Remarks on the A-polynomial of a Knot.

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Abstract

This paper reviews the two variable polynomial invariant of knots defined using representations of the fundamental group of the knot complement into $SL_2\mathbb{C}$. The slopes of the sides of the Newton polygon of this polynomial are boundary slopes of incompressible surfaces in the knot complement. The polynomial also contains information about which surgeries are cyclic, and about the shape of the cusp when the knot is hyperbolic. We prove that at least some mutants have the same polynomial, and that most untwisted doubles have non-trivial polynomial. We include several open questions.

Keywords: Knot, polynomial, boundary slope, Newton polygon, surgery.

1 Introduction

In this paper we review the two-variable A-polynomial for a knot which was introduced in [3]. Many interesting features concerned with the geometry and topology of the knot complement are reflected in this polynomial. For example the boundary slopes of some, or possibly all, of the incompressible embedded surfaces are coded by it. In the case that the knot is hyperbolic, information about the cusp shape is in this polynomial. Under certain conditions one may deduce that a knot has property P from this polynomial, and more generally which surgeries are cyclic. This polynomial seems to be unconnected with the various combinatorially defined invariants descended from the Jones polynomial. In what follows we survey some known results, discuss some new ones [(6.3),(7.1),(7.3),(8.2),(9.4), (9.6),(11.3)], and pose some open questions concerning this polynomial.

2 Definition of the A-polynomial

We will give a definition of the A-polynomial slightly different from that in [3]. But first some background. Due to Thurston's pioneering work we know that a knot complement, X, has a hyperbolic structure if and only if it is not a satellite or a torus knot. Now a hyperbolic structure determines an action of $\pi_1 X$

by isometries on hyperbolic 3-space $\mathbb{H}^3$. Actually this representation is only determined up to conjugacy corresponding to a choice of a base point and frame in $\mathbb{H}^3$. Now $Isom_+\mathbb{H}^3 \cong PSL_2\mathbb{C} = SL_2\mathbb{C}/\pm I$ thus the hyperbolic structure determines a homomorphism

$$\rho_0 : \pi_1 X \longrightarrow PSL_2\mathbb{C}.$$

It is know that this lifts to a representation, also denoted ρ_0, into $SL_2\mathbb{C}$ and the lifts are parameterized by $H^1(X, \mathbb{Z}_2) = \mathbb{Z}_2$.

Thurston showed that ρ_0 can be deformed to give a one complex parameter family of non-abelian representations of $\pi_1 X$ into $SL_2\mathbb{C}$ all inequivalent up to conjugacy. However even non-hyperbolic knots may have such families of representations, for example it is easy to see that torus knots do, and perhaps all knots do. Now a representation can be thought of as an assignment of matrices to each element of a generating set of $\pi_1 X$ and thus a point in $\mathbb{C}^{4n}$ where n is the number of generators. The relations in the group place restrictions on which points correspond to representations. In fact a relation requires that a certain product of matrices equals the identity and this in turn imposes four polynomial equations between the matrix entries. Thus the subset of $\mathbb{C}^{4n}$ corresponding to representations is precisely the set of common zeroes of a finite set of polynomials, and is thus an affine algebraic set which is called the *representation variety* of the knot complement. Actually this set is not usually a variety in the sense of algebraic geometry since it is not irreducible but typically contains various components of different dimensions. In section 8 we show that there are (hyperbolic) knots with arbitrarily large dimensional components.

In particular every knot group abelianizes to $\mathbb{Z}$ and thus every representation of $\mathbb{Z}$ into $SL_2\mathbb{C}$ induces a representation of the knot group. These are called the *abelian representations* of the knot. They carry no useful information. The abelian representations form a component of the representation variety isomorphic to $SL_2\mathbb{C}$.

Invariants of the representation variety are invariants of the knot. For example the number of components of the variety, the dimension of the variety, its (co)homology are all subtle invariants. In general the topology of this variety is likely to be complicated, however there is some extra structure we can exploit to produce something more manageable. The longitude and meridian of the knot provide a way of projecting the representation variety into $\mathbb{C}^2$ and the image is easier to understand. The image of a component of the representation variety is either a single point, or else is a complex curve minus finitely many points, see 10.1. In the latter case, the curve is the zero set of some irreducible polynomial in two variables which is unique up to scaling. It turns out that this polynomial carries a lot of information about the topology and geometry of the knot complement.

Let M be a compact 3-manifold with boundary a torus T. Pick a basis λ, μ of $\pi_1 T$ which we will refer to as the longitude and meridian. Consider the subset R_U of the affine algebraic variety $R = Hom(\pi_1 M, SL_2\mathbb{C})$ having the property that for ρ in R_U that $\rho(\lambda)$ and $\rho(\mu)$ are upper triangular. This is an algebraic subset of R since one just adds equations stating that the bottom left entries in certain matrices are zero. Furthermore, since every representation can be conjugated to have this form, we are not losing any information. The reason for considering only this subset of the representation variety is that it makes it technically easier to define the eigenvalue map. There is a well-defined *eigenvalue map*

$$\xi \equiv (\xi_\lambda \times \xi_\mu) : R_U \longrightarrow \mathbb{C}^2$$

given by taking the top left entries of $\rho(\lambda)$ and $\rho(\mu)$ (which are thus eigenvalues of $\rho(\lambda)$ and $\rho(\mu)$). Thus the closure of the image $\overline{\xi(C)}$ of an algebraic component C of R_U is an algebraic subset of $\mathbb{C}^2$. In the case that $\overline{\xi(C)}$ is a curve, there is a polynomial F_C, unique up to constant multiples, which defines this curve. The product over all components of R_U having this property of the F_C is the A-polynomial. It is shown in [3] that the constant multiple may be chosen so that the coefficients are integers. The additional requirement that there is no integer factor of the result means that the A-polynomial is defined up to sign.

Thus, with finitely many exceptions (see section 5), a pair of complex numbers L, M satisfy $A(L,M) = 0$ if and only if there is a representation ρ for which:

$$\rho(\lambda) = \begin{pmatrix} L & * \\ 0 & L^{-1} \end{pmatrix} \qquad \rho(\mu) = \begin{pmatrix} M & * \\ 0 & M^{-1} \end{pmatrix}.$$

We have adopted a different convention to [3] in that we count curves with multiplicities here, so that the A-polynomial may have repeated factors. We often ignore the abelian representations which, for the complement of a knot in a homology sphere, contributes a factor of $L - 1$.

3 Calculations

Calculations are ultimately based on using polynomial resultants, which we briefly review. Let I be any ideal in $C[x_1, x_2, \cdots, x_n, y]$, we call the set of common zeroes

$$V = \{\ (X_1, X_2, \cdots, X_n, Y) \in \mathbb{C}^{n+1}\ :\ \forall p \in I\ \ p(X_1, X_2, \cdots, X_n, Y) = 0\ \}$$

the *variety* defined by the ideal I. In algebraic geometry it is conventional to require that I be irreducible, but we will not do this. Consider coordinate projection

$$\pi : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^n$$

onto the first n coordinates. The image πV has closure (in the classical topological sense, and also in the Zariski topology) a subset $\overline{\pi V}$ of $\mathbb{C}^n$ which is a variety defined by some ideal J.

In general $\overline{\pi V}$ contains points not in πV, for example consider the variety in $\mathbb{C}^2$ defined by $xy = 1$. The projection of this variety onto the first coordinate is $\mathbb{C} - 0$. Since varieties are closed subsets of Euclidean space, in general a point of $\overline{\pi V} - \pi V$ is the limit of the image of a sequence of points in V going to infinity. In the case that $\overline{\pi V}$ is a curve $\overline{\pi V} - \pi V$ consists of finitely many points.

In our context, V will typically be a (projection of a) representation variety, and this naturally leads to the consideration of sequences of representations which are going to infinity. Generally, such a sequence yields an incompressible surface in the knot exterior, a situation discussed in section 5. Thus points of $\overline{\pi V} - \pi V$ give incompressible surfaces.

Choose a set of generators for the ideal I, and for each pair form the resultant polynomial. Assume that for each such pair that case (2) of the theorem 3.1 below does not happen. Then this set of resultant polynomials generates the ideal J. This is based on the following well known property of resultants.

Theorem 3.1 *[16],[13] If f, g are polynomials in variables $x_1, x_2, \cdots, x_n, y$ then the resultant polynomial h is obtained from f, g by eliminating y and is a polynomial in the variables $x_1, x_2, \cdots, x_n$ with the following property. If $X_1, X_2, \cdots, X_n$ are complex numbers for which $h(X_1, X_2, \cdots, X_n) = 0$ then one of two possibilities happens*

case 1 *there is a complex number Y such that*

$$f(X_1, X_2, \cdots, X_n, Y) = 0 = g(X_1, X_2, \cdots, X_n, Y).$$

case 2 *the coefficients of the highest power of y in both f and g simultaneously vanish for the specialization $X_1, X_2, \cdots, X_n$.*

Suppose that case 2 does not happen, then the theorem implies that the projection of the variety in $\mathbb{C}^{n+1}$ where f, g both vanish has closure in $\mathbb{C}^n$ equal to the variety where h vanishes.

To calculate the A-polynomial, one starts with a finite presentation of the fundamental group in which the meridian is one of the generators and assigns matrices to each of the generators. It is computationally convenient to use the observation in section 2 that one may conjugate so that the meridian is upper triangular. This makes one of the variables in the meridian matrix zero. The entries in the matrices are variables $x_1, x_2, \cdots, x_n, M$ where M is the upper left entry in the meridian matrix. The relations give polynomials whose vanishing is equivalent to the requirement that the matrices determine a representation. Thus these polynomials define the representation variety. Now one adjoins a new variable L together with a new polynomial relation:

$$L - \ell(x_1, x_2, \cdots, x_n) = 0.$$

The polynomial ℓ is the upper left entry in the matrix obtained by multiplying out the matrices corresponding to the word in the generators that gives the longitude. Thus it is an eigenvalue of the longitude. One now views this enlarged set of polynomials as defining the representation variety as the subset $V \subset \mathbb{C}^{n+2}$.

The goal now is to find the image of V under coordinate projection into $\mathbb{C}^2$ given by the coordinates L, M. More precisely one wants to find, for each component of V, the irreducible polynomial in two variables L, M which defines the image curve (in the case that the image has complex dimension one). One uses resultants to do this, repeatedly eliminating variables until only L, M remain.

The calculations have been done for the knots from 3_1 up to 8_2 together with $8_5, 9_1, 9_2, pretzel(-2, 3, 7)$, the untwisted double of the trefoil knot, and a few others. See [3]. For example

$$A_{3_1} = 1 + LM^6 \qquad A_{4_1} = -M^4 + L(M^8 - M^6 - 2M^4 - M^2 + 1) - L^2M^4.$$

Often the calculation of a resultant in the above process will take too long. One may try to manipulate the defining polynomials in an effort to shorten them and sometimes this helps. For example, instead of setting a product of matrices equal to the identity and obtaining polynomial equations from this, it is usually better to move half the matrices to the other side of the equals sign so that one is equating two words of half the length. This will usually reduce the degree of the polynomials one obtains by a factor of two in each variable. Also the order in which one eliminates variables can affect the computation time. It seems better to eliminate the variables which occur with lowest degree first.

In view of the fact that the coefficients in the corners of the Newton polygon (defined below) are ± 1, see (11.3), it suffices to do these calculations mod 2 if the goal is to find the Newton polygon and hence the boundary slopes.

4 Basic Properties

Standard questions about knot invariants are: when is the invariant non-trivial, how good is it at distinguishing knots, how does it behave under connect sum, what relations does it have to other invariants, what values can the invariant take. The A-polynomial of the unknot is $L-1$ due to the abelian representations. We call an A-polynomial non-trivial if it is distinct from that of the unknot.

Proposition 4.1 *[3] The A polynomial of a hyperbolic knot, or a torus knot is non-trivial.*

Proposition 4.2 *[3]*
(1) For every knot $A(L,M) = \pm A(L^{-1}, M^{-1})$ *up to powers of* L *and* M*.*
(2) If K *is a knot in a homology sphere, then* $A(L,M)$ *involves only even powers of* M*.*
(3) Reversing the orientation of K *does not change* A*.*
(4) Reversing the orientation of the ambient manifold changes $A(L,M)$ *to* $A(L, M^{-1})$*.*

Proof. These statements are immediate consequences of the fact that (except finitely often) zeroes of the A-polynomial are the eigenvalues of the longitude and meridian for some representation. □

Proposition 4.3 *If* K_1 *and* K_2 *are two knots and* $K_1 \# K_2$ *is their connect sum then* $A_{K_1\#K_2}$ *is divisible by* $A_{K_1}A_{K_2}(L-1)^{-1}$*.*

Proof. Let X_i be the exterior of K_i and X the exterior of $K_1\#K_2$ then $\pi_1 X$ surjects onto $\pi_1 X_1$ thus a representation ρ_1 of $\pi_1 X_1$ pulls back to a representation ρ of $\pi_1 X$. Note that ρ restricts to an abelian representation on the subgroup $\pi_1 X_2$ of $\pi_1 X$. Let ξ be the eigenvalue map for K and ξ_1 that for K_1. Then $\xi\rho = \xi_1\rho_1$ and it follows that A_{K_1} divides $A_{K_1\#K_2}$.

The reason for the $(L-1)^{-1}$ factor is that our argument glues an arbitrary representation of one knot complement to an abelian representation of the other, and this method counts the abelian representations of the composite knot twice. □

The reef and granny knots have different A-polynomials, and provide an example where equality does not hold in the above. One can say rather more than this. Given two representations ρ_1, ρ_2 of $\pi_1 X_1, \pi_1 X_2$ respectively there is a representation ρ of $\pi_1 X$ which restricts to ρ_i if and only if ρ_1, ρ_2 agree on the meridian. Thus if $\xi(\rho_i) = (L_i, M)$ then $\xi(\rho) = (L_1 L_2, M)$. In particular, if both knots have no factors of the A polynomial which do not involve L, then the number of factors of A_K is at least the product of the number of factors of A_{K_1} and A_{K_2}.

Proposition 4.4 *Suppose that in the complement of a knot K every closed incompressible surface is a boundary parallel torus. Suppose that L, M are complex numbers with $L^q M^p = \pm 1$ and that $A_K(L, M) = 0$. Suppose that either $L \neq \pm 1$ or $M \neq \pm 1$. Then Dehn filling K along $\lambda^q \mu^p$ produces a closed manifold with non-cyclic fundamental group.*

Proof. If $A_K(L, M) = 0$ then either there is a representation ρ with L, M as eigenvalues or there is a closed incompressible surface which is not boundary parallel, see section 5. In the first case, the element $\lambda^q \mu^p$ of $\pi_1(T)$ has eigenvalues $L^q M^p = \pm 1$.

Suppose that $\rho(\lambda^q \mu^p)$ is parabolic, then it commutes with both $\rho(\lambda)$ and $\rho(\mu)$ and one of these is loxodromic which contradicts parabolicity. Hence $\rho(\lambda^q \mu^p) = \pm Id$ thus the representation, thought of as mapping into $PSL_2\mathbb{C}$, kills $\lambda^q \mu^p$. Thus we have a representation of the Dehn-filled manifold. Typically this representation will have non-cyclic image, for example if $p \neq 0$ and the eigenvalue M is not a root of unity. In general one argues that there is another representation with the same eigenvalues L, M and with non-cyclic image, see [9] or [2] proposition (2.1) for details. □

In his thesis [19], Shanahan gives a necessary condition based on the Newton polygon for a Dehn-filling to give a manifold with cyclic fundamental group. Shanahan defines, for each rational direction, a *width* of the Newton polygon in that direction. For a cyclic filling, this width must be minimal over all possible directions. He also shows that there are at most three such minimal-width directions, in agreement with the cyclic surgery theorem [8].

In section (2.8) of [3] a somewhat stronger version of the following is incorrectly asserted.

Corollary 4.5 *Suppose that in the complement of a knot K every closed incompressible surface is a boundary parallel torus, then $A(L, 1) = \pm(L+1)^\alpha (L-1)^\beta L^\gamma$.*

Proof. By 4.4, we have that $A(L, 1) = C(L+1)^\alpha (L-1)^\beta L^\gamma$, and (11.3) implies that $C = \pm 1$. □

Question 4.6 *Do the integers α, β, γ have any topological significance?*

Question 4.7 *Is the hypothesis on closed incompressible surfaces necessary in (4.4) and (4.5)?*

Proposition 4.8 *[3] A knot has property P if there is no closed incompressible surface in its exterior and the degree of the A-polynomial in M is more than twice the degree in L.*

For example either of the above suffice to show that many knots (eg. the figure eight knot) have property P. There is a relation between the A-polynomial and the Alexander polynomial but as it is somewhat technical we refer the reader to [3]. However it is shown in [7] that if the Alexander polynomial of a knot is non-trivial then the A-polynomial is non-trivial.

Question 4.9 *Is there a crossing change formula for the A-polynomial? guess: no.*

5 Boundary Slopes

In [3] it is shown that the slopes of edges of the Newton polygon of the A-polynomial are the boundary slopes of incompressible surfaces in the knot complement. We will now give a brief review of this. If $p(x,y)$ is a polynomial in two variables the *Newton polygon* of p is the convex hull of the finite set of points in the plane:

$$Newt(p) = \{\ (i,j)\ :\ \ the\ coefficient\ of\ x^i y^j\ in\ p(x,y)\ is\ not\ zero\ \ \}.$$

The sides of the Newton polygon describes the geometry of the curve C defined by $p = 0$ when at least one of the coordinates is near zero or infinity. To see this, suppose that (X,Y) is a point on C and that at least one of the variables, for example X, has large modulus. The polynomial p is a linear combination of monomials of the form $x^a y^b$ and the logarithm of the modulus of this monomial at (X,Y) is $\phi(a,b) = a\log|X| + b\log|Y|$. Since $p(X,Y)$ vanishes there cannot be a single monomial which is far larger in modulus than all the other monomials. One thinks of ϕ as a linear map defined on $\mathbb{R}^2$ and in particular on the Newton polygon. The level sets of ϕ are straight lines with slope $-\log|X|/\log|Y|$. By the previous discussion there is a side of $Newt(p)$ which is nearly parallel to these lines. Similar considerations hold if X is very close to zero. From this one sees that to each topological end of the curve C one may assign an edge e of $Newt(p)$ consisting of those terms of p of approximately largest modulus for points on C near the given end.

Let X be the exterior of a knot, $T = \partial X$, and $R = Hom(\pi_1 X \longrightarrow SL_2\mathbb{C})$ the representation variety. A sequence ρ_n in R is said to *blow up* if there is an element α in $\pi_1 X$ such that $trace(\rho_n\alpha) \to \infty$. We will assume that all these representations lie on a curve in R. In this situation, after passing to a subsequence, the representations converge in a certain sense to an action on a simplicial tree [9], for a more geometric proof see [1] and also [4]. There are two possibilities.

Type 1 There is an element α associated to the blow up in $\pi_1 T$ such that $trace(\rho_n\alpha) \to \infty$. In this case there is a unique, up to taking inverses, primitive element β in $\pi_1 T$ such that $trace(\rho_n\beta)$ remains bounded as $n \to \infty$. Then β is parallel to the boundary components of a properly embedded, non-boundary parallel incompressible surface in X. Thus β is a *boundary slope.*

Type 2 For every α in $\pi_1 T$ we have that $trace(\rho_n\alpha)$ remains bounded. In this case there is a closed incompressible surface in X.

We briefly explain the connection between sides of the Newton polygon and boundary slopes. Suppose that ρ_n blows up and that both $M, L \to \infty$. There is an edge, e, of $Newt(A(L,M))$ containing the terms of largest magnitude. If $L^a M^b$ and $L^c M^d$ both lie on e then $L^a M^b \approx L^c M^d$ and so $M^{b-d} L^{a-c} \approx 1$. Thus $\rho_n(\mu^{b-d}\lambda^{a-c})$ has bounded eigenvalues and therefore trace as $n \to \infty$ and thus $\mu^{b-d}\lambda^{a-c}$ is the boundary slope. The slope of this curve on T is $\frac{b-d}{a-c}$ which equals the slope of e. A similar analysis applies if $M, L \to 0, \infty$.

In section 8 we show that type 2 degenerations can occur. We saw in section 3 that ξC is a curve minus finitely many points. These missing points are due to blow ups. To see this, consider a sequence ρ_n such that $\xi\rho_n$ converges to a point (L_0, M_0) in $\overline{\xi C} - \xi C$. The sequence ρ_n must be going to infinity in the representation variety otherwise there would be an accumulation point which maps by ξ onto (L_0, M_0). It can be shown (see [4] corollary (2.1)) that this means the sequence is blowing up. The traces of the longitude and meridian remain bounded if and only if both L_0, M_0 are non-zero. In this case we say that L_0, M_0 is a *hole.*

Question 5.1 *Do holes exist? ie. is there a knot K in S^3 and a point (L_0, M_0) on $A_K(L,M) = 0$ with non-zero coordinates and a component C of R_U such that $\xi(C)$ contains a deleted neighborhood of (L_0, M_0) but not (L_0, M_0).*

We will call the boundary slope of an incompressible surface *strongly detected* if it appears in a type 1 degeneration. In all the known examples, every boundary slope of a surface in the knot complement is strongly detected. The known examples comprise two-bridge knots, see [18], plus a handful of other examples. The curve of abelian representations, for example, produces the slope zero of a spanning surface.

Question 5.2 *Is the slope of every incompressible surface in a knot complement strongly detected?*

Error. The main result, Theorem (1.5), claimed in [5] is wrong. In that paper it is claimed that a certain boundary slope, 1/6, of a knot in a certain rational homology sphere is not (strongly) detected. The error is in the proof of (1.4) which asserts the existence of such a slope. In fact there is no incompressible, ∂-incompressible surface with this slope in the given manifold. We thank Alan Lash for pointing this out to us.

One might attempt to phrase a similar question for links. However the situation is more complicated here because the natural invariant of a link is not a polynomial, but an ideal. Consider a link of n components in the 3-sphere, and let X be its exterior. The restriction of a representation $\rho \in R \equiv Hom(\pi_1 X \longrightarrow SL_2\mathbb{C})$ to the group of one of the n torus boundary components $\pi_1 T_i$ for $1 \leq i \leq n$ is an abelian representation which gives rise to a pair of pairs of eigenvalues $(L_i, M_i)^{\pm 1}$, as in section(2).

These n pairs of pairs of eigenvalues determine 2^n points in $\mathbb{C}^{2n}$. Thus one obtains a 2^n-valued map

$$\xi : R \longrightarrow \mathbb{C}^{2n}.$$

The image of a component of R has closure an affine algebraic set of complex dimension at most n. Thus there is an ideal I for each component of R, and the product of these ideals over all components of R is the invariant. In the case $n = 1$ one obtains a principal ideal and hence a polynomial unique up to scaling by an element of $\mathbb{C}$, and this is the A-polynomial. For a hyperbolic link, Thurston has shown that the component corresponding to the complete representation has image under ξ of complex dimension n, thus this ideal is different to the ideal for the unlink.

The relation between boundary slopes in the link exterior and this ideal is more complicated. Some work has been done by Lash in his Ph.D. thesis [14]. Roughly speaking Lash shows that every boundary slope in the Whitehead link complement is strongly detected. Floyd and Hatcher [10] used combinatorial methods to determine all incompressible surfaces in two-bridge link complements. First Lash extends their procedure to calculate the boundary slopes of these surfaces. Then the delicate part is to show that these are all strongly detected. The work is made easier by the fact that R is a hyper-surface in $\mathbb{C}^3$ and is thus defined by a single polynomial.

6 Cusp Polynomials

A hyperbolic knot has a single torus cusp and associated to this cusp is a complex number called the *cusp constant.* The fundamental group of the torus is represented as a parabolic subgroup and thus acts by Euclidean isometries on a horosphere. Different choices of horospheres change the action by rescaling. Thus the quotient Euclidean torus is unique up to homothety. Identify the horosphere with $\mathbb{C}$ in such a way that the holonomy of the meridian of the knot corresponds to a translation by 1 then the longitude corresponds to a translation by some complex number α and the torus is $\mathbb{C}/\Gamma$ where Γ is the lattice generated by 1 and α. The *cusp constant* is α, and the *cusp polynomial* is the minimum polynomial for α over $\mathbb{Q}$.

We can obtain information about the cusp constant from the A-polynomial for the following reason. Geometric considerations show that there are representations near to the complete one such that the longitude and meridian are loxodromic with a common axis. The ratio of their complex translation lengths approximates the cusp constant. Thus a Taylor series expansion of the A-polynomial near the complete representation gives this ratio. We give some more details:

Lemma 6.1 *Suppose that x, y are a basis for $\mathbb{Z} \oplus \mathbb{Z}$ and that ρ_t is a sequence of representations of this group for which $\rho_t(x)$ and $\rho_t(y)$ are loxodromics converging to the parabolic representation*

$$x \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad y \to \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

Then if ζ denotes complex translation length, we have $\lim_{t\to\infty} \zeta(\rho_t y)/\zeta(\rho_t x) \to \alpha$.

Proof. After a one parameter family of conjugacies we may assume that

$$\rho_t x = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

The fixed points of this are ∞ and $(\lambda^{-1}-\lambda)^{-1}$. Since x and y commute, y also fixes ∞ thus

$$\rho_t y = \begin{pmatrix} \mu & c \\ 0 & \mu^{-1} \end{pmatrix}.$$

Now y must also fix the other fixed point of x which implies that $c = (\mu^{-1} - \mu)/(\lambda^{-1} - \lambda)$. We must have that $c \to \alpha$ as $\lambda \to 1$. Writing $\lambda = 1 + \epsilon_\lambda$ and $\mu = 1 + \epsilon_\mu$ this implies that $\epsilon_\mu/\epsilon_\lambda \to \alpha$. The complex translation length is given by $\zeta(\rho_t x) = 2\log(\lambda)$. So Taylor's theorem gives that

$$\zeta(\rho_t y)/\zeta(\rho_t x) = \epsilon_\mu/\epsilon_\lambda + o(\epsilon_\mu, \epsilon_\lambda)$$

as required. □

Question 6.2 *The complete hyperbolic structure on a hyperbolic knot determines a unique holonomy representation into* $PSL_2\mathbb{C}$ *and hence two representations into* $SL_2\mathbb{C}$*, (see [3] for a proof that the representation lifts). The trace of the longitude and meridian are* ± 2 *for the complete representation. One of these has trace of the meridian* $+2$*. Thus there are two cases depending on whether the trace of the longitude is* ± 2*. Does this sign have any significance?. (Does anyone know an example when the longitude has trace* $+2\Gamma$*.)*

Theorem 6.3 *Suppose that* K *is a hyperbolic knot with holonomy* ρ_0 *and cusp constant* α*. Suppose that* $F(L, M)$ *is the factor of* $A(L, M)$ *corresponding to the curve containing this representation. The terms of lowest total degree in*

$$F\left(L - \frac{trace(\rho_0\lambda)}{2}, M - \frac{trace(\rho_0\mu)}{2}\right)$$

can be viewed as a polynomial in one variable, $f(t)$*. Then* $f(\alpha) = 0$*.*

Proof. We will discuss the case that both longitude and meridian have trace 2 at the complete representation, the other case is similar.

Write $A(1+\epsilon_\lambda, 1+\epsilon_\mu) = 0$ as a sum of homogeneous polynomials in $\epsilon_\lambda, \epsilon_\mu$ and let $g(\epsilon_\lambda, \epsilon_\mu)$ be the homogeneous polynomial of lowest total degree, n, say. This amounts to taking the lowest order terms in the Taylor expansion around $(1,1)$. Then for $\epsilon_\lambda, \epsilon_\mu$ close to 0 we have that

$$\epsilon_\lambda^{-n} A(1+\epsilon_\lambda, 1+\epsilon_\mu) = g(1, \epsilon_\mu/\epsilon_\lambda) + o(\epsilon_\lambda, \epsilon_\mu).$$

Since $\epsilon_\mu/\epsilon_\lambda \to \alpha$ we see in the limit that $g(1, \alpha) = 0$. □

Corollary 6.4 *If* K *is a hyperbolic knot in* S^3 *then the contribution to the* A*-polynomial from the component containing the complete structure is not of the form* $c.L^a - d.M^b$*.*

Proof. Putting $M = 1$ gives $A_K(L, 1) = \pm(L-1)^{k_1}(L+1)^{k_2}L^{k_3}$, by (4.5). Thus $a = 1$ and $c = \pm d$ or $a = 2$ and $c = d$. In both cases the cusp polynomial has only real roots, but the cusp constant is not real, a contradiction. □

This can be used to show that no hyperbolic knot has the same A-polynomial as any torus knot. The idea is that the Seifert fibration of a torus knot meets the torus boundary in a curve of slope pq. Now the fiber is central in the fundamental group of the knot, and so any non-abelian representation of the group into $SL_2\mathbb{C}$ must kill the fiber. See [3] (2.7) for more of a discussion.

7 Mutation

Most knot invariants are unchanged by mutation. We do not know in general if the A-polynomial is always unchanged by mutation. However in some cases it is. A consequence of Theorem (7.3) is that the polynomial of a hyperbolic knot and a mutant of it always have at least one **Z**-irreducible factor in common.

The relation between the A-polynomial and boundary slopes leads to a purely topological corollary:

Corollary 7.1 *A hyperbolic knot and a mutant of it always have at least one nonzero boundary slope in common.*

We do not know a topological proof of this corollary, and the following is open:

Question 7.2 *Do a knot and a mutant of it always have the same set of boundary slopes?.*

Now suppose that K is a knot in S^3 which contains an incompressible four punctured sphere F meeting the knot in meridians. This is the situation in which we may perform a mutation, defined below. Our main result may then be stated:

Theorem 7.3 *Suppose that X is a component of the character variety of $S^3 \setminus N(K)$ with the property that there is at least one representation whose character lies on X whose restriction to $\pi_1(F)$ is irreducible.*

Then the $\mathbb{Z}$-irreducible factor of the A-polynomial corresponding to X appears in both K and its mutant.

In particular, the component which contains the complete structure contains a faithful representation of $\pi_1(S^3 \setminus N(K))$ so that there is always at least one factor in common between the knot and a mutant of it. This suffices to deduce Corollary 7.1. It is of course well known that all the skein invariants of a knot are preserved by mutation; however Theorem (7.3) leaves open the possibility that the A-polynomial can distinguish mutants.

In examples one can often check whether all components of the character variety satisfy the hypothesis of Theorem (7.3). For example one finds easily that the Kinoshita-Teresaka knot cannot have an irreducible representation which restricts to a reducible representation on the mutating sphere; so that this knot and its mutant have identical polynomial. We remark in passing that this does not suffice to show that these two knots have identical sets of boundary slopes, due to question (5.2).

Consider the knot exterior $X = Cl(S^3 \setminus N(K))$, we can cut X open along F and this yields two manifolds M_1 and M_2. We will refer to M_1 as the *inside* of the mutation sphere. We identify F with the unit sphere in such a way that the punctures are equally spaced points on the equator. Thus they form two antipodal pairs. The identification is chosen so that antipodal punctures are connected by the knot inside the mutation sphere. The closed genus-2 surface $F^+ = \partial M_1$ is obtained by adding to F two annuli connecting paired punctures. The mapping class group of the 4-punctured sphere has center $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by half-turns around orthogonal axes. Choose a mapping-class τ in the center and define X^τ to be the 3-manifold obtained by glueing M_1 to M_2 using τ. Thus one obtains 4 possible 3-manifolds, one of which is X and the others are the exteriors of the 3 knots obtained by mutation of K. The involution τ of F extends to an involution τ^+ of F^+.

We shall base all fundamental groups at one of the fixedpoints of τ. Notice that we have a decomposition

$$\pi_1(S^3 \setminus N(K)) \cong \pi_1(M_1) *_{\pi_1(F)} \pi_1(M_2)$$

Let ρ be a representation of $\pi_1(S^3 \setminus N(K))$ which satisfies the hypotheses of Theorem 7.3. Observe that the property that a representation of a group is irreducible can be characterized by the property that there is at least one commutator in the group whose trace is not 2 and it follows from this that all representations which are sufficiently near to ρ also satisfy the hypotheses of the theorem. The key feature of irreducible representations which we use is that such representations are determined up to $SL_2\mathbb{C}$ conjugacy by their character [9]. The following lemma is well known:

Lemma 7.4 *The map τ^+ does not change the character of a representation of $\pi_1(F^+)$.*

Proof. Since characters are class functions, there is no necessity to be concerned with basepoints. Then one easily sees using the arguments of for example [9] that the character is completely determined by its values on a (finite set of) simple closed curves. Since it is well known that τ^+ carries every such curve on F^+ either to itself or its inverse (up to conjugacy) and neither of these changes $SL_2\mathbb{C}$ trace, the result follows. □

We may use this lemma to construct a representation of the mutant manifold as follows. Define ρ_{mut} on $\pi_1(M_1)$ to be $\rho \mid \pi(M_1)$. Now we use the lemma to see that $\rho\tau$ is conjugate to ρ when restricted to $\pi_1(F)$; that is to say, there is an element C in $SL_2\mathbb{C}$ so that ρ and $C.\rho\tau.C^{-1}$ agree on $\pi_1(F)$. (Observe that this proof shows that actually they agree on $\pi_1(F^+)$.) Then on the mutant manifold we define the representation on the piece corresponding to M_2 to be $C.\rho\tau.C^{-1}$. These agree on the amalgamating subgroup and yield a representation of the mutant knot complement.

Our claim is that this construction does not change the curve of eigenvalues on a small open (classical) neighborhood of ρ so that since this neighborhood is Zariski dense in the relevant component of the eigenvalue variety, the eigenvalue varieties are the same, whence they contribute the same polynomial to A_K and $A_{K(mut)}$. This will complete the proof of Theorem 7.3. First notice that ρ and ρ_{mut} agree on the meridian. The claim will follow if we show that they agree on the longitude. However this follows since the longitude can clearly be written as a product of elements which lie entirely either in $\pi_1(M_1)$ or in $\pi_1(F^+)$ and by construction ρ and ρ_{mut} agree on these subgroups.

Question 7.5 *Do mutants always have the same A-polynomial?.*

8 High Dimensional Representation Varieties

For each integer n we give an example of hyperbolic knot in S^3 for which there is a component of the representation variety of dimension bigger than n. The idea is the following. One may obtain a non-hyperbolic knot with a representation variety of large dimension by taking the connect sum of a large number of knots. To obtain a hyperbolic knot, express this connect sum as a braid β such that removing both β and the braid axis A from S^3 gives a 2-cusp hyperbolic 3-manifold. Now Thurston tells us that for p large the orbifold obtained by killing the p'th power of the meridian of the braid axis A is hyperbolic. Thus the pre-image $\tilde{\beta}$ of β under the p-fold cover of S^3 branched over A is a knot (provided p is suitably chosen) in S^3 with hyperbolic complement. It has a component of representations of the same dimension as the one we construct for β. We will now fill in the details.

Lemma 8.1 *Let K be a knot in S^3 with hyperbolic complement, and K_n the connect sum of n copies of K. Then there is a component of $Hom(\pi_1(S^3 - K_n), SL_2\mathbb{C})$ of dimension at least n.*

Proof. The proof is by induction on n. For $n = 1$ since K is hyperbolic the result follows from Thurston's deformation argument. The process of taking a connect sum may be viewed as taking two knot complements and identifying an annulus neighborhood of the meridian in one knot complement with such an annulus in the other. Let ρ_n be a representation of $\pi_1(S^3 - K_n)$ and ρ a representation of $\pi_1(S^3 - K)$ such that they both send a generator μ of the annulus to the same element A of $SL_2\mathbb{C}$. We may suppose that A is a loxodromic element with axis γ in $\mathbb{H}^3$. Let B be any other loxodromic with axis γ thus B commutes with A. Now let $\rho^B = B^{-1}\rho B$ be a conjugate representation, then $\rho^B(\mu) = \rho(\mu) = \rho_n(\mu)$. Thus there is a well defined representation ρ_{n+1} of

$$\pi_1(S^3 - K_{n+1}) = \pi_1(S^3 - K_n) *_{<\mu>} \pi_1(S^3 - K)$$

which is given by ρ_n on $\pi_1(S^3 - K_n)$ and ρ^B on $\pi_1(S^3 - K)$. The freedom in choosing B is given by $trace(B)$ and so the complex dimension of the component of the representation variety containing ρ_{n+1} is at least 1 greater than that for ρ_n. □

We will apply the lemma with K the figure 8 knot. This knot is given as a braid $(\sigma_1\sigma_2^{-1})^2$ and the connect sum K_n is given as a braid by

$$\beta = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\sigma_3\sigma_4^{-1}\cdots\sigma_{2n-1}\sigma_{2n}^{-1}\sigma_{2n-1}\sigma_{2n}^{-1}.$$

Let A be the axis of this braid then $N = S^3 - (A \cup \beta)$ is a bundle over the circle with fiber a disc punctured $2n + 1$ times and monodromy β. By results contained in [15] this monodromy is pseudo-Anosov and so the bundle is hyperbolic. By Thurston any sufficiently large Dehn filling of one of the components yields a hyperbolic manifold or orbifold with one cusp. Thus for p large there is a hyperbolic orbifold M with cone angle $2\pi/p$ on the braid axis A and a cusp along β. Thus the p-fold cyclic cover of M branched over A

$$\pi : \tilde{M} \longrightarrow M$$

gives a hyperbolic 3-manifold $\tilde{M}$ which is topologically the result of removing the braid $\beta^p = \pi^{-1}(\beta)$ from S^3. It is easy to see that this braid is alternating. In order to arrange that β^p is connected it suffices to choose p coprime to $2n+1$. This is because the braid β is connected and so defines a permutation of the $2n+1$ strings of β which is transitive on the strings. Thus β^p is transitive on strings if and only if p is coprime to $2n+1$.

By lemma 8.1 there is a component C of $Hom(\pi_1(S^3 - K_n), SL_2\mathbb{C})$ of dimension at least n. The branched covering $\pi : S^3 - \beta^p \longrightarrow S^3 - \beta$ can be used to pull-back these representations to $\pi_1(S^3 - K_n)$.

Theorem 8.2 *Given n there is an alternating hyperbolic knot K_n in S^3 and a component, C, of $Hom(\pi_1(S^3 - K_n), SL_2\mathbb{C})$ with $dim_{\mathbb{C}} C \geq n$.*

Corollary 8.3 *There is a hyperbolic knot for which a type 2 degeneration occurs.*

Proof. If the dimension of the space of representations mod conjugacy is at least 2, the pre-image of some point in (L, M) space contains at least a curve. Going to infinity on this curve gives a type 2 degeneration. □

9 Satellites

It is known that if K is a hyperbolic knot ([3](2.6)) or torus knot ([3](2.7)) then the A-polynomial is non-trivial. Here *non-trivial* should be interpreted as distinct from the A-polynomial for the unknot which is $L-1$ due to abelian representations. There is no known example of a non-trivial knot in S^3 with trivial A-polynomial. The question of whether there is a non-trivial knot with trivial polynomial may be attacked using a torus decomposition of the knot complement into pieces. There is one piece with a single torus boundary component. It is either a torus knot or hyperbolic knot complement. One would like to take the representations of this piece and extend them over the rest of the 3-manifold. The remaining pieces are compact 3-manifolds with 2 torus boundary components. This leads to the following:

Question 9.1 *Let M be a compact 3-manifold with boundary consisting of two incompressible tori. When does a representation of the group of one torus extend to a representation of the 3-manifold ?. When is this representation non-trivial on the other torus boundary?*

Theorem 9.2 *Let K be a satellite knot with a non-zero winding number n around a knot K'. Then A_K has a factor F such that $F(L^n, M) = A_{K'}(L, M^n)$.*

Proof. The exterior X of K is $W \cup X'$ where X' is the exterior of the knot K' and W has two torus boundary components. The winding number hypothesis means that the inclusion of either boundary torus into W induces an isomorphism on rational homology. This in turn means that every representation of one boundary torus of W extends as an abelian representation of W into $SL_2\mathbb{C}$. Let λ, μ be the longitude and meridian of K and λ', μ' those for K'. Then if L, M, L', M' are the respective eigenvalues of an abelian representation of $\pi_1 W$ then $L = L'^n$ and $M' = M^n$. Thus there is a factor $F(L, M)$ of $A_K(L, M)$ such that $F(L'^n, M) = A_{K'}(L', M^n)$. $\square$

Let W be a compact 3-manifold with boundary consisting of two tori T_1, T_2 and call a representation $\rho : \pi_1 T_1 \longrightarrow SL_2\mathbb{C}$ *forbidden* if it does not extend to a representation of $\pi_1 W$. The closure of the set of forbidden representations is an affine algebraic set. To see this, the representation variety of $\pi_1 W$ is mapped by a polynomial map into the representation variety of $\pi_1 T$ and so has image a constructible set [16]. Thus the complement has closure an affine algebraic set. We will assume that every torus in W is boundary parallel, so that W is either Seifert fibered or hyperbolic. If W is Seifert fibered then W is a cable space and the discussion in the previous theorem applies. Thus we assume that W is hyperbolic. Thurston's deformation argument implies that we may deform the complete representation so that on each boundary torus it is non-hyperbolic. Thus there is at least a curve of representations on each boundary torus which extends. There is an example [17] showing that the restriction of representations on the component containing the complete representation yields only a curve of representations of either boundary torus. However it is not known if that example has other components of representations which yield a set of complex dimension 2 of representations of either boundary torus.

Question 9.3 *If W is a compact 3-manifold with boundary consisting of two incompressible tori, and ξ_1 is the map defined in section (2) for one of the torus boundary components of W does $\xi_1(Hom(\pi_1 W, SL_2\mathbb{C})) \subset (\mathbb{C} - 0)^2$ have complex dimension 2 always?.*

By a *forbidden curve* for (W, T_1) we mean an affine algebraic curve in $\xi_1(Hom(\pi_1 T_1, SL_2\mathbb{C})) = (\mathbb{C} - 0)^2$ of non-conjugate representations such that only finitely many of them extend over $\pi_1 W$. Suppose that W is a solid torus with a knot K removed. If the winding number of K round the solid torus is zero, then killing the meridian of K in $\pi_1 W$ gives $\mathbb{Z}$ and in particular this kills the longitude of K. Let T_1 be the torus boundary corresponding to K and M the eigenvalue of the meridian of K then $M = \pm 1$ are forbidden curves for such examples. We call such examples *trivial.*

Lemma 9.4 *Let W be the exterior of the Whitehead link and M, L eigenvalues of the meridian and longitude of one of the components of the Whitehead link. Let T be the boundary torus of W corresponding to this component of the Whitehead link. A representation of $\pi_1 T$ extends over $\pi_1 W$ unless it lies on one of the forbidden curves:*

$$\{\ M = 1, M = -1, L + M^2 = 0\ \}.$$

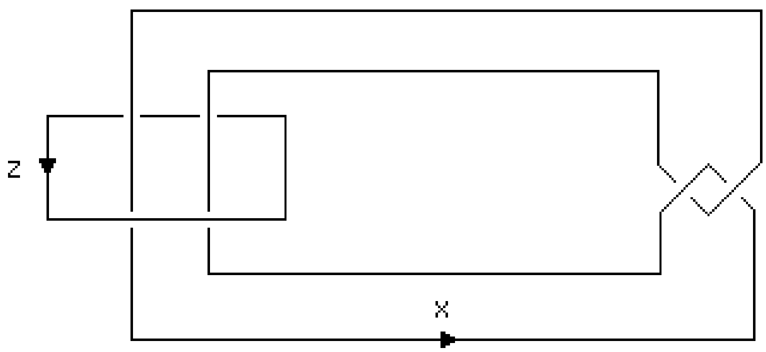

Figure 1

Proof. Refer to Figure 1 where the labels next to the components of the Whitehead link should be interpreted as generators in the Wirtinger presentation, thus x, z are meridians of the two components. The fundamental group of W has presentation

$$< x, z| \ zxz^{-1}x^{-1}zx^{-1}z^{-1}x = xz^{-1}x^{-1}zx^{-1}z^{-1}xz \ \ > .$$

We may conjugate an irreducible representation so that

$$x \mapsto \begin{pmatrix} p & 1 \\ 0 & p^{-1} \end{pmatrix} \qquad z \mapsto \begin{pmatrix} M & 0 \\ t & M^{-1} \end{pmatrix}.$$

The relation is satisfied if and only if a certain polynomial $f(p, M, t) = 0$. The highest power of t is t^3 and it has coefficient p^2M^2.

The longitude of the component of the Whitehead link labelled z is $\lambda = z^{-1}x^{-1}zxz^{-1}xzx^{-1}$ which has eigenvalue L and using resultants one deduces that:

$$0 = L(p^4+1)(M-1)(M+1)(L+M^2)+p^2(L+L^2-2LM^2+2L^2M^2-M^4-L^2M^4).$$

Given $L, M \neq 0$ there is $p \neq 0$ making this expression zero, unless the first term is zero. Furthermore, given such p, M there is t such that $f(p, M, t) = 0$ because the coefficient of the highest power of t in f is p^2M^2 hence not zero. Thus there is a representation with the given L, M. $\square$

Question 9.5 *What are the possible forbidden curves for knots in solid tori?. Are there any non-trivial examples other than the Whitehead link?*

An interesting feature of this example is that the forbidden curve, $L + M^2 = 0$, is given as the eigenvalue of $\lambda^{-1}\mu^2$ is -1. Thus the simple closed curve $\lambda^{-1}\mu^2$ on

T has a forbidden eigenvalue. Attempts to construct other examples of non-trivial forbidden curves have been unsuccessful.

Corollary 9.6 *Suppose that K is a knot with A-polynomial having a factor other than $L \pm 1$ and $M + L^2$. Then the untwisted double DK of K has a non-trivial A-polynomial.*

Proof. Let $X(K)$ be the exterior of K and W the Whitehead manifold used above. Let $T \subset \partial W$ be a torus boundary component of the Whithead link exterior corresponding to a component, γ say, of the Whitehead link. The exterior $X(DK)$ of the untwisted double of K is formed by glueing $\partial X(K)$ to T so that the longitude (resp. meridian) of K goes to the meridian (resp. longitude) of γ. Let C be a component of $R_U(K)$ (defined in section (2)) such that $\overline{\xi C}$ is a curve other than $L = \pm 1$ or $M + L^2 = 0$. Then except for finitely many choices of ρ in C there is a representation ρ' of $\pi_1(W)$ such that $\rho|\pi_1(\partial X(K))$ coincides with $\rho'|\pi_1(T)$ under the identification of $\partial X(K)$ with T used in forming $X(DK)$. Thus we may glue the representations ρ, ρ' to obtain an irreducible representation σ of the $\pi_1(X(DK))$. One checks that $\xi\sigma$ traces out a curve as σ varies. $\Box$

10 The Volume Form

In [3] (4.5) it is shown that given a representation of ρ of $\pi_1 M$ into $SL_2\mathbb{C}$ there is an associated volume, and this defines a function $V : Hom(\pi_1 M, SL_2\mathbb{C}) \longrightarrow \mathbb{R}$. Briefly, the idea is that given a representation ρ, one chooses a nice ρ-equivariant map of the universal cover $\tilde{M}$ of M into $\mathbb{H}^3$. Use this map to pull-back the volume form on $\mathbb{H}^3$ to a $\pi_1 M$-equivariant 3-form on $\tilde{M}$. This descends to a form on M and integrating this over M gives the volume of the representation.

Using a basis λ, μ for $\pi_1 T$ let L, M be the eigenvalues of the longitude and meridian ie $\xi(\rho) = (L, M)$ and set

$$log M = \ell_\mu + i\theta_\mu \qquad log L = \ell_\lambda + i\theta_\lambda.$$

Then define a 1-form ω on $(\mathbb{C} - 0)^2$ by the formula $\omega = \ell_\mu d\theta_\lambda - \ell_\lambda d\theta_\mu$. This form is not exact since $d\omega = d\ell_\mu \wedge d\theta_\lambda - d\ell_\lambda \wedge d\theta_\mu$. However pulling back to a curve C in $Hom(\pi_1 M, SL_2\mathbb{C})$ gives the form $\xi^*\omega$ which is exact since it equals dV. This formula is due to Hodgson, [12] see also [3]. Since ω is not exact on $(\mathbb{C} - 0)^2$ we obtain:

Corollary 10.1 *[3]* $\quad dim_{\mathbb{C}}(\xi(R_U)) \leq 1.$

This leads to an obstruction to a polynomial arising as the A-polynomial of a knot. Let γ be a loop on the curve $A = 0$ which lies in the image of ξ then the integral of ω around γ must be zero. The polynomial of the figure eight knot is:

$$A(L, M) = -2 + M^4 + M^{-4} - M^2 - M^{-2} - L - L^{-1}.$$

Changing this slightly gives a different polynomial:

$$f(L, M) = -2 + M^6 + M^{-6} - M^2 - M^{-2} - L - L^{-1}.$$

We will use the volume form to show this is not the A-polynomial of any knot. However it does satisfy every other condition that we know of to be a knot polynomial.

Let S be the affine curve in $\mathbb{C}^2$ where f vanishes and consider the coordinate projection $\pi : \mathbb{C}^2 \longrightarrow \mathbb{C}$ which sends $(L, M) \mapsto M$. Then $\pi|_S : S \longrightarrow \mathbb{C}$ is a 2-fold cover of the complex plane branched over a subset of the set where $D_M f$ vanishes. Given a path γ in $\mathbb{C}$ which misses the image of this set, one may uniquely lift it to a path $\tilde{\gamma}$ in S given the start point. Certain paths lift to closed paths representing non-zero homology classes. In this way a computer can calculate

$$\oint_{\tilde{\gamma}} \omega.$$

Experimentation reveals that with γ consisting of small loops linking the points $e^{5\pi/4}$ counterclockwise and $e^{3\pi/4}$ clockwise together with two copies of a straight line connecting these two loops that the integral is approximately -0.956. Since this is not zero f is not the A-polynomial of a knot. $\square$

Question 10.2 *Which affine curves C in $(\mathbb{C} - 0)^2$ satisfy the condition that ω is exact on $C\Gamma$.*

11 Further Results

We mention two more results concerning the A-polynomial. The terms of the A-polynomial appearing along an edge e of its Newton polygon may be viewed as a polynomial called an *edge polynomial* $f_e(t)$. The variable t may be identified with an eigenvalue of the loop β on the boundary torus which is the boundary slope of an incompressible surface in the knot complement. Thus $t = L^q M^p$ if the slope of β is p/q.

Theorem 11.1 *[3],[4],[6] The edge polynomial $f_e(t)$ is a product*

$$C.f_1(t).f_2(t) \cdots f_n(t)$$

where C is an integer and $f_i(t)$ is a cyclotomic polynomial. If ω is a p'th root of unity which is a zero of $f(t)$ then p divides the number of boundary components of every component of an incompressible surface associated to the action on a tree arising from a degeneration corresponding to the edge e.

Definition 11.2 *A* corner *of a polynomial $p(x, y)$ in two variables is a term appearing in a corner of the Newton polygon of p.*

One might view corners as analogous to the first and last term in a polynomial in a single variable, then the following says that in a certain sense the A-polynomial is monic.

Theorem 11.3 *[6] The coefficients of terms in the corners of the A-polynomial are* ± 1.

Corollary 11.4 *The constant C appearing in theorem (11.1) is ± 1.*

Corollary 11.5 *The edge polynomials of a 2-bridge knot are all $\pm(t-1)^k(t+1)^l$.*

Proof. It is shown in [11] that an incompressible surface in a 2-bridge knot has one or two boundary components. The above theorems now give the result. $\square$

It is shown in [1] that if a Conway sphere is strongly detected then the corresponding edge polynomial is $C.(t^2+1)^k$ and again by the above $C = \pm 1$.

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