Topology Vol. 27, No. 1, pp. 23–25, 1988
Printed in Great Britain.

0040 9383/88 $3.00 + .00
Pergamon Press plc

TRIANGULATING 3-MANIFOLDS USING 5 VERTEX LINK TYPES

D. Cooper and W. P. Thurston

(*Received* 5 *June* 1985)

We show that a closed orientable 3-manifold can be triangulated in a simple way locally. There are 5 triangulations of S^2 with the property that every such manifold has a triangulation in which the link of each vertex is one of these 5 link types. This triangulation is obtained from a paving of the manifold by cubes. In this paving, the order of every edge is 3, 4 or 5. Denoting the union of the edges of order 3 (respectively 5) by $\sum_3$ (respectively $\sum_5$), then $\sum_3$ and $\sum_5$ are disjoint 1-submanifolds. It is known that for any dimension n, there is a finite set of link types such that every n-manifold has a triangulation in which the link of each vertex is in this set. However for $n > 3$, no such set is known.

If K is a simplicial complex, we denote the barycentric subdivision of K by K', and the suspension of K by $\sum K$. The set J consists of the 5 triangulations of S^2 listed below:

(*T*1) ∂(octahedron)

(*T*2) $[\partial(\text{3-simplex})]'$

(*T*3) $[\sum\partial(\text{triangle})]'$

(*T*4) $[\sum\partial(\text{square})]' = [\partial(\text{octahedron})]'$

(*T*5) $[\sum\partial(\text{pentagon})]'$

Each of these triangulations is obtained by doubling along the boundary a suitable triangulation of a 2-disc. These triangulations of the 2-disc are shown in Fig. 1.

Theorem 1. *Let M be a closed, orientable* 3-*manifold. Then M can be triangulated so that the link of every vertex is in J.*

A *paving* of a manifold by cubes is like a triangulation, but made out of n-dimensional cubes instead of simplexes. In particular two cubes either meet in a face (which is a cube of lower dimension) or are disjoint. Suppose a 3-manifold M is paved by cubes, and e is an edge of a 3-cube C, then the *order* of e is card$\{C' : C'$ is a 3-cube in M and $e \subset C'\}$.

Theorem 2. *Let M be a closed, orientable* 3-*manifold. Then there is a paving of M by cubes such that the order of every edge is* 3, 4 *or* 5. *Furthermore $\sum_3$ and $\sum_5$ are disjoint* 1-*submanifolds.*

Proof of Theorem 2. According to [1] every closed orientable 3-manifold is obtained by taking a suitable covering of S^3 branched over the Borromean rings. Let $B \subset S^3$ denote the Borromean rings. Now the 3-torus $T^3 = S^1 \times S^1 \times S^1$ is obtained by doing 0-framed surgery along each component of B. Thus there is a link L of three components in T with $T^3 - L = S^3 - B$. The components of L can be chosen to be $S^1 \times (\theta_1, \theta_1)$, $\theta_2 \times S^1 \times \theta_2$ and $(\theta_3, \theta_3) \times S^1$

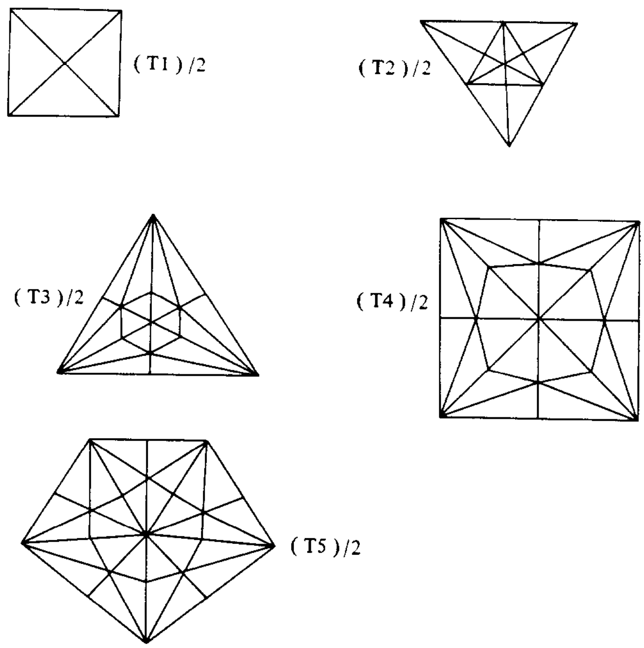

Fig. 1.

where $\theta_i \in S^1$ and all the θ_i's are distinct. T^3 can be represented as the unit cube, A, in $\mathbb{R}^3$ with opposite faces identified. Then $L \cap A$ consists of three disjoint arcs labelled γ_1, γ_2 and γ_3 which are parallel to the coordinate axes x_1, x_2 and x_3 respectively; see Fig. 2. Let P_A be the paving of A by N^3 cubes each of side length N^{-1} (where N is suitably large, e.g. $N = 10$). The corresponding paving of T^3 obtained by identifying opposite faces of A is regular, i.e. every edge has order 4. We may assume that γ_1, γ_2 and γ_3 are disjoint from the 1-skeleton of P_A. For $i = 1, 2, 3$ let N_i be the union of the cubes of P_A which intersect γ_i. We may assume that N_1, N_2 and N_3 are disjoint. In T^3, each N_i glues up to become a solid torus X_i, and X_i is paved with N cubes. This paving of X_i is the product paving of D^2 paved as a single square, and S^1 paved by N intervals. Let $D = \overline{T^3 - (X_1 \cup X_2 \cup X_3)}$. Then ∂D consists of three tori: T_1, T_2 and T_3. Each T_i is paved in an identical way by $4N$ squares. This paving is the product of a paving of S^1 by four intervals with a paving of S^1 by N intervals. Now pave $T^2 \times I$ using the product of the paving on T^2 given by the paving of any T_i, and the paving of I as a single interval. Take three such paved copies of $T^2 \times I$ and glue one copy along $T^2 \times 0$ onto each of T_1, T_2 and T_3 so that

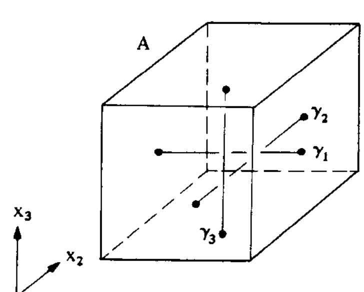

Fig. 2.

the pavings match up along the glueing. Topologically all we have done is to add a collar onto each boundary component of D. Call the resulting manifold E, and the paving of E, P_E. Then $\sum_5(P_E)$ consists of 12 S^1's, four coming from each of T_1, T_2 and T_3. Each of these S^1's on T_i runs parallel to γ_i. $\sum_3(P_E)=\varnothing$.

Let $p\colon M\to S^3$ be a covering of S^3 branched over B. Then identifying S^3-B with $\mathring{E}$ gives a paving $\tilde{P}_E$ of $p^{-1}(E)$ by lifting the paving P_E. Locally the pavings P_E and $\tilde{P}_E$ are the same. To obtain a paving of M, we must attach paved solid tori to each boundary component of $p^{-1}(E)$. Let U be a boundary component contained in $p^{-1}(T_i)$, then the paving of U is obtained by taking some abelian cover of the paving of T_i. The framing of the solid torus Y to be glued to U is determined by $\tilde{\gamma}_u=U\cap p^{-1}(\gamma_i)$ (γ_i may be regarded as a simple closed curve on T_i). Thus U is paved as a product of a paving of $\tilde{\gamma}_u=S^1$ by N_p intervals and a paving of S^1 by $4q$ intervals where the positive integers p, q depend on the abelian cover $U\to p(U)$. Now choose $N=4M$, M an integer. Then pave Y as (S^1 paved by $4q$ intervals) $\times$ (D^1 paved by M_p intervals) $\times$ (D^1 paved by M_p intervals). Then Y may be glued onto U so that the pavings match up. This introduces four components to $\sum_3$, each component is a circle on U which projects down to a meridian of γ_i in T^3. Doing this for each boundary component U gives the required paving of M. □

Remark. This paving puts a Euclidean cone-manifold structure on M, in which the singular locus is a link, and the cone angles are $3\pi/2$ and $5\pi/2$.

Proof of Theorem 1. The cube may be triangulated as [(3-simplex)]′, see Fig. 1($T2$). Each face of the cube is triangulated as cone(∂ square). Thus a triangulation of M is obtained from a paving by cubes using this triangulation for each cube. The link of a vertex at the centre of a cube is ($T2$). The link of a vertex at the centre of a face of a cube is ($T1$). The link of a vertex v at a corner of a cube is ($T3$) if v lies on $\sum_3$, is ($T5$) if v lies on $\sum_5$, and is ($T4$) otherwise. □

Note added in Proof

Kevin Walker has shown independently, using a variant of our method, that 3 vertex link types suffice.

REFERENCE

1. H. M. Hilden, M. T. Lozano and J. M. Montesinos: The Whitehead Link, the Borromean rings and the knot 9_{46} are universal. *Coll. Math.* **34** (1983), 19–28.

Mathematics Institute,
Warwick University,
Coventry, CV4 7AL, U.K.

Department of Mathematics,
Princeton University,
Princeton, NJ 08540, U.S.A.