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# ROOTS OF UNITY AND THE CHARACTER VARIETY OF A KNOT COMPLEMENT

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**Abstract**

Using elementary methods we give a new proof of a result concerning the special form of the character of the bounded peripheral element which arises at an end of a curve component of the character variety of a knot complement.

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## 1. Introduction

In [3] the following theorem is proved:

THEOREM 1.1. *Suppose that $\rho_n$ is a sequence of representations of the fundamental group of a knot which blows up on the boundary torus $T$, and which converge to a simplicial action on a tree. Suppose that there is an essential simple closed curve $C$ on $T$ whose trace remains bounded. Then* $\lim_{m\to\infty} \mathrm{tr}(\rho_m(C)) = \lambda + 1/\lambda$ *where $\lambda^n = 1$ whenever there is a component $S$ of a reduced surface associated to the degeneration so that $S$ has $n$ boundary components.*

Precise definitions of the terms will be given below, but the rough description is as follows. If one has a curve of characters of representations of a manifold



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with a single torus boundary component, then the method of [5] for producing boundary slopes is to go to some end of the character variety. Two things can happen on the boundary torus when one does this; either all the characters remain bounded and the surface produced from the resulting splitting can be chosen to be closed, or there is a particular simple closed curve whose character remains bounded. We shall focus on this latter behaviour. This simple closed curve gives the boundary slope and a natural question to ask is what the value of the character of the closed curve at the ideal point is. The point of the theorem is that the character has a special form and that some information about this form is carried by the topology of a splitting surface coming from the degeneration.

As an aside, we note that the theorem shows that for a two-bridge knot only the numbers $\pm 1$ can occur, since it is known ([6]) that the essential surfaces in such knot complements have either one or two boundary components. It is also known that other values are possible — the untwisted double of the trefoil contains an ideal point where the bounded character takes on the value $\omega + \omega^{-1}$ where $\omega$ is an eleventh root of unity. But other than this, little is known. For example, it still seems to be an open question whether a nontrivial root of unity can arise in this way in the character variety of a hyperbolic knot.

In this paper we shall give a new proof of Theorem 1.1. In fact it is a geometric version of one of the proofs of [3], but the fact that it avoids both algebraic $K$-theory and algebraic geometry and provides a somewhat new perspective should hopefully yield some new insights.

The point of view of this proof is that the action on a tree produced by the techniques of [5] is approximated in a geometrical sense by the action of the representations $\rho_m$ for $m$ large. This is the idea used in [1] and also [2].

## 2. Main results

LEMMA 2.1. *Given $L > 0$ and $n > 0$ there is a constant $K_n > 0$ such that for any set of matrices $A_1, A_2, \ldots, A_n \in \mathrm{SL}_2(\mathbf{C})$ with $|\mathrm{tr}(A_i A_j)| < L$, for all $1 \leq i, j \leq n$ then there is a point $x \in \mathbf{H}^3$ which is moved a distance of at most $K_n$ by $A_i$ for every $i$.*

PROOF. The proof is by induction on $n$. For $n = 1$, the result follows from the relationship between trace and translation length. For $n = 2$, suppose that we are given a pair of matrices $A, B$ in $\mathrm{SL}_2(\mathbf{C})$. The proof proceeds by showing that we can simultaneously conjugate $A$ and $B$ so that they are in the compact subset $\Omega$ of $\mathrm{SL}_2(\mathbf{C})$, where:

$$\Omega = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{C}) : \ |a|, |b|, |c|, |d| \leq 2L + 2 \right\}$$

If $A$ and $B$ have a common fixed point on the sphere at infinity, then we may perform a simultaneous conjugacy on them to put the common fixed point at infinity in the upper half space, then:

$$A = \begin{pmatrix} a & c \\ 0 & 1/a \end{pmatrix}, \qquad B = \begin{pmatrix} b & d \\ 0 & 1/b \end{pmatrix}.$$

Furthermore, by conjugating by a diagonal matrix we may ensure that $|c|, |d| \leq 1$. The hypothesis implies that $1/(L+1) < |a|^2, |b|^2 < (L+1)$ so that $A, B$ lie in $\Omega$.

If $A$ and $B$ do not have a common fixed point then there is a point $z$ on the sphere at infinity which is fixed by $A^{-1}B$. By means of a conjugacy we can arrange that $z = 0$ and that $Bz = \infty$. Thus $Az = \infty$ also and thus $A, B$ are conjugate to:

$$A = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} b & c \\ -1/c & 0 \end{pmatrix}.$$

The hypothesis implies that $|a|^2, |b|^2 < L + 2$. Observe that $\mathrm{tr}(AB) = ab - c - (1/c)$, thus $|c + (1/c)| < 2L + 2$, and so $A$ and $B$ lie in $\Omega$.

Given a point $x$ in $\mathbf{H}^3$ the function which assigns to a pair of matrices $(A, B)$ the maximum of the hyperbolic distance of $x$ from $Ax$ and from $Bx$ is continuous, and therefore bounded on $\Omega \times \Omega$. Since the existence of a point $x$ satisfying the conclusion is invariant under conjugacy, the result for $n = 2$ follows.

Suppose inductively that the result is true for any set of $(n-1)$ matrices with a constant $K_{n-1}$. Given a set of $n \geq 3$ matrices satisfying the hypothesis, let $x_i$ be a point moved a distance at most $K_{n-1}$ by the matrices $\{A_j \mid 1 \leq j \leq n,\ j \neq i\}$. Define $C_i$ to be the convex hull of the finite set $\{x_j | j \neq i\}$ and consider the goedesic triangle $T$ with vertices $\{x_1, x_2, x_3\}$. The radius of the largest circle which may be inscribed in a geodesic triangle is $2\ln[(1+\sqrt{5})/2]$ thus there is a point $y$ which lies within this distance of each side of $T$. Each $C_i$ contains at least two vertices of $T$, and therefore at least one edge of $T$. Therefore $y$ lies within a distance of $2\ln[(1+\sqrt{5})/2]$ of $C_i$. The vertices of $C_i$ are moved at most a distance of $K_{n-1}$ by the matrix $A_i$, therefore every point of $C_i$ is moved at most a distance $K_{n-1}$ by $A_i$. This uses the fact that the distance between a point and its image under an isometry, is a convex function. Thus the distance of $y$ from $A_i y$ is at most $K_{n-1} + 4\ln[(1+\sqrt{5})/2]$.

COROLLARY 2.1. *Suppose that $G$ is a finitely generated group and that we are given that $\rho_n : G \longrightarrow \mathrm{SL}_2(\mathbf{C})$ is a sequence of representations which have characters $\chi_n = \mathrm{trace} \circ \rho_n$ which converge weakly to a function $\chi$. Then there is a subsequence $\rho_{n_i}$ and matrices $A_i \in \mathrm{SL}_2(\mathbf{C})$ such that $A_i.\rho_{n_i}.A_i^{-1} \to \rho$ and* $\mathrm{trace} \circ \rho = \chi$.

PROOF. Choose a finite set of elements $\{\alpha_1, \alpha_2, \cdots, \alpha_p\}$ which generate $G$, then by Lemma 2.1 we have that for $n$ sufficiently large there is $x_n$ in $\mathbf{H}^3$ which is moved a distance at most $K_p$ by $\rho\alpha_i$ for $i = 1, 2, \cdots, p$. After conjugating each $\rho_n$, we may arrange that $x_n = x$ for every $n$. The subset $\Omega$ of $\mathrm{SL}_2(\mathbf{C})$ consisting of elements which move $x$ a distance of at most $K_p$ is compact. Thus there is a subsequence as claimed.

The set $X(G, \mathrm{SL}_2(\mathbf{C}))$ of characters of representations of a group $G$ into $\mathrm{SL}_2(\mathbf{C})$ is given the weak topology. This coincides with the topology induced by an embedding of $X(G, \mathrm{SL}_2(\mathbf{C}))$ into a finite dimensional Euclidean space given by using the traces of a (large enough) finite set of elements of $G$. If $G$ is finitely generatated it follows from Corollary 2.1 that $X(G, \mathrm{SL}_2(\mathbf{C}))$ is a closed subset of Euclidean space. (See [5].)

LEMMA 2.2. *Suppose that $G$ is a finitely generated group and $\rho_n : G \longrightarrow \mathrm{SL}_2(\mathbf{C})$ is a sequence of representations with the property that for every $\alpha \in G$, $\mathrm{tr}(\rho_n\alpha) \to \pm 2$ as $n \to \infty$. Then after changing each $\rho_n$ by a suitable conjugacy, a subsequence of $\{\rho_n\}$ converges to an abelian representation.*

PROOF. By Corollary 2.1, we can conjugate a subsequence of the $\rho_n$ so that this subsequence converges to a representation $\rho$ for which $\mathrm{tr}(\rho\alpha) = \pm 2$ for every $\alpha$ in $G$. The image of $\rho$ consists entirely of parabolic elements and $\{\pm I\}$. If two of these parabolics have distinct fixed points, then a large power of one times a large power of the other is hyperbolic, which contradict the hypothesis. Thus $\rho$ is reducible, and so can be conjugated to be upper triangular. Now a sequence of conjugacies by suitable diagonal matrices makes $\rho$ converge to a diagonal representation.

We now study degenerations of knot complements. Let $M$ be the complement of a knot and $\rho_n : \pi_1(M) \longrightarrow \mathrm{SL}_2(\mathbf{C})$ be a sequence of representations. We say that this sequence *blows up* if there is an element $\alpha \in \pi_1(M)$ such that $\mathrm{trace}(\rho_n\alpha) \to \infty$. We assume that the projectivized length functions which they determine converge to some projectivized length function and further, that all these representations lie on a curve in the representation variety. The consequence of this assumption is that the limiting projectivized length function comes from an action of $\pi_1(M)$ on a simplicial tree $\Gamma$ rather than an **R**-tree.

We shall assume that $\pi_1(M)$ acts on $\Gamma$ without inversions and that if an edge $e$ in $\Gamma$ is incident to a vertex $v$ then $\mathrm{stab}(e)$ is contained in but not equal to $\mathrm{stab}(v)$. Let $e$ be an edge of $\Gamma$; we construct a properly embedded surface $S$ in $M$ from $e$ as follows. Let $\tilde{M}$ be the universal cover of $M$ and choose an equivariant map

$$f : \tilde{M} \longrightarrow \Gamma.$$

Make $f$ transverse to the midpoint of $e$. Then $\tilde{S} = f^{-1}(e)$ is a properly embedded 2-sided surface in $\tilde{M}$, possibly not connected. After performing compressions on $\tilde{S}$ by equivariantly homotoping $f$ we may assume that all components of $\tilde{S}$ are planes. We assume that the action of $\pi_1(M)$ on $\Gamma$ has no common fixed point. Let $F$ be a component of $S$ which separates $M$ into two components $M_+$ and $M_-$. Then:

$$\pi_1(M) = \pi_1(M_+) *_{\pi_1(F)} \pi_1(M_-).$$

For some choice of $F$ this decomposition is non-trivial. We have that $\pi_1(F)$ is contained in $\mathrm{stab}(e)$ and there are finite collections of vertices $v_i^{\pm}$ of $\Gamma$ with $\pi_1(M_+)$ contained in the group generated by the union of the $\mathrm{stab}(v_i^+)$, and similarly for the minus sign.

PROPOSITION 2.1. *With the above assumptions, each $\rho_n$ may be replaced by a conjugate so that there is a subsequence of $\rho_n|\mathrm{stab}(e)$ which converges to an abelian representation.*

PROOF. If $\gamma \in \pi_1(M)$ has the property that $\mathrm{trace}(\rho_n(\gamma))$ is bounded as $n \to \infty$, we will say that $\gamma$ *remains bounded.* By [4], $\gamma$ remains bounded if and only if $\gamma$ stabilizes some vertex of $\Gamma$.

We apply Corollary 2.1 to the sequence of representations $\rho_n|\mathrm{stab}(v_+)$ to get a representation $\rho_+$ of $\mathrm{stab}(v_+)$ and $A_i \in \mathrm{SL}_2(\mathbf{C})$ such that

$$A_i.(\rho_{n_i}|\mathrm{stab}(v_+)).A_i^{-1} \to \rho_+.$$

Apply Corollary 2.1 to the sequence of representations $\rho_{n_i}|\text{stab}(v_-)$ to get a representation $\rho_-$ of stab$(v_-)$ and $B_j \in \text{SL}_2(\mathbf{C})$ such that

$$B_j.(\rho_{n_{i_j}}|\text{stab}(v_-)).B_j^{-1} \longrightarrow \rho_-.$$

If every element $\gamma \in \text{stab}(e)$ has $\text{tr}(\rho_+\gamma) = \pm 2$ then Lemma 2.2 gives the result.

So we may assume there is an element $\gamma$ in stab$(e)$ with $\text{trace}(\rho_+\gamma) = c \neq \pm 2$. We claim that there is an element $\alpha_+ \in \text{stab}(v_+) - \text{stab}(e)$ with the property that $\text{trace}(\rho_+\alpha_+) = a \neq \pm 2$. Choose an element $\alpha \in \text{stab}(v_+) - \text{stab}(e)$. Note that $\alpha, \alpha\gamma, \alpha\gamma^{-1} \in \text{stab}(v_+) - \text{stab}(e)$ so that if any of these elements have $\text{trace}(\rho) \neq \pm 2$ we are done. Otherwise set $A = \rho_+\alpha$ and $C = \rho_+\gamma$; then from

$$\text{tr}(AC) + \text{tr}(AC^{-1}) = \text{tr}(A)\text{tr}(C)$$

we see that $\pm 2\text{tr}(C) = \pm 2 \pm 2$, but $\text{tr}(C) = c \neq \pm 2$ hence $\text{tr}(C) = 0$. Now

$$\text{tr}(A^2C) + \text{tr}(C) = \text{tr}(A)\text{tr}(AC),$$

thus $\text{tr}(A^2C) = \pm 4$. If $\alpha^2\gamma$ is not in stab$(e)$ we are done. Otherwise we can replace $\gamma$ in the above argument by $\alpha^2\gamma$ to conclude that $\alpha^3\gamma$ will do.

Similarly, we can assume there is $\alpha_- \in \text{stab}(v_+) - \text{stab}(e)$ with $\text{tr}(\rho_-\alpha_-) \neq \pm 2$.

Consider the action of $\alpha_+$ and $\alpha_-$ on the tree $\Gamma$. These elements stabilize different vertices of $\Gamma$ and do not stabilize the edge between them, so the element $\alpha_+.\alpha_-$ acts as a non-trivial hyperbolic translation on $\Gamma$ (see [4]). Thus $\text{tr}(\rho_n(\alpha_+.\alpha_-)) \to \infty$ as $n \to \infty$. It follows that the non-parabolic elements $\rho_n(\alpha_+)$ and $\rho_n(\alpha_-)$ have axes which are moving away from each other in $\mathbf{H}^3$. By a sequence of conjugacies, we may arrange that $\text{Fix}(\rho_n(\alpha_+))$ is converging to 0 and $\text{Fix}(\rho_n(\alpha_-))$ is converging to $\infty$.

Now consider any element $\beta \in \text{stab}(e)$. Then $\beta\alpha_\pm\beta^{-1} \in \text{stab}(v_\pm)$ since $\text{stab}(e) \subset \text{stab}(v_\pm)$. This implies that the axes of $\rho_n(\beta\alpha_+\beta^{-1})$ and $\rho_n(\alpha_+)$ remain within a bounded distance of each other, since they are elements with trace bounded away from $\pm 2$ and their product is in stab$(v_+)$ and so has bounded trace. Similarly for the minus sign. It follows that for $n$ large, $\rho_n(\beta)$ moves 0 and $\infty$ by a very small amount. Thus $\rho_n(\beta)$ converges to a diagonal matrix as $n \to \infty$.

Suppose that $M$ is a connected 3-manifold and $F$ is a surface properly embedded, but possibly not connected, in $M$. We do not assume that either $M$ or

$F$ is orientable; we do not assume that $F$ is incompressible. We show how to construct an action of $\pi_1(M)$ on a tree from this data.

Let $F_1, F_2, \cdots, F_n$ be the components of $F$ and $M_1, M_2, \cdots, M_m$ be components of $M - F$. Let $\pi : \tilde{M} \longrightarrow M$ be the universal cover of $M$. We construct a graph $\Gamma$ by assigning one vertex to each component of $\pi^{-1}(M - F)$ and one edge to each component of $\pi^{-1}(F)$. The edge corresponding to the component $\tilde{F}_i$ of $\pi^{-1}(F_i)$ is incident to the vertex corresponding to the component $\tilde{M}_j$ of $\pi^{-1}(M_j)$ if the closure of $\tilde{M}_j$ contains $\tilde{F}_i$. We must show that every edge is incident to precisely 2 vertices. To see this, note that there are either one or two components of $M - F$ adjacent to $F_i$. If there are two components, the result is clear. If there is only one component of $M - F$ adjacent to $F_i$, say $M_j$, then there is a loop in $M_j \cup F_i$ which meets $F_i$ once transversely. Thus this loop is essential, and hence in $\tilde{M}$ there are two distinct components of $\pi^{-1}(M_j)$ whose closure contains $\tilde{F}_i$. It is clear that the action of $\pi_1(M)$ on $\tilde{M}$ by covering transformations induces a simplicial action on $\Gamma$.

Next we show that $\Gamma$ is a tree. There is an embedding $i : \Gamma \longrightarrow \tilde{M}$ such that the image of each vertex of $\Gamma$ lies in the component of $\pi^{-1}(M - F)$ to which it corresponds, and so that the image of each edge of $\Gamma$ intersects once transversely $\pi^{-1}F$ in the component to which it corresponds. Observe that there is a neighborhood of $\tilde{F}_i$ in $\tilde{M}$ which is a product $I \times \tilde{F}_i$. This is because $\tilde{F}_i$ is properly embedded in $\tilde{M}$, and if $\tilde{F}_i$ is one-sided in $\tilde{M}$ then there is a loop in $\tilde{M}$ which meets $\tilde{F}_i$ once transversely, which implies that this loop is non-zero in $H_1(\tilde{M}; \mathbf{Z}_2)$. However $\tilde{M}$ is simply connected, giving a contradiction. Therefore $\tilde{F}_i$ is 2-sided in $\tilde{M}$. There is a retraction $r : \tilde{M} \longrightarrow i(\Gamma)$ defined by sending a product neighborhood $I \times \tilde{F}_i$ of $\tilde{F}_i$ onto the edge to which it corresponds by projection onto the $I$ factor, and sending a component of $\pi^{-1}(M - F)$ with these product neighborhoods removed to the vertex of $i(\Gamma)$ to which it corresponds. Since $\tilde{M}$ is simply connected, it follows that $i(\Gamma)$ is simply connected.

Now suppose that the boundary of $M$ contains an incompressible torus $T$ and that some component $S$ of $F$ meets $T$ in an essential loop $\alpha$. We now assume that $F$ is incompressible and contains no boundary parallel disc. The incompressibility of $F$ means that every component of $S \cap T$ is essential in $T$, and therefore parallel to $\alpha$. Let $\tilde{T}$ be a component of $\pi^{-1}T$. This implies that each $\pi_1(M_i)$ injects into $\pi_1(M)$, and thus $\tilde{M}_i$ is simply connected. It follows that $\tilde{M}_i$ meets $\tilde{T}$ in a connected, but possibly empty, set.

We choose a base point $\tilde{x} \in \tilde{T}$ and set $x = \pi(\tilde{x})$ in order to identify $\pi_1(M, x)$ with the covering transformations of $\tilde{M}$; then $\tilde{T}$ is stabilized by $\pi_1(T, x)$. Let $C_1, C_2, \cdots, C_n$ be the components of $T \cap S$ which are all parallel to $\alpha$, labelled in the order they go round $T$. The components of $\pi^{-1}(C_1 \cup \cdots \cup C_n)$ are parallel lines on $\tilde{T}$. It follows that each component of $\tilde{T} - \pi^{-1}(C_1 \cup \cdots \cup C_n)$ meets a distinct component $\tilde{M}_j$ of $\tilde{M} - \tilde{F}$ and thus corresponds to a distinct vertex in $\Gamma$. Thus the image of $\tilde{T}$ under $r$ is a line $\ell$ in $\Gamma$, and $r : \tilde{T} \longrightarrow \ell$ can be chosen to be a submersion.

We now assume that $S$ can be transversely oriented, that is, $S$ is 2-sided in $M$. Choose two arcs, one in $S$ and the other in $T - S$, from $C_i$ to $C_{i+1}$ with the same end points. The union of these two arcs is a loop $\gamma_i$. Push $\gamma_i$ off $S$ using the transverse orientation. We now assume that $[S] = 0 \in H_2(M, \partial M; \mathbf{Z}_2)$. From this it follows that when the loop $\gamma_i$ is pushed off $S$ it must intersect $S$ an even number of times. This implies that if $C_i$ is isotoped along $T$ to $C_{i+1}$ then the transverse orientations of $S$ along $C_i$ and $C_{i+1}$ are *opposite*.

Each line $\tilde{C}_i$ in $\tilde{T} \cap \pi^{-1}S$ lies in a component of $\pi^{-1}(S)$, thus the edges of $\Gamma$ corresponding to the family of lines $\tilde{T} \cap \pi^{-1}S$ are in the same orbit under $\pi_1(M, x)$. Given a pair of adjacent lines $\tilde{C}_i$, $\tilde{C}_{i+1}$ in $\tilde{T} \cap \pi^{-1}(C_i \cup C_{i+1})$, let $e_i, e_{i+1}$ be the corresponding edges in $\Gamma$. Orient $\ell$ and use this to orient each edge on $\ell$. Let $\tau : \pi_1(M, x) \longrightarrow \mathrm{Aut}(\Gamma)$ be the action of $\pi_1(M, x)$ on $\Gamma$. We will write $\tau_\gamma$ for $\tau(\gamma)$. Then for some $\delta_i \in \pi_1(M, x)$,

$$\tau_{\delta_i}(e_i) = -e_{i+1}, \tag{1}$$

where the minus sign means with orientation reversed. This follows from the discussion of transverse orientations of surfaces above because an orientation of an edge $e_i$ corresponds to a transverse orientation of the corresponding surface $\tilde{S}$. We remark for later use that $\delta_i$ is in the free homotopy class of the loop $\gamma_i$ constructed above.

Now suppose that

$$\tau' : \pi_1(M, x) \longrightarrow \mathrm{Aut}(\Gamma')$$

is a simplicial action without inversions on a simplicial tree $\Gamma'$. Then there is an equivariant map

$$f : \tilde{M} \longrightarrow \Gamma'$$

which is transverse to the midpoints of all edges of $\Gamma'$, and this map may be chosen so that every component of the pre-image under $f$ of the midpoints of the edges of $\Gamma'$ is an incompressible 2-sided surface $\tilde{F}$ in $\tilde{M}$. This surface $\tilde{F}$ in $\tilde{M}$ projects to a 2-sided incompressible surface $F$ in $M$. (The condition that the action is without edge inversions is equivalent to $F$ being 2-sided.) We may apply the construction above to $F$ to get an action

$$\tau : \pi_1(M, x) \longrightarrow \mathrm{Aut}(\Gamma)$$

on a tree $\Gamma$. Clearly the map $f$ factors as $f = \overline{f} \circ r$, where $\overline{f} : \Gamma \longrightarrow \Gamma'$ is an equivariant map.

Suppose now that $\pi_1(T, x)$ stabilizes no vertex of $\Gamma'$. There is a line $\ell'$ in $\Gamma'$ which is stabilized by $\pi_1(T, x)$. We claim that $f$ may be chosen so that $\overline{f}|\ell$ is injective. $\tilde{T}$ is a plane on which $\pi_1(T, x)$ acts freely with quotient the torus $T$. $f|\tilde{T}$ covers a map $T \longrightarrow \ell'/\pi_1(T, x) = S^1$, which is homotopic to a submersion. Lifting this homotopy gives a $\pi_1(T, x)$-equivariant homotopy of $f|\tilde{T}$. This can be used to give a homotopy of $f$ on all of $\tilde{M}$ by using a small collar neighborhood of $\tilde{T}$. This homotopy may then be done equivariantly to each component of $\pi^{-1}(T)$.

Let $e_i' = f(e_i)$ and $e_{i+1}' = f(e_{i+1})$, then since $f|\ell \longrightarrow \ell'$ is an an equivariant simplicial homeomorphism it follows from equation (1) that

$$\tau_{\delta_i}'(e_i') = -e_{i+1}'. \tag{2}$$

Let $S$ be a component surface of $F$ which we assume is oriented. Use this orientation to orient the boundary components $C_1, C_2, \cdots, C_n$ of $S$. The base point $x$ is chosen on $C_1$, and let $c_1, c_2, \cdots, c_n$ be elements of $\pi_1(S, x)$ which correspond to the oriented boundary components of $S$. Thus $c_1.c_2.\cdots.c_n$ is a commutator in $\pi_1(S, x)$. Since $C_i$ and $C_{i+1}^{-1}$ are isotopic on $T$, the elements $c_i$ and $c_{i+1}^{-1}$ are conjugate so there is an element $\delta_i \in \pi_1(M, x)$ with $\delta_i.c_i.\delta_i^{-1} = c_{i+1}^{-1}$. Clearly the covering transformation of $\tilde{M}$ corresponding to $\delta_i$ sends $\tilde{S}_i$ to $\tilde{S}_{i+1}$ and thus $\delta_i$ satisfies (1) and hence (2).

DEFINITION. Suppose that $\pi_1(M)$ acts on a tree, then a surface $F$ in $M$ is called a *reduced surface associated to the action* if it is associated to the action and has the minimal number of boundary components.

We can now give a proof of Theorem 1.1.

PROOF OF THEOREM 1.1. Let $S$ be a component of a reduced surface associated to the limiting action on a tree. We continue to use the notation $c_1, c_2, \cdots, c_n$ used above for elements of $\pi_1(S, x)$ corresponding to the boundary components of $S$. Thus $\pi_1(S, x)$ is a subgroup of stab$(e)$ for some edge $e$ of $\Gamma$. Let $\lambda$ be a limiting eigenvalue of $\rho_n(C)$.

If the surface $S$ does not separate, then $S$ must be a Seifert surface for the knot and have a single boundary component. The last sentence of this proof gives the result in this case. Thus we may assume that $S$ separates, and thus has an even number of boundary components. By Proposition 2.1 we may assume that $\rho_m|\text{stab}(e)$ converges to a diagonal representation $\rho$ so:

$$\rho(c_i) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}^{\epsilon_i} \qquad \text{for all } i$$

where $\epsilon_i = \pm 1$. If $\lambda = \pm 1$ there is nothing to prove. Otherwise for $m$ large $\rho_m(c_i)$ has trace bounded away from $\pm 2$ and therefore the endpoints of the axis of $\rho_m(c_i)$ are converging to $0, \infty$. Now there is $\delta_i \in \pi_1(M)$ with $\delta_i.c_i.\delta_i^{-1} = c_{i+1}^{-1}$, and $\delta_i$ satisfies (2), hence $\rho_m(\delta_i)$ almost switches 0 and $\infty$. It follows that $\rho_m(c_i)$ and $\rho_m(c_{i+1})$ are almost equal, and hence that all the $\epsilon_i$ are equal.

The homotopy class $\gamma = c_1.c_2.\cdots.c_n$ is a commutator; therefore by Proposition 2.1, $\rho(\gamma) = I$, hence $\lambda^n = 1$.

## References

[1] M. Bestvina, 'Degenerations of the hyperbolic space', *Duke J. Math.* **56** (1988), 142–161.
[2] D. Cooper, 'Degenerations of representations in SL(2,C)', preprint.
[3] M. Culler, D. Cooper, H. Gillett, D. D. Long and P. B. Shalen, 'Plane curves associated to character varieties of knot complements', preprint.
[4] M. Culler and J. Morgan, 'Group actions on **R**-trees', *Proc. London Math. Soc.* **55** (1987), 571–604.
[5] M. Culler and P. B. Shalen, 'Varieties of group representations and splittings of 3-manifolds', *Ann. of Math.* **117** (1983), 109–145.
[6] A. Hatcher and W. P. Thurston, 'Incompressible surfaces in two bridge knot complements', *Invent. Math.* **79** (1985), 225–246.

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