The universal abelian cover of a link

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1. Introduction

Given a Seifert surface for a classical knot, there is associated a linking form from which the first homology of the infinite cyclic cover may be obtained. This article considers classical links of two components and shows how to obtain a pair of linking forms from the analogue of a Seifert surface. From these the first homology of the universal abelian $(\mathbb{Z} \oplus \mathbb{Z})$ cover is obtained, thus giving a practical method for calculating the Alexander polynomial. Also obtained is a new signature invariant for links. The method generalises to links of any number of components; however this is not done here.

In this paper, unless otherwise stated, a link will mean a piecewise-linear embedding of two oriented circles in the three sphere S^3. The main results are (2.1) and (2.4). The former provides a square matrix presenting the first homology of the cover obtained from the Hurewicz homomorphism of the link complement. The latter gives a signature invariant, obtained from this matrix, which vanishes for strongly slice links. On the way some known results are obtained, namely Torres' conditions on a link polynomial, and a result of Kawauchi and independently Nakagawa on the (reduced) Alexander polynomial of a strongly slice link.

The paper is organised as follows. Section 2 contains the method of obtaining the matrix used in (2.1) and states the main results. The reader not interested in the proofs need read no further. Section 3 contains a proof of (2.1) and also a statement of the Isotopy lemma (3.2). It concludes with a derivation of the Torres conditions. Section 4 is devoted to proving (2.4) and the result on polynomials of slice links.

The material presented here arose out of a study of the method Conway used in [C] to calculate potential functions. A proof of Conway's identities for the Alexander polynomial in a

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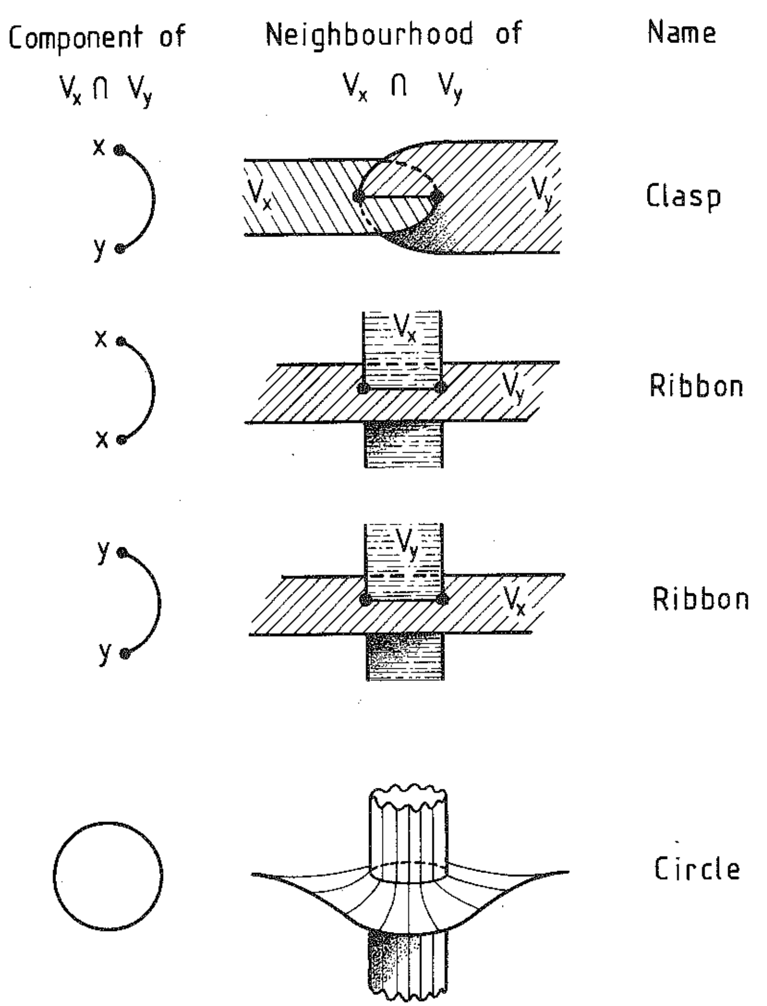

Figure 1.

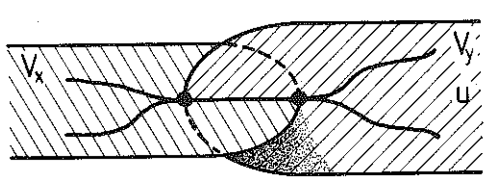

Figure 2. A loop near an intersection

single variable comes from manipulating Seifert surfaces as in [Co], see also Kauffman [K]. The present work enables this proof to be pushed through in the many variable case.

I wish to express my gratitude to Raymond Lickorish and John Conway for their encouragement, and especially to Bill Brakes. This work will form a part of the author's Ph.D. thesis at Warwick University.

2. The Algorithm

In this section it is shown how to obtain a pair of matrices from a link. These matrices are used to describe the first homology of the universal abelian cover of the link, to calculate the Alexander polynomial, and to define a new numerical invariant.

Let V_x and V_y be compact pl embedded surfaces in S^3 and suppose V_x is disjoint from V_y and that V_x meets V_y transversely. The components of $V_x \cap V_y$ are of three types called *clasp* (or C), *ribbon* (or R) and *circle*, see Fig. 1. The 2-complex $S = V_x \cup V_y$ is called a C-*complex* if all intersections are clasps, and R-*complex* if all intersections are ribbon, and an RC-*complex* if ribbon and clasp intersections are allowed. An *orientation* for such a 2-complex is an orientation for each of the component surfaces. The *boundary* of S written ∂S is $(\partial V_x, \partial V_y)$, and the *singularity* of S written $\varepsilon(S)$ is $V_x \cap V_y$.

Given a C-complex S , we define two bilinear forms

$$\alpha, \beta : H_1(S) \times H_1(S) \longrightarrow \mathbb{Z}$$

as follows. A 1-cycle u in S is called a loop if whenever an ant walking along u meets $\varepsilon(S)$, it does so at an endpoint of some component of $\varepsilon(S)$. That is every component of $u \cap \varepsilon(S)$ has a neighbourhood in S of the form shown in Fig. 2. Given two elements of $H_1(S)$ represent them by loops (this may always be done) u and v and define

$$\alpha([u], [v]) = Lk(u^{--}, v),$$

$$\beta([u], [v]) = Lk(u^{-+}, v),$$

where Lk denotes linking number. u^{-+} is the cycle in S^3 obtained by lifting u off S in the negative normal direction off V_x and the positive normal direction off V_y . Similarly u^{--} is obtained by using the negative direction for both V_x and

V_y . That u is a loop ensures this can be done continuously along $\varepsilon(S)$ where the only difficulty might arise.

Choose a basis $\{\gamma_1,\dots,\gamma_g\}$ of $H_1(V_x)$ and a basis $\{\gamma_{g+1},\dots,\gamma_{g+h}\}$ of $H_1(V_y)$ and, identifying via inclusion, extend to a basis $\{\gamma_1,\dots,\gamma_{g+h+k}\}$ of $H_1(S)$. Let A and B be the integral matrices of α and β respectively using this basis.

Suppose now that L is a link of two oriented circles pl embedded in S^3 called L_x and L_y ; we write $L = (L_x, L_y)$. A *C-complex* for L is a connected oriented C-complex S such that $\partial S = L$. (Lemma (3.1) says that any pair of Seifert surfaces for L may be deformed into a C-complex for L). The Hurewicz homomorphism $\pi_1(S^3 - L) \longrightarrow H_1(S^3 - L)$ induces a regular cover $\tilde{X}$ of $S^3 - L$, the universal abelian cover, and $H_1(\tilde{X})$ has a natural Λ-module structure where $\Lambda = \mathbb{Z}[x,y,x^{-1},y^{-1}]$ is the integral group-ring in two variables x and y representing the deck transformations induced by the meridians of L_x and L_y . Define a $(g+h+k) \times (g+h+k)$ matrix J over the field of fractions of Λ by

$$\begin{aligned} J_{r,s} &= 0 && 1 \le r \ne s \le g+h+k \\ J_{r,r} &= (y-1)^{-1} && r \le g \\ &= (x-1)^{-1} && g+1 \le r \le g+h \\ &= 1 && g+h+1 \le r . \end{aligned}$$

THEOREM 2.1. *If* S *is connected,* $H_1(\tilde{X})$ *is presented as a* Λ*-module by the matrix* $J(xyA - A^T - xB - yB^T)$. *In particular this matrix has entries in* Λ .

J. Bailey has obtained a presentation for $H_1(\tilde{X})$ by different means, see [B].

COROLLARY 2.2. *The Alexander polynomial of* L *is*

$$\Delta(x, y) = (y-1)^{-g} (x-1)^{-h} \det (xyA + A^T - xB - yB^T)$$

where $g = 2 \times \text{genus}(V_x)$ $\quad h = 2 \times \text{genus}(V_y)$.

The Alexander polynomial as given in (2.2) may vanish, in which case the determinant of a presentation mstrix for the torsion submodule of $H_1(\tilde{X})$ I call the *reduced Alexander polynomial* written $\Delta_{red}(x, y)$.

A link is *strongly slice* if its components bound disjoint locally flat discs properly embedded in the 4-ball.

THEOREM 2.3. [Kaw], [N]. *If* L *is strongly slice then* $\Delta(x, y) = 0$ *and* $\Delta_{red}(x, y) = F(x, y)\, F(x^{-1}, y^{-1})$ *for some* $F(x, y) \in \mathbb{Z}[x, y]$.

Let ω_1, ω_2 be complex numbers of modulus 1 and let M be the hermitian matrix $(1 + \bar{\omega}_1\bar{\omega}_2)(\omega_1\omega_2 A + A^T - \omega_1 B - \omega_2 B^T)$.

Define
$$\sigma(\omega_1, \omega_2) = \text{signature } (M)$$
$$n(\omega_1, \omega_2) = \text{nullity } (M)$$

THEOREM 2.4. *(i)* σ *and* n *are invariants of* L *provided* $(1 + \bar{\omega}_1 \bar{\omega}_2) \ne 0$ *and* $\omega_1, \omega_2 \ne 1$. *(ii) If* L *is strongly slice then* $\Delta_{red}(\omega_1, \omega_2) \ne 0 \Longrightarrow \sigma(\omega_1, \omega_2) = 0$.

Conway has suggested that it is more natural to consider

$$\text{signature } (\omega_1\omega_2 A + \bar{\omega}_1\bar{\omega}_2 A^T - \omega_1\bar{\omega}_2 B - \bar{\omega}_1\omega_2 B^T)$$

in place of the above. This has the advantage of removing the jump in σ at $1 + \bar{\omega}_1\bar{\omega}_2 = 0$ at the expense of replacing the connection with the Alexander polynomial by his potential function.

3. Homology of the cover

First it is proved that any pair of Seifert surfaces for a link may be deformed into a C-complex. The idea in the proof can be used to prove the isotopy lemma (3.2) for C-complexes which gives a pair of moves by means of which two C-complexes with isotopic component surfaces may be transformed into each other. This result is used in Section 4 to provide a proof of invariance of the signature introduced in Section 2. Theorem 2.1 is proved, and finally a new derivation of the Torres conditions is given.

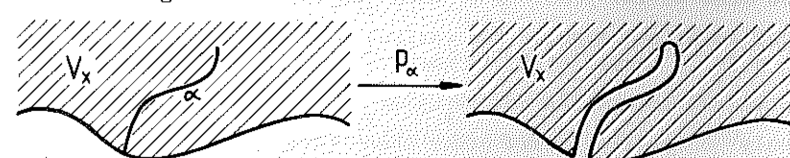

Figure 3. Pushing along an arc.

DEFINITION. Given a surface V with boundary, and an arc $\alpha : [0, 1] \to V$ with $\alpha(0)$ the only point on ∂V, a *push* along α is an embedding $p_\alpha : V \to V$ defined by choosing two regular neighbourhoods of α, N_1 and N_2, meeting ∂V regularly, with $N_1 \subset \text{Int } N_2$. Then $p_\alpha | (V - \text{Int } N_2) = \text{identity}$ and p_α maps N_2 homeomorphically onto $N_2 - \text{Int } N_1$, see Fig. 3. Given a pair of Seifert surfaces V_x and V_y for a link, a push along an arc α in V_x is allowed only if $N_2 \cap \partial V_y = \emptyset$. Similarly for a push in V_y. That is you are not allowed to push one boundary component through the other.

LEMMA 3.1. *Any pair of Seifert surfaces for a link may be isotoped keeping their boundaries fixed to give a C-complex.*

Proof. First make the surfaces transverse, and then remove an outermost-on-V_x circle component of $V_x \cap V_y$ by pushing in along an arc in V_x going from ∂V_x to that circle. This transforms the circle into a ribbon intersection. Continue in this way until all circles have been removed; note that this process does not introduce new circles. Next remove the ribbon intersections, in any order, by pushing along an arc from the boundary of one of the surfaces to the ribbon intersection to replace it by two clasps. The resulting isotopy has moved the link, but only by an ambient isotopy.

ISOTOPY LEMMA 3.2. *Suppose* $S = V_x \cup V_y$ *and* $S' = V'_x \cup V'_y$ *are C-complexes for a link and that* V_x *is isotopic* rel ∂V_x *to* V'_x *and* V_y *is isotopic* rel ∂V_y *to* V'_y. *Then* S *may be transformed into* S' *by the following moves and their inverses:*

(I0) *Isotope* S

(I1) *Add a ribbon intersection between* V_x *and* V_y, *see Fig. 4*

(I2) *Push in along an arc in* V_x *or* V_y.

The idea of the proof is to make the isotopies of V_x and V_y critical level and examine the various possible critical points of $V_x \cap V_y$. Move (I2) is used to change the isotopies so that critical points lie on $\partial V_x \cup \partial V_y$. There are now essentially only 3 possibilities other than (I1) and (I2), and these may be replaced by combinations of (I1) and (I2). The details appear in [Co].

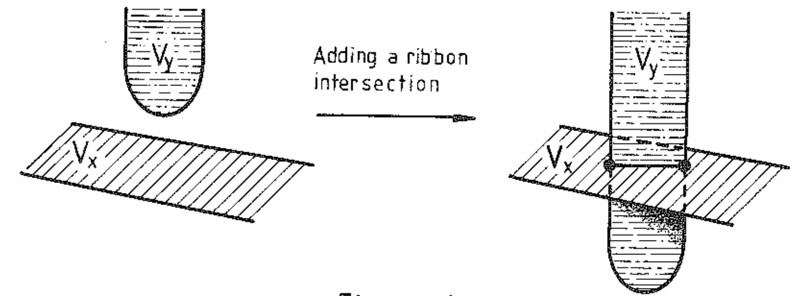

Figure 4.

It is well known that any two Seifert surfaces for a knot are equivalent under adding handles and isotopy. Combining this with the above gives the equivalence relation between C-complexes with the same boundary.

We turn now to homology. Let S be an oriented connected C-complex, then a neighbourhood of a clasp has a cross-section as in Fig. 5. Cut each clasp as shown in Fig. 6(i) to yield an oriented surface V_{--} homotopy equivalent to S by inclusion. Let $V_{--} \times [-1, 1]$ be a bicollar of V_{--} with the $+1$ side as in Fig. 6(i), and let j be the inclusion $V_{--} \longrightarrow V_{--} \times 1 \longrightarrow S^3$. Define a homomorphism i_{--} by requiring the following diagram to commute

$$\begin{array}{ccc} H_1(S) & \xrightarrow{\ i_{--}\ } & H_1(S^3 - S) \\ (\text{incl})_* \uparrow \cong & & \cong \downarrow (\text{incl})_* \\ H_1(V_{--}) & \xrightarrow{\ j_*\ } & H_1(S^3 - V_{--}) . \end{array}$$

Similarly define i_{-+}, i_{+-} and i_{++} by using Figs. 6(ii), (iii) and (iv) respectively. The linking forms

$$\alpha , \beta : H_1(S) \times H_1(S) \to \mathbb{Z}$$

are defined by

$$\alpha(e_1 , e_2) = \text{Lk } (i_{--}e_1 , e_2) ,$$
$$\beta(e_1 , e_2) = \text{Lk } (i_{-+}e_1 , e_2) .$$

Suppose now that S is a C-complex for a link L and let $p : \tilde{X} \to S^3 - L$ be the universal abelian cover. Then $p^{-1}(S)$ separates $\tilde{X}$ into components homeomorphic to $S^3 - S$. If S

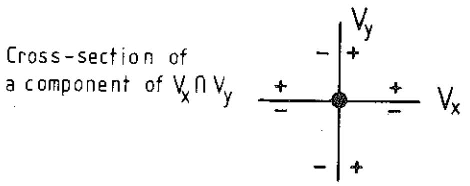

Figure 5.

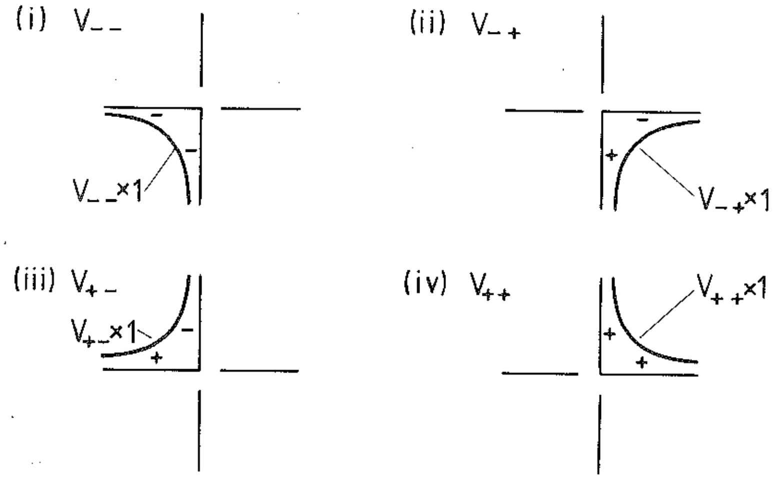

Figure 6.

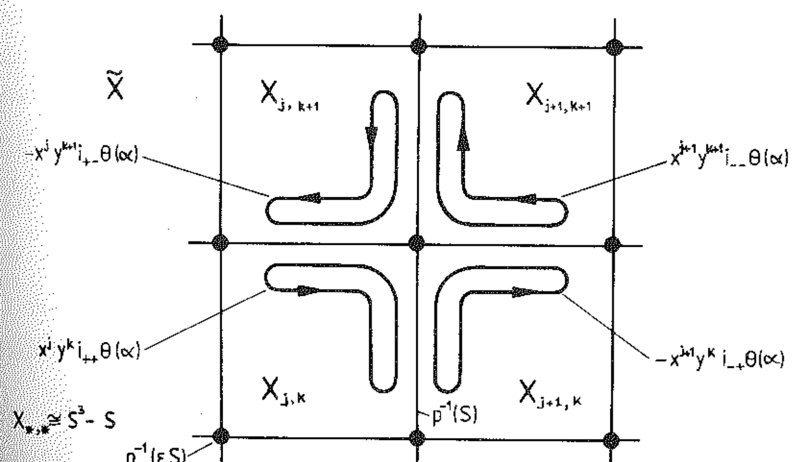

Figure 7.

is connected, $p^{-1}(S)$ is connected, and so $H_1(\tilde{X})$ is generated, as a Λ-module, by (lifts of) $H_1(S^3 - S)$. By duality $H_1(S^3 - S) \cong H_1(S)$ and by Mayer-Vietoris

$$H_1(S) \cong H_1(V_x) \oplus H_1(V_y) \oplus \hat{H}_0(V_x \cap V_y)$$

where $\hat{H}_*$ is reduced homology. Let θ be the isomorphism

$$\theta : H_1(V_x) \oplus H_1(V_y) \oplus \hat{H}_0(V_x \cap V_y) \longrightarrow H_1(S^3 - S) .$$

Regarding $H_1(\tilde{X})$ as a Λ-module generated by $H_1(S^3 - S)$ a complete set of relations is:

For $\alpha \in H_1(V_x)$ $\quad i_{++}\theta(\alpha) = x.i_{--}\theta(\alpha)$

For $\alpha \in H_1(V_y)$ $\quad i_{++}\theta(\alpha) = y.i_{--}\theta(\alpha)$

For $\alpha \in \hat{H}_0(V_x \cap V_y)$ $\quad i_{++}\theta(\alpha) = x.i_{-+}\theta(\alpha) + y.i_{+-}\theta(\alpha) - xy.i_{--}\theta(\alpha)$

The proof of this is by a double application of the Mayer-Vietoris sequence, see [Co]. The only relations it is hard to visualise is the third set which is suggested by Fig. 7.

It is obvious that for $\alpha \in H_1(V_x)$

$$i_{--}\theta(\alpha) = i_{-+}\theta(\alpha) \quad \text{and} \quad i_{+-}\theta(\alpha) = i_{++}\theta(\alpha)$$

Similarly for $\alpha \in H_1(V_y)$

$$i_{--}\theta(\alpha) = i_{+-}\theta(\alpha) \quad \text{and} \quad i_{-+}\theta(\alpha) = i_{++}\theta(\alpha)$$

The relations may thus be rewritten

For $\alpha \in H_1(V_x)$ $\quad (y-1)^{-1}(xy.i_{--} + i_{++} - x.i_{-+} - y.i_{+-})\theta(\alpha) = 0$

For $\alpha \in H_1(V_y)$ $\quad (x-1)^{-1}(xy.i_{--} + i_{++} - x.i_{-+} - y.i_{+-})\theta(\alpha) = 0$

For $\alpha \in \hat{H}_0(V_x \cap V_y)$ $\quad (xy.i_{--} + i_{++} - x.i_{-+} - y.i_{+-})\theta(\alpha) = 0$

Using the basis of $H_1(S)$ given in Section 2 proves Theorem 2.1. □

THEOREM 2.1. (Torres). *The Alexander polynomial of a link* L *of two components satisfies*

(i) $\Delta(x, y) \doteq \Delta(x^{-1}, y^{-1})$

(ii) $\Delta(x, 1) \doteq \Delta(x) . (1 - x^{|\ell|})/(1 - x)$

where $\doteq$ *denotes equal up to multiplication by* $\pm x^r y^s$ *, and* ℓ *is the linking number of the two components.*

Torres' proof [Tor] made use of Fox's calculus. Here is a proof using theorem 2.1; (i) is immediate. For (ii), using the basis of $H_1(S)$ given in Section 2, the linking matrices A, B have the form

$$A = \begin{array}{c} H_1(V_x) \\ H_1(V_y) \\ H_1(\varepsilon S) \end{array} \overset{\begin{array}{ccc} H_1(V_x) & H_1(V_y) & H_1(\varepsilon S) \end{array}}{\begin{bmatrix} C & D & E \\ D^T & F & G \\ H & J & K \end{bmatrix}} \qquad B = \begin{bmatrix} C & D & E \\ D^T & F^T & J^T \\ H & G^T & L \end{bmatrix}$$

Restrict the basis of $H_1(S)$ by requiring that the loops representing the basis of $\hat{H}_0(\varepsilon S)$ are disjoint from those representing the basis of $H_1(V_y)$. This gives $G = J^T$, and

$$(x\,,1) = \det \begin{bmatrix} xC - C^T & (x-1)D & xE - H^T \\ 0 & F - F^T & 0 \\ 0 & 0 & x(K-L) + (K-L)^T \end{bmatrix}$$

$$= \det(xC - C^T)\,\det(F - F^T)\,\det(xM + M^T)$$

where $M = K - L$.

Now C is a Seifert matrix for the x-component so $\det(xC - C^T)$ = Alexander polynomial for x-component. F is a Seifert matrix for the y-component so $\det(F - F^T) = 1$. Finally it is shown below that $\det(xM + M^T)$ depends only on the linking number of the two components, and evaluating for a simple link gives $(1 - x^{|\ell|}) / (1 - x)$.

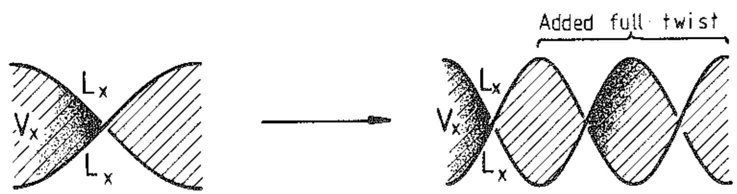

Figure 8.

It is well known that any knot may be changed into the unknot by changing crossovers. This is easily extended to: any link may be changed into the simple link of the same linking number by changing crossovers at which both strings belong to the same component. Let L' be the link L with a single such crossover changed. Choose a C-complex S for L; then a C-complex S' for L' is obtained by adding a full twist to S next to the changed crossover, see Fig. 8. The matrix M is the matrix of the form $(\alpha - \beta)$ restricted. Adding a twist to S changes α and β by adding to each a (symmetric) form γ. Thus $(\alpha - \beta)$ is unchanged, and so $\det(xM + M^T)$ is unchanged, completing the proof. □

The Torres conditions are known to characterise link polynomials when the linking number of the two components is 0 or 1, see [B], [L]. On the other hand, Hillman has shown in [H] that they are not sufficient for linking number 6.

4. Cobordism Invariance of Signature

First, linking forms are introduced for an RC-complex and then the Isotopy lemma is used to show signature is a link invariant. Next is a well known result on the rank as a Λ-module of $H_1(\tilde{X})$. This is used in the proofs of theorems 2.3 and 2.4 for ribbon links. Finally the proofs are extended to slice links. The method of proof of 2.4 is similar to that used by Murasugi in [M] and Tristram in [T].

DEFINITION. Let F be an RC-complex and choose a ribbon intersection r in F. Remove from F a small disc centred on the mid point of r and lying in the component surface of F in which the endpoints of r are interior points, see Fig. 9. Let S be a C-complex obtained from F by the above construction at each ribbon intersection, then S is said to be *obtained by puncturing* F. The linking forms α, β for S are *associated to* F.

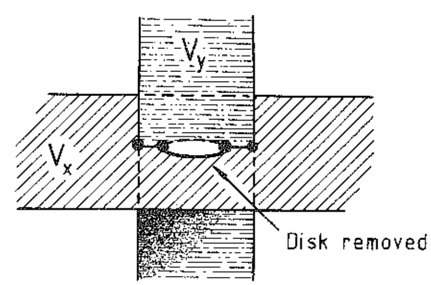

Figure 9.

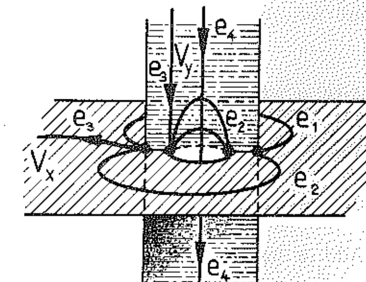

Figure 10.

Let F be an RC-complex and F_1 an RC-complex obtained from F by pushing in along an arc α to convert some ribbon intersection r into two clasps. Let S be a C-complex obtained by puncturing F, and S_1 a C-complex obtained by puncturing F_1; we may suppose $S_1 = S \cap F_1$. Choose a neighbourhood U of r in S of the form shown in Fig. 10. Pick loops $\{e_1, \ldots, e_n\}$ representing a basis of $H_1(S)$ such that e_i misses U for $i > 4$ and $e_i \cap U$ is as shown in Fig. 10 for $i \le 4$. The loops $\{e_2, \ldots, e_n\}$ represent a basis of $H_1(S_1)$. The matrix $(1 + \bar{\omega}_1 \cdot \bar{\omega}_2)(\omega_1\omega_2 A + A^T - \omega_1 B - \omega_2 B^T)$ for S using this basis is

$$Q = \left[\begin{array}{cc|ccc} 0 & 0 & \theta & \phi & -0- \\ 0 & 0 & -\bar{\omega}_1\theta & -\bar{\omega}_1\phi & -0- \\ \hline \bar{\theta} & -\omega_1\bar{\theta} & & & \\ \bar{\phi} & -\omega_1\bar{\phi} & & * & \\ | & | & & & \\ 0 & 0 & & & \\ | & | & & & \end{array}\right] \qquad \begin{array}{l} \theta = \omega_1 + \bar{\omega}_2 \\ \\ \phi = (1 - \omega_2)\,\theta \end{array}$$

Let Q_1 be the matrix obtained from Q by omitting the first row and column, so it is the corresponding matrix for S_1. Then

$$\text{Signature }(Q) = \text{Signature }(Q_1)$$

$$\text{nullity }(Q) = \text{nullity }(Q_1) + 1 .$$

By the remark after the Isotopy lemma, in order to complete the proof of theorem 2.4(i) it suffices to calculate the effect of:

(I1) Add a ribbon intersection between V_x and V_y

(I2) Push in along an arc

(H_x) Add a handle to V_x

(H_y) Add a handle to V_y

The above calculation shows (I2) has no effect on signature.

If P is an hermitian matrix and

$$P_1 = \left[\begin{array}{c|cc} P & v & 0 \\ \hline v^{\dagger} & u & w \\ -0- & \bar{w} & 0 \end{array}\right]$$

where w is a non-zero complex number, u a real, v a complex column vector and $v^{\dagger}$ its conjugate transpose, then P_1 is called an *elementary enlargement* of P. The effect of (I1), (Hx) and (Hy) on Q is an elementary enlargement with for

(I1) $w = 1 + \bar{\omega}_1 \cdot \bar{\omega}_2$ or $\omega_1 + \omega_2$

(Hx) $w = |1 + \bar{\omega}_1 \cdot \bar{\omega}_2|^2 \, (1 - \bar{\omega}_2)$

(Hy) $w = |1 + \bar{\omega}_1 \cdot \bar{\omega}_2|^2 \, (1 - \bar{\omega}_1)$

Thus signature and nullity are independent of the C-complex used for a link, provided $w \neq 0$ in the above, thus proving 2.4(i). □

LEMMA 4.1. *Let* $L = (L_x, L_y)$ *be a link of two components. Let* M *be the* Λ*-module which is the first homology of the universal abelian cover. Then rank* $(M) = 0$ *or* 1.

Proof. If $\Delta_L(x, y) \neq 0$ then M is a torsion module so rank $M = 0$. Otherwise notice that $\Delta_L(1, 1) = \mathrm{Lk}(L_x, L_y) = 0$. Choose a C-complex S for L and let S_1 be obtained from S by removing one clasp so that S_1 is a C-complex for a link L_1 with $\Delta_{L_1}(1, 1) = \pm 1$. Thus the module for L_1, M_1 is a torsion module. Putting back the clasp corresponds to adding a single row and column to a presentation matrix for M_1 giving a presentation matrix for M. This latter matrix thus has nullity (equal to rank M) at most 1. □

DEFINITION. Let D_x and D_y be 2-discs immersed in S^3 without triple points, with $\partial D_x \cap \partial D_y = \emptyset$, and so that the only intersections self and mutual are of ribbon type. Then $(\partial D_x, \partial D_y)$ is called a *ribbon link*.

Suppose that L is a ribbon link, the immersed discs D_x and D_y may be cut along their self intersections to give

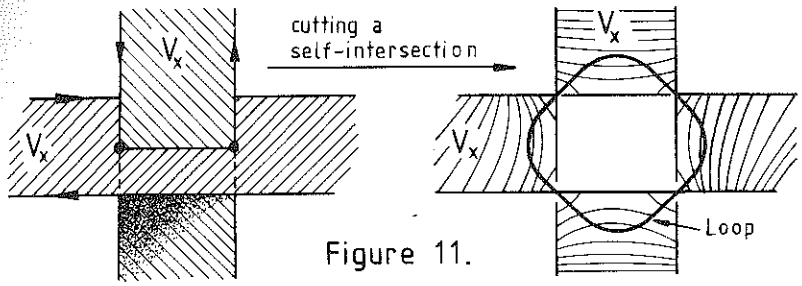

Figure 11.

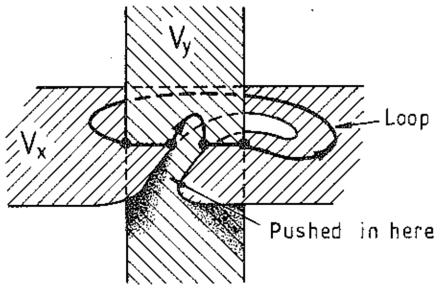

Figure 12.

orientable surfaces V_x and V_y with $L = (\partial V_x , \partial V_y)$, see Fig. 11. $F = V_x \cup V_y$ is an R-complex for L ; push in to get a C-complex S . Pick loops in S representing an ordered basis of $H_1(S)$ as follows:

(1) For each self intersection of D_x pick a loop going around that intersection-cut-open in V_x , see Fig. 11.

(2) Do the same for V_y .

(3) For each ribbon intersection of F pick a loop in S going through the two resulting clasps in S , see Fig. 12.

(4) Complete the basis by picking a further n loops.

The dimension of $H_1(S)$ is easily seen to be $2n + 1$. The matrices A and B using this basis have the shape

$$\begin{array}{c} \\ n+1 \\ n \end{array}\begin{array}{c} \begin{array}{cc} n+1 & n \end{array} \\ \left[\begin{array}{c|c} 0 & * \\ \hline * & * \end{array}\right] \end{array}$$

Thus $\Delta(x , y) = 0$, and by lemma 4.1

$$\text{nullity } (xyA + A^T - xB - yB^T) = 1 .$$

Using the method of proof for 4.1 we may assume the first row and column of A and B are zero. Then it follows that

$$\Delta_{red}(x , y) = F(x , y) . F(x^{-1} , y^{-1})$$

for some $F(x , y) \in \mathbb{Z}[x , y]$, and also that

$$\Delta_{red}(\omega_1 , \omega_2) \neq 0 \Rightarrow \sigma(\omega_1 , \omega_2) = 0 .$$

This proves 2.3 and 3.4(i) for L a ribbon link.

DEFINITION. $L = (L_x , L_y)$ is a link in S^3 and $U_1 , \ldots , U_n$ are unknots in S^3 separated from L and from each other by 2-spheres. $b_1 , \ldots , b_n$ are bands, that is disjoint embeddings $I \times I \hookrightarrow S^3$ with $b_i(0 \times I) \subset U_i$ and $b_i(1 \times I) \subset L$. The link $L' = (L'_x , L'_y)$ defined by

$$L' = L \cup \cup U_i - \cup b_i(\partial I \times I) \cup \cup b_i(I \times \partial I)$$

is said to be *obtained from* L *by band-summing an unlink.*

It is well known that a knot is slice if and only if it may be made into a ribbon knot by band-summing an unlink, see for example Tristram [T]. It follows that a link is strongly slice if and only if it may be made into a ribbon link by band-summing an unlink.

Let $L = (L_x , L_y)$ be a link and $L' = (L'_x , L'_y)$ be obtained from L by band-summing an unlink. Choose a C-complex S for L and discs $D_1 , \ldots , D_n$ spanning the unlink $U_1 , \ldots , U_n$ disjoint from each other and from S , and such that $S \cup D_i$ is transverse to the bands used in band-summing $\cup b_i$. From the RC-complex $S \cup \cup D_i \cup \cup b_i$ form a C-complex S' for L' in a manner similar to that used for making the C-complex for a ribbon link above.

Let A , B be the linking matrices for S . Using the same methods as in the proof for ribbon links above we obtain linking matrices A_1 , B_1 for S' of the shape

$$A_1 = \begin{bmatrix} 0 & C & 0 \\ D & * & * \\ 0 & * & A \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 & E & 0 \\ F & * & * \\ 0 & * & B \end{bmatrix}$$

The matrices C, D, E, F being square and of the same size. It follows that

$$\Delta_{L'}(x, y) = F(x, y) \, . \, F(x^{-1}, y^{-1}) \, . \, \Delta_L(x, y)$$

where $$F(x, y) = \det\,(xyC + D' - xE - yF') \, .$$

Also $$F(\omega_1, \omega_2) \neq 0 \Rightarrow \sigma_{L'}(\omega_1, \omega_2) = \sigma_L(\omega_1, \omega_2)$$

$$n_{L'}(\omega_1, \omega_2) = n_L(\omega_1, \omega_2)$$

This completes the proof of 2.3 and 2.4(ii).

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