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A Kleinian group with contractible quotient not simply connected at infinity

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Abstract. We give an example of a co-compact Kleinian group Γ which contains a subgroup Γ_0 having the property that $\mathbb{H}^3/\Gamma_0$ is contractible but not simply connected at infinity.

1. Introduction

The purpose of this article is to prove the following theorem:

THEOREM 1.1. *There is a hyperbolic 3-orbifold $\tilde{X}$ homeomorphic to a contractible 3-manifold without boundary that is not simply connected at infinity. The singular locus of the orbifold $\tilde{X}$ is a circle at which the cone angle is π. Furthermore $\tilde{X}$ is an orbifold covering of a closed hyperbolic orbifold X which is homeomorphic to S^3 and the singular locus of X is a link of two components at which the cone angle is π.*

We recall that a hyperbolic 3-orbifold is the quotient of $\mathbb{H}^3$ by a discrete group of hyperbolic isometries. The theorem may thus be reformulated as:

REFORMULATION. *There is a co-compact Kleinian group Γ which contains an infinitely generated subgroup Γ_0 having the property that $\mathbb{H}^3/\Gamma_0$ is contractible but not simply connected at infinity. There are two conjugacy classes of torsion element in Γ and each has order two.*

This result is perhaps somewhat surprising. Of course Thurston [Th2] has shown that many closed 3-manifolds have hyperbolic structures. Furthermore, the fact that there is a universal hyerbolic link [Th3, HLM] implies that every closed orientable 3-manifold has a hyperbolic orbifold structure. However such general

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results do not seem to predict the existence of an example of this type. The orbifold $\tilde{X}$ is an irregular orbifold covering of a closed hyperbolic orbifold X which is S^3 with a singular locus the link of two components shown in Fig. 1. The cone angle around each component is π. It is an unresolved question whether a closed 3-manifold can be covered by a contractible manifold other than Euclidean space. However, it has been shown that many contractible manifolds cannot do this [My, Wr]. Our examples shows that this can almost happen in the sense that the closed orbifold X has such an orbifold cover. Perhaps the most surprising feature of our example is that we could prove that is exists at all. It will be seen in the construction that several fortuitous accidents combine to enable the construction to succeed. For a more general definition of orbifold, see [Mo]. The authors thank the referee for finding errors in the original proof of 1.2(2) and for other helpful comments.

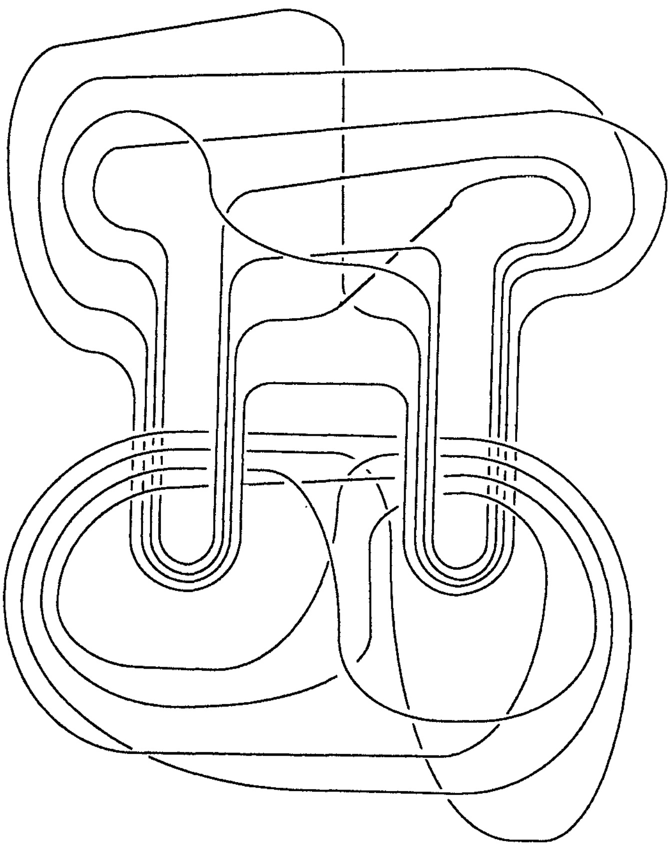


Figure 1

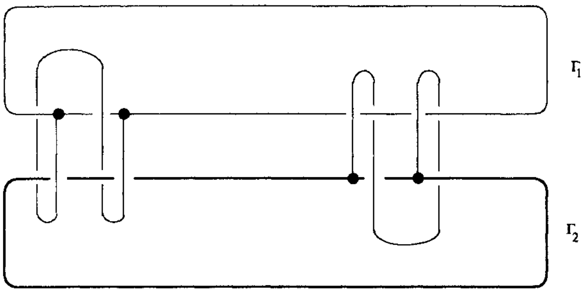


Figure 2

Let Γ_1 and Γ_2 be the pair of graphs embedded in S^3 shown in Fig. 2. Each graph is homeomorphic to the graph shown in Fig. 3, which we call a theta-curve. We will denote by M the compact 3-manifold $S^3 - int(N_1 \cup N_2)$ where N_i is a regular neighborhood of Γ_i. Thus ∂M consists of two genus 2 surfaces $\partial_i M = \partial N_i$, for $i = 1$, 2. The proof of the theorem depends on the following technical result the proof of which is deferred to section 2.

PROPOSITION 1.2.
(1) *M has incompressible boundary.*
(2) *$\pi_1(M)$ contains no $\mathbb{Z} \times \mathbb{Z}$ subgroup.*
(3) *Every properly embedded annulus A in M is isotopic rel ∂A into ∂M.*
(4) *M contains no essential 2-sphere.*

There is an involution τ of S^3 given by rotation around the circle C shown in Fig. 4 which exchanges Γ_1 and Γ_2. The restriction of this to M gives an involution, also called τ, of M which exchanges the boundary components of M.

Let $\phi: \partial_1 M \to \partial_1 M$ be a diffeomorphism with ϕ^2 the central element in the mapping class group of $\partial_1 M$ and such that ϕ exchanges the un-oriented meridians of Γ_1 with the un-oriented longitudes. To be precise we require that $\phi(l_i^1) = m_i^1$ and $\phi(m_i^1) = l_i^{1^{-1}}$ for $i = 1$, 2, where m_1^1, m_2^1 are the meridians of Γ_1 and l_1^1, l_2^1 are the longitudes of Γ_1 shown in Fig. 5. Similarly we define meridians m_1^2, m_2^2 and the longitudes l_1^2, l_2^2 of Γ_2 to be the images under τ of the corresponding loops for Γ_1.

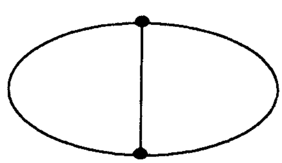


Figure 3

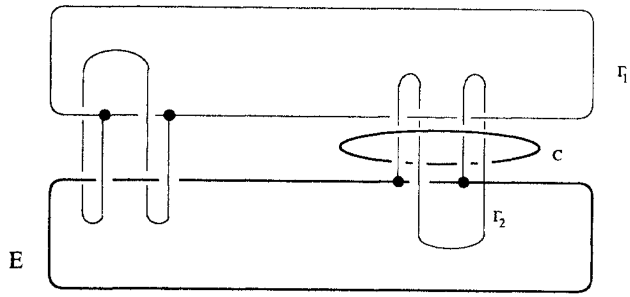


Figure 4

To see that such ϕ exists, consider the genus 2 surface $\partial_1 M$ as the union of two punctured tori. A punctured torus is a punctured square with opposite sides identified. A quarter rotation of the square gives an order 4 symmetry of the punctured torus, see Fig. 6. Then ϕ is the map of $\partial_1 M$ which restricts to the above map on each punctured torus.

Take 2 copies of M which are denoted by M and $h(M)$ where $h\colon M \to h(M)$ is a diffeomorphism. Define an involution $\bar{\tau}$ on the disjoint union of M and $h(M)$ by $\bar{\tau} \mid M = \tau$ and $\bar{\tau} \mid h(M) = h\tau h^{-1}$. Now construct a closed 3-manifold N by identifying the boundary of M with the boundary of $h(M)$ as follows. Identify $\partial_1 M$ with $h(\partial_1 M)$ via $\phi_1 = h\phi$. Identify $\partial_2 M$ with $h(\partial_2 M)$ via $\phi_2 = \bar{\tau}h\phi\bar{\tau}$. Then the involution $\bar{\tau}$ passes to the quotient to give a well defined involution, also denoted $\bar{\tau}$, of N. See Fig. 7.

Then proposition 1.2 implies that N is Haken. Suppose that $\pi_1 N$ contains a $\mathbb{Z} \times \mathbb{Z}$ subgroup. The Torus theorem implies that N contains an essential torus T,

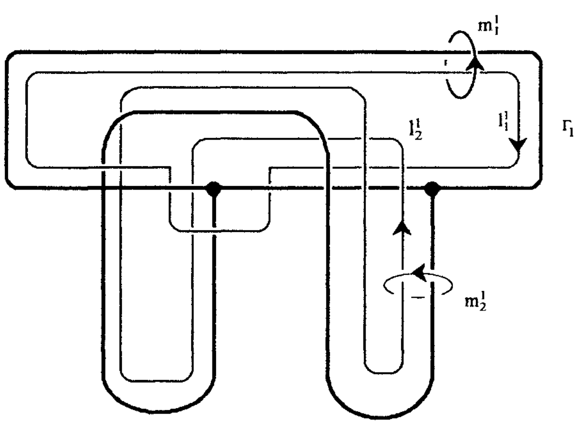


Figure 5

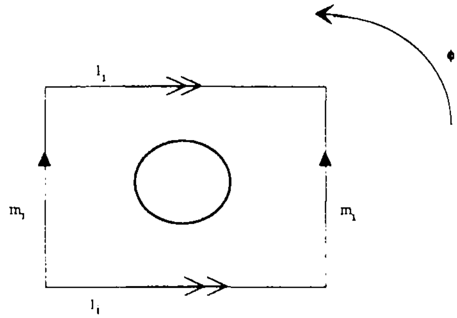


Figure 6

by 1.2(2) T cannot be isotoped into either copy of M. Thus $T \cap M$ contains an essential non-boundary parallel annulus which is impossible by 1.2(3). Thus N contains no $\mathbb{Z} \times \mathbb{Z}$ subgroup. Thus Thurston's uniformization theorem implies that N has a hyperbolic structure. It follows from Mostow rigidity that $\bar{\tau}$ is homotopic to an isometry of N. A complete proof of Thurston's Uniformization theorem has been published by McMullen [McM1, McM2]. In fact it can can be shown that N does not fiber over the circle, and so the particular case of the uniformization theorem which we appeal to is Haken manifolds that don't fiber.

If we knew that $\bar{\tau}$ was *conjugate* to an isometry by a diffeomorphism isotopic to the identity then we could conclude that $N/\bar{\tau}$ was a hyperbolic orbifold. Instead we argue as follows. The involution, $\bar{\tau}$ of N has 1 dimensional fixed locus $C \cup h(C)$, and

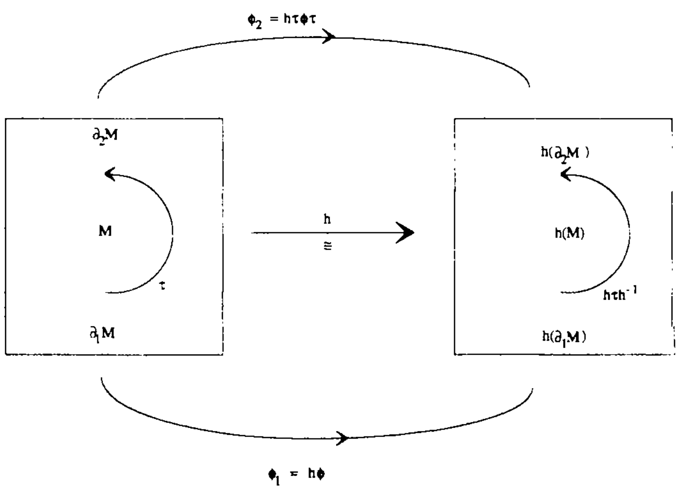


Figure 7

so by Thurston's Orbifold Theorem [Th, Ho], the quotient has a geometric decomposition. However since the 2-fold orbifold (branched) cover gives N back, the quotient $N/\bar{\tau}$ must in fact be a hyperbolic orbifold. Set $X = N/\bar{\tau}$, a closed, orientable, hyperbolic orbifold.

The referee has pointed out that we may avoid appealing to the Orbifold Theorem as follows. By a result of Tollefson [To] two involutions of a Haken 3-manifold that are homotopic are in fact conjugate by a diffeomorphism isotopic to the identity provided that the manifold is not a Seifert fiber space and $H_1(M)$ is infinite. We may apply this to the manifold N and to $\bar{\tau}$ and the isometry provided by Mostow rigidity.

Now $X = (M/\tau)\bigcup_{\bar{\phi}_1} h(M/\tau)$ identified along $\partial(M/\tau)$ by the map

$$\bar{\phi}_1 : \partial(M/\tau) \to \partial(h(M/\tau))$$

which is covered by ϕ_1. Let $\pi: N \to N/\tau$ be the projection; we will also use π for the restriction $\pi: M \to M/\tau$. Now N/τ is S^3, and Fig. 8 shows $\pi(\Gamma_1) = \pi(\Gamma_2)$ and $\pi(C)$. The graph $\pi(\Gamma_1)$ is easily seen to be isotopic in S^3 to an un-knotted theta curve, thus $\pi(M) = S^3 - N(\pi\Gamma_1)$ is a genus 2 handlebody H. The branch locus $\pi(C)$ is shown in a standard handlebody in Fig. 9. The following result is crucial to our construction, and appears to be a fortuitous accident:

LEMMA 1.3. *$\pi(l_1^1)$ and $\pi(l_2^1)$ bound discs in H.*

Proof. We sketch two proofs. First the curves $\pi(l_1^1)$ and $\pi(l_2^1)$ are shown in $H = S^3 - N(\pi\Gamma_1)$ in Fig. 10. A little manipulation shows that these curves are unlinked from $\pi(\Gamma_1)$ and are unknotted. The second proof is to calculate the (free) homotopy classes of l_1^1, l_2^1. One then adds the relations which identify an element of

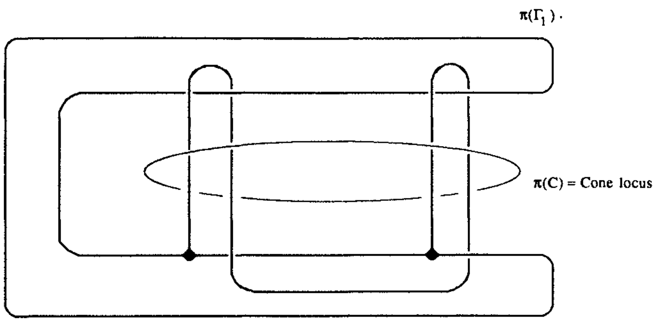


Figure 8

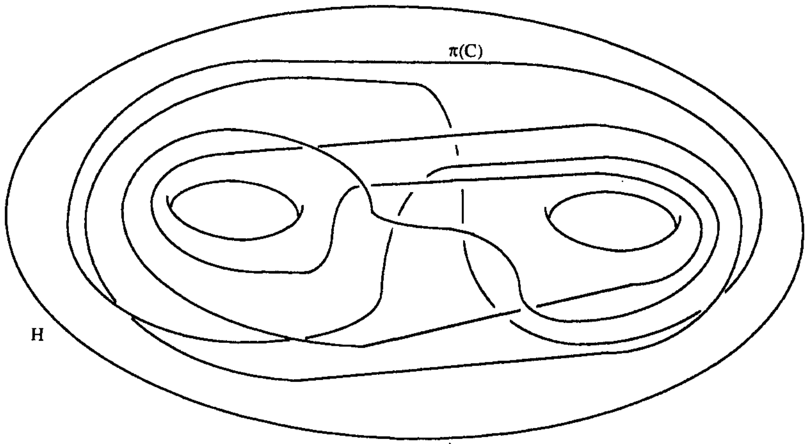


Figure 9

$\pi_1(M)$ with its image under τ_* and checks that l_1^1, l_2^1 are killed by this. This calculation is shown in Fig. 11 where we have made the identifications induced by τ_* writing down the Wirtinger presentation of $\pi_1(M)$. Thus $\pi(l_1^1)$, $\pi(l_2^1)$ are simple closed curves in the boundary of the handlebody H which are inessential in H and thus bound discs in H. □

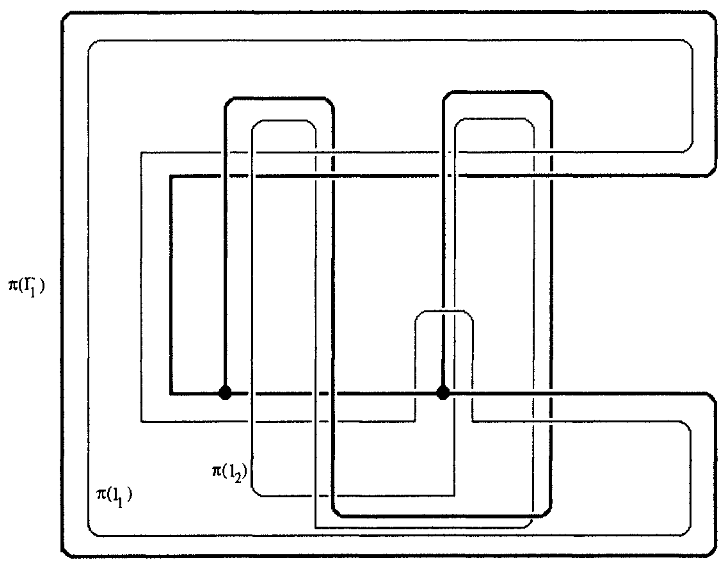
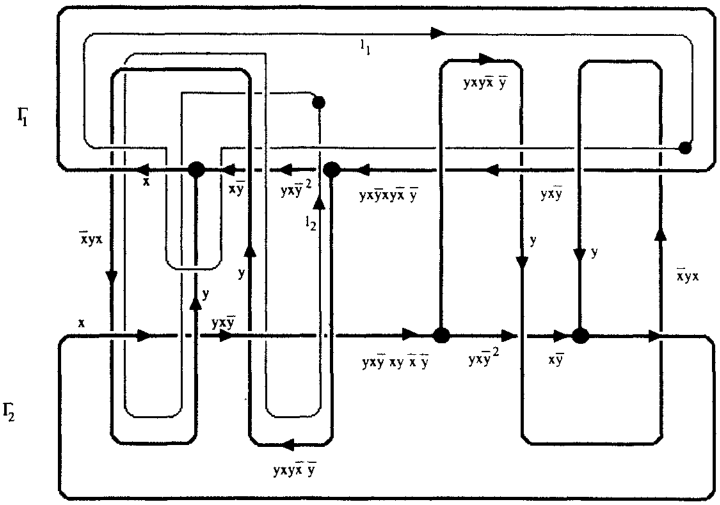


Figure 10



$l_1 = \bar{y}(yxy\bar{x}\bar{y})y(x\bar{y})\bar{x} = 1$

$l_2 = yxx\bar{x}\bar{y}(y\bar{x}\bar{y}) = 1$

Figure 11

The curves $\pi(m_1^1)$, $\pi(m_2^1)$ are longitudes of H, and it follows from (1.3) that X is topologically S^3 since the handlebodies M/τ and $h(M/\tau)$ are glued together by identifying meridians to longitudes via ϕ_1. As a hyperbolic orbifold, H contains a singular locus, a topological circle, with cone angle π, shown in Fig. 8 and also in Fig. 9. Thus X has singular locus a link of 2 components $C_1 \cup C_2$ each with a cone angle of π, this link is shown in Fig. 1. The linking number of C_1 with C_2 is zero, in fact since C_1 bounds a Seifert surface in H, we see that $C_1 \cup C_2$ is a boundary link in S^3. Thus there is a homomorphism from $\pi_1(S^3 - (C_1 \cup C_2))$ onto the free group of rank 2. This in turn maps onto $\mathbb{Z}_2 * \mathbb{Z}_2$ where the meridians of C_1 and C_2 map to the generators of order 2 in $\mathbb{Z}_2 * \mathbb{Z}_2$. This determines a homomorphism $G \to \mathbb{Z}_2 * \mathbb{Z}_2$ where G is the orbifold fundamental group of X. Now let $\tilde{X}$ be the irregular orbifold covering space of X corresponding to the subgroup $\langle \alpha_1 \rangle$ of order 2 in $\mathbb{Z}_2 * \mathbb{Z}_2$ generated by the meridian α_1 of C_1. Thus $\tilde{X}$ is a hyperbolic orbifold.

LEMMA 1.4. *Denoting the normal closure by* $\langle \cdot \rangle_N$ *we have*:
(1) l_1^1 *and* l_2^1 *are trivial in* $\pi_1 M / \langle m_1^2, m_2^2 \rangle_N$.
(2) l_1^2 *and* l_2^2 *are trivial in* $\pi_1 M / \langle m_1^1, m_2^1 \rangle_N$.

Proof. Referring to Figs. 2 and 5, the manifold obtained from M by filling in

$$\boxed{H} \xrightarrow{\phi_1} \boxed{h(M)} \xleftarrow{\phi_2} \boxed{M} \xrightarrow{\phi_1} \boxed{h(M)} \xleftarrow{\phi_2} \boxed{M} \xrightarrow{\phi_1}$$

Figure 12

$N(\Gamma_2)$ is seen to be a handlebody in which l_1^1, l_2^1 bound discs. From this it follows that after attaching 2-handles to $\partial_1 M$ along meridians m_1^2, m_2^2 that l_1^1, l_2^1 bound discs, this proves (1). Applying the involution τ of M proves (2). □

Proof of Theorem. The orbifold $\tilde{X}$ is obtained by glueing copies of M to a single copy of H using ϕ_1 and ϕ_2 to do the glueing, as shown in Fig. 12. We calculate the topological (not orbifold) fundamental group $\pi_1(\tilde{X})$ by applying Van Kampen's theorem to this decomposition to show that X is simply connected. For each positive integer n let M_n be a copy of M and let H_n denote the union of H and the first n copies of M with boundaries identified appropriately. Then $\tilde{X}$ is the union of the increasing family of submanifolds H_n. The boundary ∂H_n is a component of M_n, a genus two surface with copies l_1^n, l_2^n of l_1, l_2 marked on it.

Note that H is attached to M_1 by the map ϕ_1 which identifies the longitudes $\pi l_1^2, \pi l_2^2$ in H with m_1^1, m_2^1 in M, but $\pi l_1^2, \pi l_2^2$ are trivial in $\pi_1(H)$ by the lemma 1.3, and so m_1^1, m_2^1 are trivial in $\pi_1(H \cup_{\phi_1} M)$. By lemma 1.4, l_1^2, l_2^2 are trivial in $\pi_1(H \cup_{\phi_1} M)$, and these are identified by ϕ_2^{-1} to m_1^1, m_2^1 in the second copy of M in $\tilde{X}$. Thus these loops are trivial in $\pi_1(H \cup_{\phi_1} M \cup_{\phi_2^{-1}} M)$. Continuing in this way, we see that $\pi_1(\tilde{X})$ is trivial. A detailed argument will now be given.

We claim that H_n is a handlebody and that l_1^n, l_2^n bound discs in H_n. Indeed Lemma (1.3) implies this for the case that $n = 0$. Suppose inductively this is true for H_n then since l_1^n, l_2^n bound discs in H_n it follows that H_{n+1} is obtained from M_{n+1} by attaching 2-handles to M_{n+1} along the curves m_1^{n+1}, m_2^{n+1} in ∂M_{n+1} to which l_1^n, l_2^n are identified. One then caps off the resulting two-sphere boundary component with a 3-handle to obtain H_{n+1}. This proves the claim.

Thus there is a homeomorphism $\theta: H_{n+1} \to S^3 - int[N(\Gamma_1)]$ taking H_n to $N(\Gamma_2)$ and taking M_{n+1} onto $S^3 - int[N(\Gamma_1) \cup N(\Gamma_2)]$. We show below that the map induced by inclusion

$$(i_n)_*: \pi_1(H_n) \to \pi_1(H_{n+1})$$

has infinite cyclic image contained in the commutator subgroup of $\pi_1(H_{n+1})$. It follows from this that $(i_{n+1} \circ i_n)_* = 0$ and thus that $\tilde{X}$ is simply connected.

Since H_n is a handlebody in which l_1^n, l_2^n bound discs it follows that $\pi_1(H_n)$ is freely generated by the copies m_1^n, m_2^n of m_1, m_2 on ∂H_n. These are identified to copies of l_1, l_2 on ∂M_{n+1}. Now $\theta(m_1^n)$, $\theta(m_2^n)$ are l_1^2, l_2^2 (recall the identification of

∂H_n with a component of ∂M_{n+1} swaps meridians and longitudes.) Referring to Figs. 2 and 5 (with τ applied which relabels Γ_1 as Γ_2), one sees that the loops l_1^2, l_2^2 in $S^3 - N(\Gamma_1)$ are both homotopic rel basepoint to the loop E shown in Fig. 4. One also sees that E is homologically unlinked from Γ_1 and thus lies in the commutator subgroup of $\pi_1(S^3 - N(\Gamma_1))$. This proves the claim and completes the proof that $\tilde{X}$ is simply connected.

We next show that $\pi_1(\tilde{X} - int(H))$ is not finitely generated. Now $\tilde{X} - int(H)$ is obtained by glueing copies of M together using the maps ϕ_1, ϕ_2. M has incompressible boundary, and it is clear that $incl_*\colon \pi_1(\partial_1 M) \to \pi_1(M)$ is not surjective, otherwise it would be an isomorphism. This proves the claim. If $\tilde{X}$ is simply connected at infinity then there is an open set U disjoint from the compact set H and which has compact complement and such that $\pi_1(U)$ maps to zero in $\pi_1(\tilde{X} - H)$. Thus $\pi_1(\tilde{X} - H)$ is the image of π_1 of some compact submanifold of $\tilde{X} - int(H)$, and is thus finitely generated, a contradiction. □

2. Proof of 1.2

We now turn to proving proposition 1.2 We will consider a particular 2-fold branched convering $p\colon S^3 \to S^3$ branched over the circle E contained in Γ_2 shown in Fig. 4. The restriction of p to $\tilde{M} = p^{-1}(M)$ gives an unbranched 2-fold cover $p\colon \tilde{M} \to M$. Set $\tilde{\Gamma}_i = p^{-1}(\Gamma_i)$ and $\tilde{N}_i = p^{-1}(N_i)$ then $\tilde{N}_i$ is a regular neighborhood of $\tilde{\Gamma}_i$ and the graphs $\tilde{\Gamma}_i$ embedded in S^3 are shown in Fig. 13. Now $\tilde{N}_2$ is a genus-3 handlebody and $\tilde{N}_1$ is the disjoint union of genus-2 handlebodies. The two

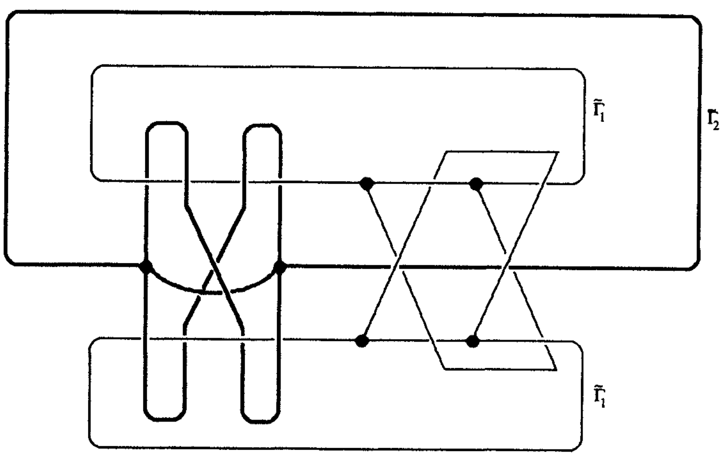


Figure 13

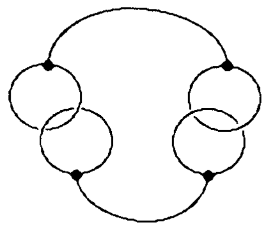


Figure 14

components of $\partial_1 \tilde{M}$ will be denoted by G_1 and G_2, each of which is a closed genus-2 surface. Note that $\tilde{M} = S^3 - int(\tilde{N}_1 \cup \tilde{N}_2)$.

LEMMA 2.1. *$\tilde{M} \cup \tilde{N}_2$ is diffeomorphic to $G_1 \times I$.*

Proof. Slide $\tilde{\Gamma}_1$ around to obtain the configuration in Fig. 14, which clearly gives $G_1 \times I$. □

From the lemma we see that $\pi_1(G_1)$ injects into $\pi_1(\tilde{M} \cup \tilde{N}_2)$ under the map induced by inclusion, and therefore also injects into $\pi_1(\tilde{M})$. Since $\partial_1 M$ lifts to G_1 in $\tilde{M}$, it follows that $\pi_1(\partial_1 M)$ injects into $\pi_1(M)$. Thus $\partial_1 M$ is incompressible, and by using the involution τ of M, one sees that $\partial_2 M$ is also incompressible, proving 1.2(1).

If M contains an essential 2-sphere S then S must separate Γ_1 from Γ_2 otherwise by the Schönflies theorem S would bound a ball. Now S lifts to a 2-sphere $\tilde{S}$ in $\tilde{M}$ which separates $\tilde{\Gamma}_1$ from $\tilde{\Gamma}_2$. However inspection of Fig. 13 reveals that each component of $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are algebraically linked in S^3 thus $\tilde{S}$ cannot separate them. This proves S cannot exist, establishing 1.2(4).

Consider the sphere S in S^3 shown in Fig. 20, which meets $(\Gamma_1 \cup \Gamma_2)$ in 4 points. Then S separates S^3 into two closed balls B_1 and B_2 and S may be chosen so that τ exchanges these balls. We may arrange that S meets $N(\Gamma_1 \cup \Gamma_2)$ standardly in 4 discs, each of which contains one pont of $(\Gamma_1 \cup \Gamma_2)$. Set $S_- = M \cap S$, a 4-punctured sphere, $Q_i = M \cap B_i$ for $i = 1, 2$. Then $S_- = \partial Q_1 \cap \partial Q_2$.

LEMMA 2.2. *S_- is incompressible in both Q_1 and Q_2.*

Proof. Suppose D is a properly embedded disc in Q_1 with $\partial D \subset S_-$. Then D separates B_1 into two balls and if D compresses S_- then Γ_1 must lie on one side of D and Γ_2 on the other side of D. Thus $\pi_1(Q_1)$ splits as a free product. Now there is a loop γ in a neighborhood of Γ_2 which is a commutator of meridians in Γ_1 and Γ_2. Thus γ lies on the same side of D as Γ_2 but such a commutator cannot be disjoint from D. Thus there is no compressing disc for S_-. Since S_- is incompressible in Q_1, applying the involution τ we see that S_- is also incompressible in Q_2. □

LEMMA 2.3. *Q_1 is a genus-3 handlebody.*

Proof. Q_1 is the complement in S^3 of an open regular neighborhood of the graph in S^3 shown in Fig. 21. By sliding this graph, one obtains the graph in Fig. 22, the complement of which is clearly a genus-3 handlebody. □

Now suppose that M contains an essential torus T. Then we may assume T is transverse to S_- and has the least possible number of circles of intersection with

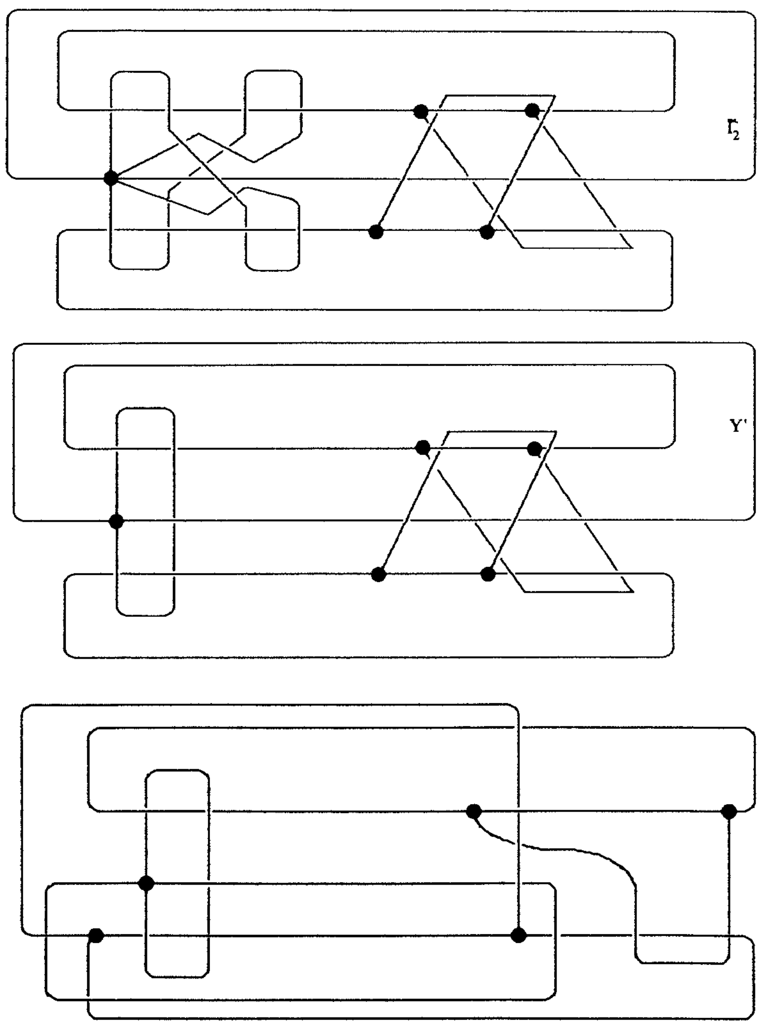


Figure 15(a)–(c)

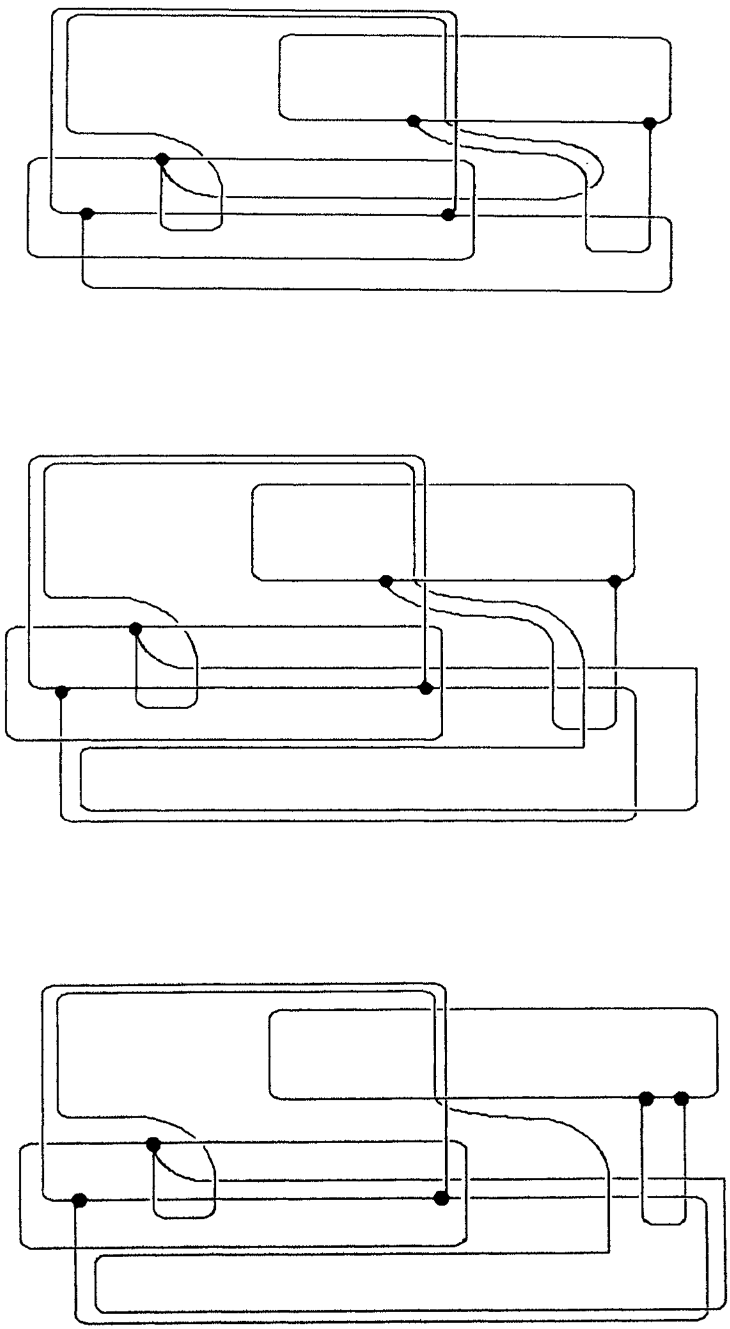


Figure 15 (d–f)

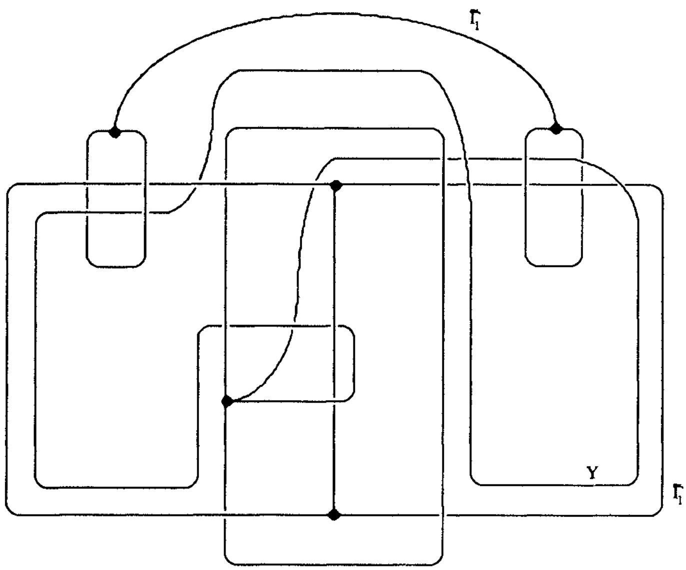


Figure 15(g)

S_-. Since S_- is incompressible it follows that every circle of intersection is essential in T. Since a handlebody contains no essential torus, by (2.3) T must have non-empty intersection with S_-. Thus S_- separates T into components each of which is an annulus and none of these annuli can be isotoped rel boundary into S_-. Let A be such an annulus properly embedded in Q_1 with boundary $\partial A = \alpha_1 \cup \alpha_2$ two disjoint circles in the four punctured sphere S_-. These circles are essential in S_-. They cannot be isotopic in S_- because this would give a torus K consisting of the union of A and an annulus in S_-. But Q_1 is a handlebody so K compresses and thus A can be isotoped into S_- a contradiction.

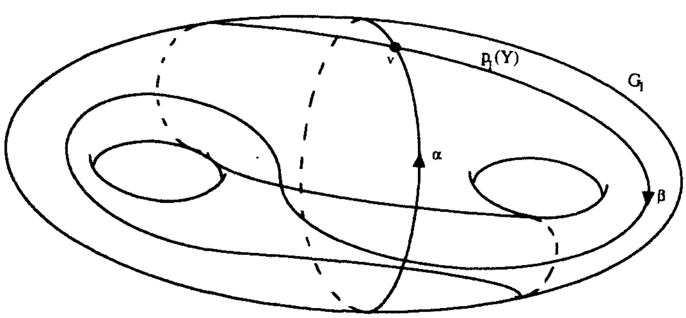


Figure 16

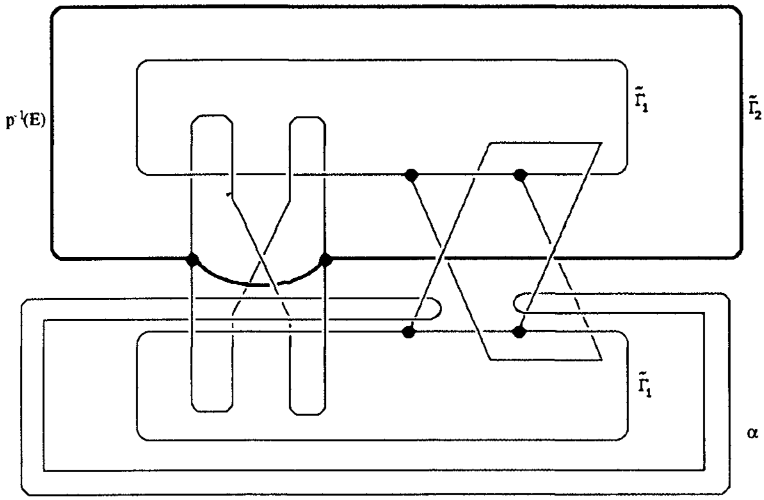


Figure 17

Now α_1 is a simple closed curve on the 4 punctured sphere S_- and if α_1 has 2 punctures on either side then since $\alpha_1 = \alpha_2$ in $H_1(Q_1)$ one sees that α_2 must also have 2 punctures on either side. But since α_1 and α_2 are disjoint this means that they are isotopic, a contradiction. It follows that α_1 has one puncture on one side and 3 punctures on the other side. Again considering $H_1(Q_1)$ one sees that α_2 must also have one punctured on one side and that there are only two possibilities for α_1, α_2 up to isotopy. Either they are the two meridians of Γ_1 on S_- or they are the two meridians of Γ_2 on S_-. Referring to Fig. 20 we see that the first case is possible, there is an annulus in a neighborhood of Γ_1 in Q_1. However the second case is impossible. One way to see this is to observe that the annulus provides a free homotopy in Q_1 between the two meridians of Γ_2 on S_-. One calculates these two

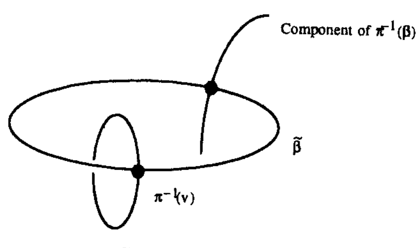


Figure 18

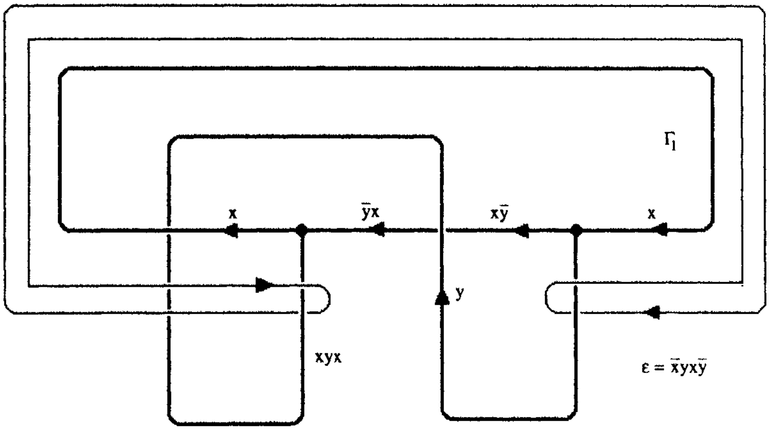


Figure 19

meridians using the Wirtinger presentation and since $\pi_1 Q_1$ is a free group the fact that these two elements are not conjugate is visible.

It follows that every component of $T \cap S_-$ is a meridian of Γ_1 but using the involution τ the above analysis applied to Q_2 implies that these curves must also be meridians of Γ_2 and so $T \cap S_-$ is empty, a contradiction. This proves 1.2(2)

Suppose now that M contains a properly embedded non-boundary parallel annulus A. Using the involution τ we may assume that A meets $\partial_1 M$. Then $p^{-1}(A)$ consists of either one or two components each of which is a non-boundary parallel annulus properly embedded in $\tilde{M}$. Choose a component $\tilde{A}$ of $p^{-1}(A)$, and note that

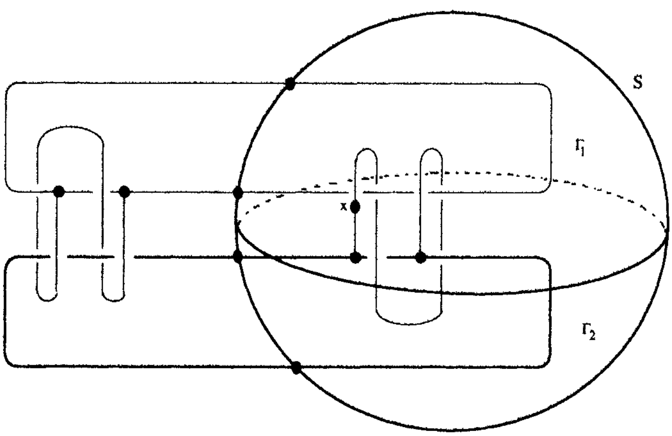


Figure 20

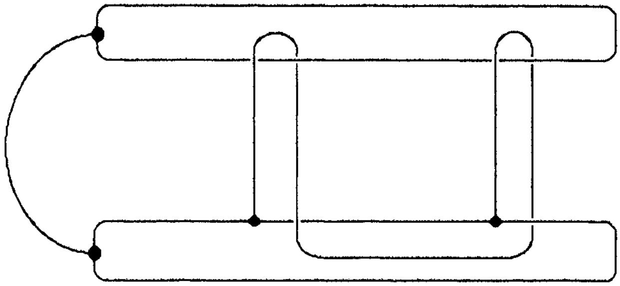


Figure 21

$\tilde{A}$ meets $\partial_1 \tilde{M}$. The covering $p: \tilde{M} \to M$ is regular and so there is a covering transformation exchanging G_1 and G_2. Thus we may assume that a boundary component of $\tilde{A}$ lies in G_1. The boundary of $\tilde{A}$ consists of 2 disjoint essential simple closed curves, γ, δ in $\partial\tilde{M}$ and we label them so that γ lies in G_1. We will now distinguish 3 cases, according to whether the second boundary component δ of $\tilde{A}$ lies in g_1, G_2 or $\partial_2 \tilde{M}$.

First suppose that δ is contained in G_2. By lemma 2.1, $\tilde{M} \cup \tilde{N}_2 = G_1 \times I$ and we may do an ambient isotopy of $G_1 \times I$ so that $\tilde{A} = \gamma \times I$ is vertical in $G_1 \times I$, where γ is some essential simple closed curve in G_1. The image of $\tilde{\Gamma}_2$ under this isotopy must be disjoint from $\gamma \times I$. Let Y be the graph in $G_1 \times I$ shown in Fig. 15(g), and $p_1: G_1 \times I \to G_1$ be projection onto the first factor.

LEMMA 2.4. *$P_{1*}\Pi_1(Y)$ is conjugae to $P_{1*}\Pi_1(\tilde{\Gamma}_2)$ in $\Pi_1(G_1)$.*

Proof. This is done in the sequence of figures 15(a) to 15(g). First, $\tilde{\Gamma}_2$ is homotoped from the position in Fig. 13 to that in Fig. 15(a). Now observe that there are 2 distinct loops in $\tilde{\Gamma}_2$ which are homotopic to each other in $G_1 \times I$. Let Y' be the graph in $G_1 \times I$ shown in Fig. 15(b). Then $p_{1*}\pi_1(Y') = p_{1*}\pi_1(\tilde{\Gamma}_2)$. Perform

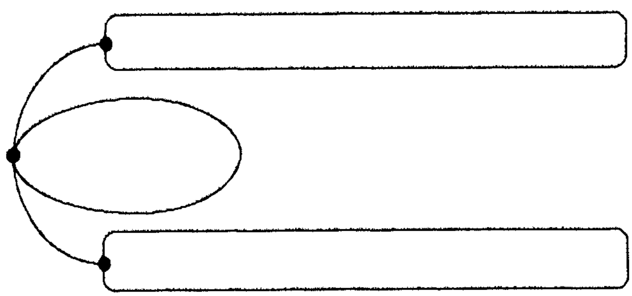


Figure 22

the sequence of homotopies of Y' in $G_1 \times I$ shown in Figs. 15(c) to 15(g) to transform Y' into Y. □

The graph Y shown in Fig. 15(g) lies in a regular neighborhood of a component of $\tilde{\Gamma}_1$. The image of Y and G_1 under the projection p_1 is shown in Fig. 16. Topologically Y is a wedge of two circles, the projection of which are the two loops α, β in G_1 shown in Fig. 16. The vertex of Y projects to the point v in Fig. 16 on the intersection of α and β. Thus $p_1(\tilde{\Gamma}_2)$ contains 2 loops which are homotopic to the 2 loops α and β in G_1 shown in Fig. 16. The loops α and β fill G_1 and so cannot be homotoped to be disjoint from any essential closed curve such as γ. This contradicts the disjointness of $\tilde{A}$ and $\tilde{\Gamma}_2$, proving that no annulus $\tilde{A}$ can exist in this case.

The next case that we consider is that δ is contained in G_1. Since $\tilde{M} \cup \tilde{N}_2 = G_1 \times I$, there is an annulus A' in G_1 with the same boundary as $\tilde{A}$. It follows that the torus $\tilde{A} \cup A'$ bounds a solid torus T in $G_1 \times I$ on one side. We may perform an isotopy of $G_1 \times I$ so that $T = A' \times [0, 1/2]$. If T contains $\tilde{\Gamma}_2$ then $\gamma \times I$ is an essential annulus disjoint from $\tilde{\Gamma}$ which cannot exist by the previous case. Otherwise if T does not contain $\tilde{\Gamma}_2$ then T is a solid torus in $\tilde{M}$ and so $\tilde{A}$ is boundary parallel in $\tilde{M}$. But this implies that A is boundary parallel in M, a contradiction.

The last case is that δ is contained in $\partial_2 \tilde{M}$.

LEMMA 2.5. *γ is isotopic in G_1 to the curve labelled α in* Fig. 16.

Proof. We first observe that δ is an essential $G_1 \times I$ and that δ can be homotoped in $G_1 \times I$ into $\tilde{\Gamma}_2$, and thus homotoped into an essential loop in Y. It follows that $p_1 \delta$ is freely homotopic into $p_1(Y)$. Let v be the point in G_1, shown in Fig. 16, which is the image under p_1 of the vertex in the graph Y. We claim that the only non-trivial element of $p_{1*}\pi_1(Y)$ which is homotopic to an essential simple closed curve is $\alpha^{\pm 1}$. To see this, let $\pi: \tilde{G}_1 \to G_1$ be the covering of G_1 corresponding to the subgroup $p_{1*}\pi_1(Y)$ of $\pi_1(G_1)$. Then $\tilde{G}_1$ is a punctured torus, on which there are unique lifts $\tilde{\alpha}, \tilde{\beta}$ of α, β. Now $\tilde{\alpha}, \tilde{\beta}$ intersect in a single point lying over v as shown in Fig. 18. Also γ is homotopic to $p_1 \delta$ and therefore lifts to a loop $\tilde{\gamma}$ on $\tilde{G}_1$. If $\tilde{\gamma}$ cannot be homotoped in $\tilde{G}_1$ into $\tilde{\alpha}$, then $\tilde{\gamma}$ runs around $\tilde{\beta}$ and intersects other components of $\pi^{-1}(\beta)$ because β has an essential self-intersection on G_1, and therefore $\tilde{\gamma}$ intersects other components of $\pi^{-1}(\gamma)$. But this contradicts the simplicity of γ and proves the lemma. □

We have shown that γ is isotopic in G_1 to α and thus the boundary component of A on $\partial_1 M$ is isotopic to $\varepsilon = p(\alpha)$. By tracing the loop α back through the Figs. 15(g) to 15(a), we see that α is homotopic in $G_1 \times I$ to the loop $p^{-1}(E)$ shown in Fig. 17. Thus α is homotopic in G_1 to the loop labelled α in Fig. 17. Hence $\varepsilon = p(\alpha)$

is homotopic in ∂M_1 to the loop labelled ε in Fig. 19. Applying the involution τ we see that the other boundary component of A must be isotopic in $\partial_2 M$ to $\tau\varepsilon$. From Fig. 19 one sees that ε is contractible in $M \cup N_1$ and hence that $\tau\varepsilon$ is contractible in $M \cup N_2$. The annulus A provides a free homotopy from ε to $\tau\varepsilon$, and thus ε is contractible in $M \cup N_2$ also. We compute the homotopy class $[\varepsilon] \in \pi_1(M \cup N_2)$ from Fig. 19, and see that it is non-trivial. This contradicts the existence of the annulus A in this last case, and proves 1.2(3), completing the proof of the proposition. □

REFERENCES

[He] J. HEMPEL, 3-*Manifolds*, Ann. Math. Studies *86*, Princeton University Press.

[HLM] H. M. HILDEN, M. T. LOZANO and J. M. MONTESINOS, Universal Knots, Knot Theory and Manifolds, *1144*, Springer-Verlag.

[Ho] C. HOGDSON, *Geometric Structures on 3-dimensional Orbifolds: Notes on Thurston's proof*, preprint.

[McM1] C. MCMULLEN, *Iteration on Teichmuller Space*. Invent. Math. *99* (2), 425–454.

[McM2] C. MCMULLEN, *Riemann Surfaces and the Geometrization of* 3-*Manifolds*, Bull. Am. Math. Soc. *27* (2), 207–216.

[Mo] J. MONTESINOS, *Classical Tesselations & Three-Manifolds*, Springer-Verlag.

[My] R. MYERS, *Contractible Open* 3-*Manifolds which are not Covering Spaces*, Topology *27* (1988), 27–35.

[Th1] W. P. THURSTON, *Preprint on the Orbifold Theorem.*

[Th2] W. P. THURSTON, *Three dimensional manifolds, Kleinian Groups and hyperbolic geometry*, Bull. Am. Math. Soc. *6*, 357–381.

[Th3] W. P. THURSTON, *Universal Links, preprint.*

[To] J. L. TOLLEFSON, *Involutions of Sufficiently Large* 3-*manifolds*, Topology *20* (1981), 323–353.

[Wh] J. H. C. WHITEHEAD, *A Certain Open Manifold whose Group is Unity*, Quaterly J. Math (1935), 113–129.

[Wr] D. G. WRIGHT, *Contractible Open Manifolds which are not Covering Spaces*, Topology *31* (2), 281–291.

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