $\mathcal{C}\Omega$ . Since  $\psi$  takes arbitrarily large positive values on  $\mathcal{C}\Omega$  it follows that  $\phi \geq 0$  on  $\mathcal{C}\Omega$  and hence  $K = \phi(z) > 0$ . This implies  $\phi(v) \geq K$  for all  $v \in C$ . Since  $\Gamma$  preserves  $C$  it follows that for every  $\gamma^* \in \Gamma^*$  that  $\gamma^*\phi \geq K$  everywhere on  $C$ .

We claim that  $\mathcal{L} \cap V = \emptyset$ . For, suppose not and that  $0 \neq x \in \mathcal{L} \cap V$ .

First, observe that  $\mathbb{P}(x)$  cannot be a parabolic fixed point of  $\Gamma$ , otherwise points high on this ray are in  $C$ , and so  $\phi$  of these high points must be at least  $K > 0$ . However,  $V$  and  $W$  are parallel hyperplanes, so that these high points, which are all in  $V$ , must be disjoint from, and therefore below  $W$ , which is a contradiction.

However, consideration of stabilizers now implies that  $\mathbb{P}(\phi) \in \partial\Omega^*$  is not a (dual) parabolic fixed point. Hence by Lemma 1.7, there is a sequence  $\gamma_k^* \in \Gamma^*$  such that  $\gamma_k^*\phi \rightarrow 0$ . Thus for large  $k$  we have  $\gamma_k^*\phi(z) < K$ . This contradicts  $\gamma_k^*\phi \geq K$  everywhere on  $C$ .

This proves the assertion that  $\mathcal{L} \cap V = \emptyset$ . Our main claim that  $W \cap \mathcal{C}(\overline{\Omega})$  is compact and convex now follows: It is clear that this set is convex. Note that  $\phi(W) = \phi(V + z) = \phi(z) = K$  is constant. However, since  $\mathcal{L} \cap V = \emptyset$ , for any ray  $(\mathbb{R}_{>0}) \cdot v$  in  $\mathcal{C}\overline{\Omega}$ ,  $\phi(v) > 0$ , so that very high points on that ray take values  $> K$ . It follows that  $W$  meets  $\mathcal{C}\overline{\Omega}$  in a compact set.

- Every point in  $\partial C \cap \mathcal{C}\Omega$  is contained in a supporting hyperplane that contains at least  $(n + 1)$  points in  $\mathcal{P}$ .

Given a supporting hyperplane  $H$ , rotate it around  $H \cap C$  until it meets another point of  $\mathcal{P}$ . Since this set is discrete, there is a first rotation angle with this property. This process stops when  $H \cap C$  contains an open subset of  $H$ . See [4] for more details.

- The set of codimension-1 faces is locally finite inside  $\mathcal{C}\Omega$ .

Let  $K \subset \mathcal{C}\Omega$  be a compact set meeting faces  $F_1, F_2, \dots$  and suppose that these faces are defined by affine hyperplanes  $A_1, A_2, \dots$ . Pick  $x_i \in K \cap F_i$  and subconverge so that  $x_i \rightarrow x$  and  $A_i \rightarrow A$ , an affine plane containing  $x$ . The  $A_i$ 's are all support planes, whence so is  $A$ , thus it meets  $\mathcal{C}\overline{\Omega}$  in a compact convex set. Move  $A$  upwards a small distance to obtain  $A^+$ . Then all but finitely many of  $A_i \cap \mathcal{C}\overline{\Omega}$  lie below  $A^+ \cap \mathcal{C}\overline{\Omega}$ . Hence  $\mathcal{P} \cap (\cup A_i)$  is finite and it follows that there were only a finite number of faces meeting  $K$ .

The locally finite cell structure on  $\partial C \cap \mathcal{C}\Omega$  is  $\Gamma$ -equivariant and projects to a locally finite cell structure on  $M = \Omega/\Gamma$ . This completes the proof of Corollary 1.2. □

*Proof of Corollary 1.3.* In the case that  $M$  has only one cusp, the convex hull  $C$  is defined by the orbit of a single vector, which in turn is uniquely defined up to scaling. It follows that  $C$  is defined up to homothety and this is invisible when one projects into  $\Omega/\Gamma$ . □

*Remarks.* This work originated with one approach to extending the result of Koszul on the openness of the set of representations of strictly convex structures from the compact to the finite volume case, although in the end we chose a different approach.



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