

STOCHASTIC AVERAGING CORRECTORS FOR A NOISY HAMILTONIAN SYSTEM WITH DISCONTINUOUS STATISTICS

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ABSTRACT. We construct here certain perturbed test functions for stochastic averaging of a noisy planar Hamiltonian system containing a homoclinic orbit. The noise is assumed to be small and have skewness at the homoclinic orbit. Following Sowers, we center our efforts on a singular perturbations problem in a boundary layer near the homoclinic orbit. At the heart of this analysis is the solution of a set of heat equations, coupled through their boundary data. We identify the glueing conditions, which are sufficient conditions ensuring solvability of the above problem. Probabilistically, the glueing conditions give the relative likelihoods, in the averaged picture, of diffusing into the various regions of phase space when one starts at the homoclinic orbit.

1. INTRODUCTION

A large number of phenomena in the natural sciences and engineering can be modelled effectively using (deterministic) dynamical systems, either by ordinary differential equations or by maps. In reality, however, these phenomena are almost always subject to random *noise*. The noise is typically small and its effect over short times is consequently negligible. Over long times, however, the noise can lead to macroscopic *random transitions*. A more accurate picture that captures this behavior can be obtained by using the noise to *regularize* the system. The evolution is now described by a stochastic differential equation (SDE), or more generally, by a second-order differential operator.

An important technique in the asymptotic analysis of such systems is *stochastic averaging*. The underpinning of stochastic averaging is a separation of time scales. There is a coordinate which varies slowly and a coordinate which varies quickly. As the ratio of the speed of the slow coordinate to the speed of the fast coordinate goes to zero, it is often possible to approximate the dynamics of the slow coordinate by a closed set of equations, obtained by carrying out a long-term average in the dynamics of the fast coordinate.

The underlying (deterministic) dynamical system in our investigation models the motion of a particle in a double-well potential and is described by a planar Hamiltonian ordinary differential equation (ODE) containing a homoclinic orbit. The focus of our study is a stochastic process arising as a result of small *skew* random perturbations of this Hamiltonian ODE. The skewness can be thought of as a coin flip: when the process hits a loop of the homoclinic orbit, a coin (possibly biased, possibly depending on the loop) is flipped to decide whether the next excursion will be outside or inside the corresponding loop. We also have here a separation of time scales: the fast motion corresponds to rotation along the orbits of the Hamiltonian ODE, while the slow motion (due to noise) is transversal diffusion *across* orbits.

Stochastic averaging for noisy Hamiltonian systems was considered first by Has'minskii ([Has68]) for the case of a single-well Hamiltonian. The calculations were extended to the case of Hamiltonians with multiple wells by Freidlin and Wentzell in [FW94] (see also [FW98], [FW99]). It was shown in [FW94] that the averaged motion can be represented as a diffusion on a graph with so-called glueing conditions at the vertices. A different perspective was provided by the work of Sowers ([Sow03], [Sow05]) where the problems of identification of the glueing conditions and averaging were recast in terms of constructing *correctors* for a natural singular perturbations problem in the glueing region. We will adopt here this latter approach.

The work presented here was the subject of the author's doctoral dissertation. The author would like to extend his deepest gratitude to his doctoral advisor Richard Sowers for his guidance and insight.

2. PROBLEM STATEMENT AND MAIN RESULT

2.1. Hamiltonian ODE. Let's start by fixing a two-well potential U .

Definition 2.1 (Potential). Fix a function $U \in C^\infty(\mathbb{R})$ with three consecutive zeroes $d_L \in (-\infty, 0)$, 0 and $d_R \in (0, \infty)$. Suppose further that $U'(d_L) < 0$, $U'(d_R) > 0$ and that there exists $0 < \varpi_0 < 1$ such that

$$U(x_1) \equiv -x_1^2/2 \quad \text{for } |x_1| \leq \varpi_0.$$

This last assumption implies in particular that $U'(0) = 0$ and $U''(0) = -1$.

We want to consider the motion of a particle subject to the potential U . Define next the *Hamiltonian* (total energy) of the particle by

$$\mathbf{H}(x_1, x_2) \stackrel{\text{def}}{=} U(x_1) + \frac{x_2^2}{2} \quad \text{for } (x_1, x_2) \in \mathbb{R}^2.$$

The function \mathbf{H} has a saddle point at $\mathfrak{o} \stackrel{\text{def}}{=} (0, 0)$ where $\mathbf{H}(\mathfrak{o}) = 0$. Let $\nabla^\perp \mathbf{H}$ be the symplectic gradient of \mathbf{H} , i.e. $\nabla^\perp \mathbf{H} = \left(\frac{\partial \mathbf{H}}{\partial x_2}, -\frac{\partial \mathbf{H}}{\partial x_1} \right)$. The motion of the particle is described by the flow

$$(1) \quad \begin{aligned} \dot{\mathfrak{z}}_t(x) &= \nabla^\perp \mathbf{H}(\mathfrak{z}_t(x)) \\ \mathfrak{z}_0(x) &= x \end{aligned}$$

for $t \in \mathbb{R}$, $x \in \mathbb{R}^2$. The phase portrait of the flow $\mathfrak{z}_t(x)$ has a homoclinic orbit in the shape of a ‘‘figure eight’’ contained in the level set $\mathbf{H}^{-1}(0)$. We are interested in small random perturbations of the system (1). In particular, we would like to understand the asymptotics of the random motion when we have *skewness* at the homoclinic orbit. We will restrict our analysis to the bounded set $\mathbf{S} \subset \mathbb{R}^2$ defined as follows:

Definition 2.2 (Restricted State Space). Fix $\mathfrak{u} > 0$ such that U' is negative on $[d_L - \mathfrak{u}, d_L + \mathfrak{u}]$ and positive on $[d_R - \mathfrak{u}, d_R + \mathfrak{u}]$ and such that U'' is negative on $[-\mathfrak{u}, \mathfrak{u}]$. Let $\mathfrak{h} > 0$ be such that

$$\begin{aligned} \mathfrak{h} &< \min\{U(d_L - \mathfrak{u}), U(d_R + \mathfrak{u})\} \quad \text{and} \\ -\mathfrak{h} &> \max\{U(d_L + \mathfrak{u}), U(d_R - \mathfrak{u}), U(-\mathfrak{u}), U(\mathfrak{u})\}. \end{aligned}$$

Let \mathbf{S} be the connected component of

$$\{x \in [d_L - \mathfrak{u}, d_R + \mathfrak{u}] \times \mathbb{R} : |\mathbf{H}(x)| < \mathfrak{h}\}$$

which contains \mathfrak{o} and let

$$\begin{aligned} \mathbf{S}_O &\stackrel{\text{def}}{=} \{x \in \mathbf{S} : 0 < \mathbf{H}(x) < \mathfrak{h}\}, \\ \mathbf{S}_L &\stackrel{\text{def}}{=} \{x \in \mathbf{S} : x \in \mathbb{R}_- \times \mathbb{R}, -\mathfrak{h} < \mathbf{H}(x) < 0\}, \\ \mathbf{S}_R &\stackrel{\text{def}}{=} \{x \in \mathbf{S} : x \in \mathbb{R}_+ \times \mathbb{R}, -\mathfrak{h} < \mathbf{H}(x) < 0\}. \end{aligned}$$

Finally, define the index set

$$\Lambda \stackrel{\text{def}}{=} \{O, L, R\}.$$

Note that, by construction, the Hamiltonian \mathbf{H} has *precisely* one critical point in $\bar{\mathbf{S}}$, namely the saddle point at \mathfrak{o} . Also, for $x = (x_1, x_2) \in \mathbf{S}$, we must have

$$|x_2| < \sqrt{2 \left(\mathfrak{h} - \min_{x_1 \in [d_L - \mathfrak{u}, d_R + \mathfrak{u}]} U(x_1) \right)}.$$

Hence, $\bar{\mathbf{S}}$ is a compact subset of \mathbb{R}^2 . Define, for each $\ell \in \Lambda$,

$$\mathcal{C}_\ell \stackrel{\text{def}}{=} \partial \mathbf{S}_\ell \cap \mathbf{H}^{-1}(0).$$

Thus, \mathcal{C}_ℓ is the part of the homoclinic orbit that borders the region \mathbf{S}_ℓ .

2.2. Pre-limit Process. We start by considering, for $\varepsilon \in (0, 1)$, the stochastic differential equation (SDE)

$$(2) \quad \begin{aligned} dx_t^{1,\varepsilon} &= \frac{\partial H}{\partial x_2}(x_t^{1,\varepsilon}, x_t^{2,\varepsilon})dt + \varepsilon dW_t^1 \\ dx_t^{2,\varepsilon} &= -\frac{\partial H}{\partial x_1}(x_t^{1,\varepsilon}, x_t^{2,\varepsilon})dt + \varepsilon dW_t^2 \end{aligned}$$

where $W = (W^1, W^2)$ is a standard two-dimensional Brownian motion. Moving to the slower time scale $t \mapsto \varepsilon^2 t$ (or equivalently, speeding up the process) and using the explicit form of the Hamiltonian H , we get the SDE

$$(3) \quad \begin{aligned} dx_t^{1,\varepsilon} &= \frac{1}{\varepsilon^2} x_t^{2,\varepsilon} dt + dW_t^1 \\ dx_t^{2,\varepsilon} &= -\frac{1}{\varepsilon^2} U'(x_t^{1,\varepsilon}) dt + dW_t^2. \end{aligned}$$

Let \mathcal{L} be the operator

$$\mathcal{L} \stackrel{\text{def}}{=} \frac{1}{2} \Delta \quad \text{where} \quad \Delta \stackrel{\text{def}}{=} \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

Then the generator of the process $(x_t^{1,\varepsilon}, x_t^{2,\varepsilon})$ has domain containing $C_c^2(\mathbb{R}^2)$ ¹ and is given by

$$\mathcal{L}^\varepsilon \stackrel{\text{def}}{=} \frac{1}{\varepsilon^2} (\nabla^\perp H, \nabla) + \mathcal{L}$$

on $C_c^2(\mathbb{R}^2)$.

The random perturbation of (1) that we have in mind is a stochastic process on \mathbb{R}^2 with initial distribution δ_{x_0} , $x_0 \in \mathbf{S} \setminus \{\mathbf{o}\}$, whose behavior away from $H^{-1}(0)$ is governed by the operator \mathcal{L}^ε , with *skewness conditions* at $H^{-1}(0)$ completing the picture². We will characterize this process as a probability measure $\mathbb{P}_{x_0}^\varepsilon$ on $C([0, \infty); \mathbb{R}^2)$ - the space of continuous functions taking $[0, \infty)$ to \mathbb{R}^2 , equipped with the topology of uniform convergence on bounded intervals. Rigorously, the effect of the skewness is to impose requirements (at $H^{-1}(0)$) on functions in the domain of the corresponding generator.

We start by spelling out the skewness conditions. Fix three positive numbers β_L , β_R and β_O (the skewness coefficients) satisfying

$$(4) \quad \boxed{\beta_O^2 = \beta_L \beta_R}$$

Definition 2.3 (Skewness Conditions). Denote by $\nu(x)$ the outward unit normal to $H^{-1}(0) \setminus \{\mathbf{o}\}$ at x . For a function $f \in C_c(\mathbb{R}^2) \cap C^2(\mathbb{R}^2 \setminus H^{-1}(0))$, the skewness conditions take the form

$$(5) \quad \begin{aligned} \beta_L \lim_{\substack{y \rightarrow x \\ y \in \mathbf{S}_L}} (\nabla f(y), \nu(x)) &= \beta_O \lim_{\substack{y \rightarrow x \\ y \in \mathbf{S}_O}} (\nabla f(y), \nu(x)) && \text{for } x \in \mathcal{C}_L \setminus \{\mathbf{o}\} \\ \beta_R \lim_{\substack{y \rightarrow x \\ y \in \mathbf{S}_R}} (\nabla f(y), \nu(x)) &= \beta_O \lim_{\substack{y \rightarrow x \\ y \in \mathbf{S}_O}} (\nabla f(y), \nu(x)) && \text{for } x \in \mathcal{C}_R \setminus \{\mathbf{o}\} \end{aligned}$$

where (\cdot, \cdot) is the standard inner product on \mathbb{R}^2 .

The intuition behind (5) is the following: on hitting loop \mathcal{C}_ℓ , $\ell \in \{L, R\}$, the process makes an excursion outside \mathcal{C}_ℓ with probability $\frac{\beta_O}{\beta_O + \beta_\ell}$ and inside \mathcal{C}_ℓ with probability $\frac{\beta_\ell}{\beta_O + \beta_\ell}$. In terms of the transition probability density (if one exists), the skewness corresponds to a discontinuity in the density at the homoclinic orbit; one might say the process has *discontinuous statistics*. One can think of the β_ℓ 's as being *control parameters* that can be tuned to regulate the relative likelihoods of diffusing into the various \mathbf{S}_ℓ 's. The requirement (4) is needed to carry out certain boundary layer calculations near the origin³. Its significance will be seen later.

¹For $k \in \mathbb{Z}^+$, $C_c^k(\mathbb{R}^2)$ denotes the space of C^k functions on \mathbb{R}^2 with compact support.

²Admittedly, we are being a little cavalier in identifying the homoclinic orbit with the level set $H^{-1}(0)$. Indeed, $H^{-1}(0)$ can, in general, include other connected components as well. However, since our analysis is restricted to $\bar{\mathbf{S}}$, we can safely assume that U has suitable growth outside $[d_L - u, d_R + u]$ to ensure that $H^{-1}(0)$ is *precisely* the homoclinic orbit.

³See Lemmas 3.7 and 3.13.

Define a linear operator \mathcal{A}^ε with domain $\mathcal{D}(\mathcal{A}^\varepsilon)$ as follows:

$$\mathcal{D}(\mathcal{A}^\varepsilon) \stackrel{\text{def}}{=} \left\{ f \in C_c(\mathbb{R}^2) \cap C^2(\mathbb{R}^2 \setminus \mathbf{H}^{-1}(0)) : f \text{ satisfies the skewness conditions (5),} \right. \\ \left. \mathcal{L}^\varepsilon f \text{ has a continuous extension to } \mathbb{R}^2 \right\},$$

$$(\mathcal{A}^\varepsilon f)(x) \stackrel{\text{def}}{=} \lim_{\substack{y \rightarrow x \\ y \in \mathbb{R}^2 \setminus \mathbf{H}^{-1}(0)}} (\mathcal{L}^\varepsilon f)(y) \quad \text{for } f \in \mathcal{D}(\mathcal{A}^\varepsilon), x \in \mathbb{R}^2.$$

Define $\Omega \stackrel{\text{def}}{=} C([0, \infty); \mathbb{R}^2)$ and let $\mathcal{P}(\Omega)$ be the set of Borel probability measures on Ω . Let X_t be the coordinate process on Ω , i.e. $X_t(\omega) \stackrel{\text{def}}{=} \omega(t)$ for all $t \geq 0$, $\omega \in \Omega$. Define the filtration $\{\mathcal{F}_t : t \geq 0\}$ by $\mathcal{F}_t \stackrel{\text{def}}{=} \sigma\{X_s : 0 \leq s \leq t\}$ for all $t \geq 0$, and define a σ -algebra on Ω by $\mathcal{F} \stackrel{\text{def}}{=} \bigvee_{t \geq 0} \mathcal{F}_t$. Define also the stopping time

$$\zeta \stackrel{\text{def}}{=} \inf\{t \geq 0 : X_t \notin \mathbf{S}\}.$$

Finally, fix an initial condition $x_0 \in \mathbf{S} \setminus \{\mathbf{o}\}$. The pre-limit process can now be characterized as a probability law $\mathbb{P}_{x_0}^\varepsilon$ on Ω via the martingale problem (see [EK86] and [SV79]).

Definition 2.4 (Pre-limit process). For $\varepsilon \in (0, 1)$, let $\mathbb{P}_{x_0}^\varepsilon \in \mathcal{P}(\Omega)$ be a solution to the stopped martingale problem for $(\mathcal{A}^\varepsilon, \delta_{x_0}, \mathbf{S})$. This means the following:

- (1) $\mathbb{P}_{x_0}^\varepsilon\{X_0 = x_0\} = 1$,
- (2) $\mathbb{P}_{x_0}^\varepsilon\{X_t = X_{t \wedge \zeta} \text{ for all } t \geq 0\} = 1$,
- (3) For $f \in \mathcal{D}(\mathcal{A}^\varepsilon)$, the process

$$f(X_{t \wedge \zeta}) - f(X_0) - \int_0^{t \wedge \zeta} (\mathcal{A}^\varepsilon f)(X_s) ds$$

is a $\mathbb{P}_{x_0}^\varepsilon$ -martingale, or equivalently, for any $0 \leq r_1 < r_2 < \dots < r_n \leq s < t$, $\{\varphi_j : j = 1, \dots, n\} \subset C_b(\mathbb{R}^2)$, we have

$$\mathbb{E}_{x_0}^\varepsilon \left[\left\{ f(X_{t \wedge \zeta}) - f(X_{s \wedge \zeta}) - \int_{s \wedge \zeta}^{t \wedge \zeta} (\mathcal{A}^\varepsilon f)(X_u) du \right\} \prod_{j=1}^n \varphi_j(X_{r_j}) \right] = 0$$

where $\mathbb{E}_{x_0}^\varepsilon$ is the expectation operator associated with $\mathbb{P}_{x_0}^\varepsilon$.

2.3. Main Result. Let's now understand how the ideas of averaging apply to this problem. First, note that for $f \in C^2(\mathbb{R})$, $x = (x_1, x_2) \in \mathbb{R}^2$, we have

$$\langle \nabla^\perp \mathbf{H}, \nabla(f \circ \mathbf{H}) \rangle \equiv 0$$

and

$$(6) \quad \mathcal{L}(f \circ \mathbf{H})(x) = \dot{f}(\mathbf{H}(x))(\mathcal{L}\mathbf{H})(x) + \frac{1}{2} \ddot{f}(\mathbf{H}(x)) \|\nabla \mathbf{H}\|^2(x).$$

For $f \in C_c^2(\mathbb{R})$ with $\text{supp}(f \circ \mathbf{H}) \subset \mathbf{S}_\ell$, $\ell \in \Lambda$,

$$(7) \quad f(\mathbf{H}(X_{t \wedge \zeta})) = f(\mathbf{H}(X_0)) + \int_0^{t \wedge \zeta} \dot{f}(\mathbf{H}(X_s))(\mathcal{L}\mathbf{H})(X_s) ds + \frac{1}{2} \int_0^{t \wedge \zeta} \ddot{f}(\mathbf{H}(X_s)) \|\nabla \mathbf{H}\|^2(X_s) ds + M_t$$

where M_t is a $\mathbb{P}_{x_0}^\varepsilon$ -martingale with quadratic variation

$$\langle M \rangle_t = \int_0^{t \wedge \zeta} \|\nabla(f \circ \mathbf{H})\|^2(X_s) ds.$$

Away from $\mathbf{H}^{-1}(0)$, X_t makes several rotations (under the law $\mathbb{P}_{x_0}^\varepsilon$) around the orbits of the Hamiltonian flow before $\mathbf{H}(X_{t \wedge \zeta})$ changes significantly. Hence, as $\varepsilon \searrow 0$, the integrals in (7) should be replaced by

$$\int_0^{t \wedge \zeta} \dot{f}(\mathbf{H}(X_s)) b_\ell(\mathbf{H}(X_s)) ds \quad \text{and} \quad \int_0^{t \wedge \zeta} \ddot{f}(\mathbf{H}(X_s)) a_\ell(\mathbf{H}(X_s)) ds$$

where b_ℓ and a_ℓ are obtained by taking a long-term average over the fast motion (the Hamiltonian flow):

$$b_\ell(h) \stackrel{\text{def}}{=} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (\mathcal{L}H)(\mathfrak{z}_t(x)) dt \quad \text{and} \quad a_\ell(h) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \|\nabla H\|^2(\mathfrak{z}_t(x)) dt$$

where $H(x) = h$, $x \in \mathbf{S}_\ell$. Thus, as $\varepsilon \searrow 0$, the dynamics of $H(X_{t \wedge \zeta})$ in region \mathbf{S}_ℓ should be approximated by a one-dimensional diffusion with effective drift and diffusion coefficients given by b_ℓ and a_ℓ respectively.

The behavior at $H^{-1}(0)$ (in the limit as $\varepsilon \searrow 0$) is more complicated. The work of Freidlin and Wentzell ([FW94]) established that one should look for so-called *glueing conditions*, which describe the relative likelihoods of moving into the various \mathbf{S}_ℓ 's as $\varepsilon \searrow 0$.⁴ The insight of Sowers was that glueing (which involves choosing one \mathbf{S}_ℓ over the other) is essentially a *boundary layer phenomenon* and this idea was extensively developed in [Sow03], [Sow05], [Sow07]. Here, the glueing conditions show up as sufficient conditions for solvability of a singular perturbations problem in a boundary layer near $H^{-1}(0)$. With solvability, one proceeds to construct certain correctors for averaging. It is the approach of Sowers that we follow here.

The probabilistic significance is the following. The heart of the stochastic averaging problem is *identifying* a "sufficiently large" subset $\mathcal{D}(\mathcal{A}) \subset C_0(\mathbb{R}^2)$ as the domain for a natural (averaging) operator \mathcal{A} on $C_0(\mathbb{R}^2)$ such that for $f \in \mathcal{D}(\mathcal{A})$, $0 \leq s < t$, we have

$$(8) \quad \lim_{\varepsilon \searrow 0} \mathbb{E}_{x_0}^\varepsilon \left[f(X_{t \wedge \zeta}) - f(X_{s \wedge \zeta}) - \int_{s \wedge \zeta}^{t \wedge \zeta} (\mathcal{A}f)(X_u) du \middle| \mathcal{F}_s \right] = 0.$$

The operator \mathcal{A} is very closely related to the generator of the limiting graph-valued process⁵ and its domain $\mathcal{D}(\mathcal{A})$ encodes the glueing conditions. The requirement that $\mathcal{D}(\mathcal{A})$ be "sufficiently large" is related to uniquely characterizing the limiting process. A natural starting point to establish (8) is the martingale characterization of the pre-limit process. The problem, however, is that f is typically *not* in $\mathcal{D}(\mathcal{A}^\varepsilon)$. This prompts the use of a *perturbed test function* methodology (see [Eva89], [Kur73], [Kur76], [Kus84]). One attempts to find a family of perturbed test functions (f^ε) with $f^\varepsilon \in \mathcal{D}(\mathcal{A}^\varepsilon)$ such that, in a suitable sense,

$$\lim_{\varepsilon \searrow 0} f^\varepsilon = f \quad \text{and} \quad \lim_{\varepsilon \searrow 0} \mathcal{A}^\varepsilon f^\varepsilon = \mathcal{A}f.$$

Then,

$$(9) \quad \mathbb{E}_{x_0}^\varepsilon \left[f^\varepsilon(X_{t \wedge \zeta}) - f^\varepsilon(X_{s \wedge \zeta}) - \int_{s \wedge \zeta}^{t \wedge \zeta} (\mathcal{A}^\varepsilon f^\varepsilon)(X_u) du \middle| \mathcal{F}_s \right] = 0$$

and letting $\varepsilon \searrow 0$, we get (8).

To make the foregoing precise, define, for $\ell \in \Lambda$,

$$\mathbf{G}_\ell \stackrel{\text{def}}{=} \int_{z \in \mathcal{C}_\ell} \|\nabla H\|(z) \mathcal{H}^1(dz),$$

where \mathcal{H}^1 denotes (normalized) 1-dimensional Hausdorff measure on \mathbb{R}^2 (see [Fol99]). Note that $\mathbf{G}_O = \mathbf{G}_L + \mathbf{G}_R$. For $\mathbf{v} \stackrel{\text{def}}{=}} (\mathbf{v}_\ell : \ell \in \Lambda) \in \mathbb{R}^\Lambda$, define

$$F(x) \stackrel{\text{def}}{=} \sum_{\ell \in \Lambda} \mathbf{v}_\ell H(x) \chi_{\bar{\mathbf{S}}_\ell}(x) \quad \text{for } x \in \bar{\mathbf{S}}.$$

The function F gives the dominant part of $f \in \mathcal{D}(\mathcal{A})$ near $H^{-1}(0)$, with the \mathbf{v}_ℓ 's describing the specific f near $H^{-1}(0)$. As stated above, we now want to approximate, for a sufficiently large class of vectors $\mathbf{v} \in \mathbb{R}^\Lambda$,

⁴Rigorously, one defines an equivalence relation \equiv on $\bar{\mathbf{S}}$ via *chain equivalence*. The quotient $\mathbf{M} \stackrel{\text{def}}{=} \bar{\mathbf{S}} / \equiv$ is a graph consisting of three legs (one for each \mathbf{S}_ℓ) that meet at a vertex (corresponding to the homoclinic orbit). Averaging yields a model reduction in the form of a limiting Markov process on \mathbf{M} . The glueing conditions are now restrictions (at the vertex) on functions in the domain of the limiting generator.

⁵Let $\pi : \bar{\mathbf{S}} \rightarrow \mathbf{M}$ be the projection map that takes $x \in \bar{\mathbf{S}}$ to its equivalence class (under \equiv) $[x]$. If the linear operator \mathcal{A}^\dagger with domain $\mathcal{D}(\mathcal{A}^\dagger) \subset C(\mathbf{M})$ is the generator of the limiting graph-valued Markov process, then we set $\mathcal{D}(\mathcal{A}) \stackrel{\text{def}}{=} \{f \circ \pi : f \in \mathcal{D}(\mathcal{A}^\dagger)\}$ and define, for $f \circ \pi \in \mathcal{D}(\mathcal{A})$, $\mathcal{A}(f \circ \pi) \stackrel{\text{def}}{=} (\mathcal{A}^\dagger f) \circ \pi$. Functions in $\mathcal{D}(\mathcal{A})$ and $\mathcal{R}(\mathcal{A})$ (the range of \mathcal{A}) are thus constant along orbits of the Hamiltonian flow.

the function F by the perturbed test function $F^\varepsilon \in \mathcal{D}(\mathcal{A}^\varepsilon)$ using additive correctors. In other words, one would like to find, for some $\bar{\varepsilon} \in (0, 1)$, a family of functions $\{\Pi^\varepsilon : \varepsilon \in (0, \bar{\varepsilon})\} \subset C(\bar{\mathbf{S}}) \cap C^2(\bar{\mathbf{S}} \setminus \mathbf{H}^{-1}(0))$ such that

- (1) $F^\varepsilon \stackrel{\text{def}}{=} F + \Pi^\varepsilon \in \mathcal{D}(\mathcal{A}^\varepsilon)$ for $\varepsilon \in (0, \bar{\varepsilon})$,
- (2) Π^ε and $\mathcal{L}^\varepsilon \Pi^\varepsilon$ are small, except perhaps near $\mathbf{H}^{-1}(0)$.

Our main result is that if the \mathbf{v}_ℓ 's satisfy the glueing conditions

$$(10) \quad \beta_O \mathbf{G}_O \mathbf{v}_O = \beta_L \mathbf{G}_L \mathbf{v}_L + \beta_R \mathbf{G}_R \mathbf{v}_R,$$

then one can construct the desired correctors $\{\Pi^\varepsilon : \varepsilon \in (0, \bar{\varepsilon})\}$.

We will need a little more notation to state the main theorem. Define

$$\mathcal{E}(z) \stackrel{\text{def}}{=} \exp \left[-\frac{1}{2} \sqrt{\frac{\pi}{\mathbf{G}_O}} \sqrt{z^2 + 1} \right] \quad \text{for } z \in \mathbb{R}.$$

Let's also define some cutoff functions.

Definition 2.5 (Cutoff functions). Let $\mathbf{c}_+ \in C^\infty(\mathbb{R}; [0, 1])$ be such that \mathbf{c}_+ is non-decreasing, $\mathbf{c}_+(x) = 0$ for $x \leq 1$ and $\mathbf{c}_+(x) = 1$ for $x \geq 2$. Define $\mathbf{c}_-(x) \stackrel{\text{def}}{=} \mathbf{c}_+(-x)$ for all $x \in \mathbb{R}$. Also define $\mathbf{c}_\vee(x) \stackrel{\text{def}}{=} \mathbf{c}_+(x) + \mathbf{c}_-(x)$ and $\mathbf{c}_0(x) \stackrel{\text{def}}{=} 1 - \mathbf{c}_\vee(x)$ for $x \in \mathbb{R}$.

Theorem 2.6 (Main Theorem). Fix $\mathbf{v} \in \mathbb{R}^\Lambda$ satisfying the glueing conditions (10). Then there exists $\bar{\varepsilon}_{2.6} \in (0, 1)$ (independent of \mathbf{v}), a family of functions $\{\Pi^\varepsilon : \varepsilon \in (0, \bar{\varepsilon}_{2.6})\} \subset C(\bar{\mathbf{S}}) \cap C^2(\bar{\mathbf{S}} \setminus \mathbf{H}^{-1}(0))$ and a constant $K > 0$ such that

- (1) $F + \Pi^\varepsilon \in \mathcal{D}(\mathcal{A}^\varepsilon)$ for $\varepsilon \in (0, \bar{\varepsilon}_{2.6})$,
- (2)

$$\begin{aligned} |\Pi^\varepsilon(x)| &\leq K\varepsilon, & x \in \bar{\mathbf{S}}, \\ |(\mathcal{L}^\varepsilon \Pi^\varepsilon)(x)| &\leq \frac{K}{\varepsilon^{1/4}} \mathcal{E} \left(\frac{\mathbf{H}(x)}{\varepsilon} \right) + K\varepsilon \\ &\quad + K \left\{ \frac{1}{\varepsilon^{3/4}} + \frac{1}{\varepsilon} \exp \left[-\frac{1}{K\varepsilon^{3/4}} \right] \right\} \mathbf{c}_0 \left(\frac{\mathbf{H}(x)}{\sqrt{2}\varepsilon^{5/4}} \right), & x \in \bar{\mathbf{S}} \setminus \mathbf{H}^{-1}(0). \end{aligned}$$

We will prove this theorem at the end of section 4.

We will construct the correctors $\{\Pi^\varepsilon : \varepsilon \in (0, \bar{\varepsilon})\}$ in two stages. First, we look for a family $\{\Psi^\varepsilon : \varepsilon \in (0, \bar{\varepsilon})\} \subset C(\bar{\mathbf{S}}) \cap C^2(\bar{\mathbf{S}} \setminus \mathbf{H}^{-1}(0))$ such that

- (1) Ψ^ε is small,
- (2) $F + \Psi^\varepsilon$ satisfies the skewness conditions (5),
- (3) Ψ^ε is approximately harmonic for the operator \mathcal{L}^ε in a boundary layer near $\mathbf{H}^{-1}(0)$, i.e. Ψ^ε solves the PDE $\mathcal{L}^\varepsilon \Psi^\varepsilon \approx 0$ near $\mathbf{H}^{-1}(0)$.

The basis for this construction is the solution of a collection of heat equations (one for each $\ell \in \Lambda$) coupled through their boundary data at $\mathbf{H}^{-1}(0)$ - the boundary data encoding the skewness conditions (5). In section 5, we shall address this problem from a functional-analytic standpoint; we shall see that the glueing conditions are *sufficient* conditions ensuring solvability of the coupled collection of PDE.

In section 3, we will use the results of section 5 to actually construct the Ψ^ε 's and establish estimates on Ψ^ε and $\mathcal{L}^\varepsilon \Psi^\varepsilon$. This will require some delicate analysis. Indeed, the natural form of the corrector suggested by the boundary layer analysis will in general have singularities at the origin. It will take a fair bit of work to establish a suitable approximation (near the origin) and then splice the two different expressions together.

In general, $F + \Psi^\varepsilon$ will *not* belong to $\mathcal{D}(\mathcal{A}^\varepsilon)$ either (but will be "close"). Indeed, \mathcal{A}^ε is a linear operator on $C_0(\mathbb{R}^2)$, while $\mathcal{L}^\varepsilon(F + \Psi^\varepsilon)$ may not have a continuous extension to all of \mathbb{R}^2 (though it will be continuous on $\mathbb{R}^2 \setminus \mathbf{H}^{-1}(0)$). It will require some further work to correct for these discontinuities and find a family $\{\Phi^\varepsilon : \varepsilon \in (0, \bar{\varepsilon})\}$ such that

- (1) Φ^ε and $\mathcal{L}^\varepsilon \Phi^\varepsilon$ are small,
- (2) $F + \Psi^\varepsilon + \Phi^\varepsilon$ satisfies the skewness conditions, and

(3) $\mathcal{L}^\varepsilon(F + \Psi^\varepsilon + \Phi^\varepsilon)$ has a continuous extension to all of \mathbb{R}^2 .

Of course, one then takes $\Pi^\varepsilon \stackrel{\text{def}}{=} \Psi^\varepsilon + \Phi^\varepsilon$. The construction of the family $\{\Phi^\varepsilon : \varepsilon \in (0, \bar{\varepsilon})\}$ will be described in section 4.

The correctors Ψ^ε and Φ^ε will be obtained by significantly different calculations. Indeed, as will be seen in the sequel, the glueing conditions ensure that we can in fact construct the Ψ^ε 's and the analysis here will be similar to that in [Sow03], [Sow05]. The Φ^ε 's will be obtained by *smoothing* and serve to correct small discontinuities in $\mathcal{L}^\varepsilon(F + \Psi^\varepsilon)$ at $\mathbf{H}^{-1}(0)$.

2.4. Notation. Let's now record some notation and conventions to be used in the sequel.

For $x = (x_1, x_2) \in \mathbb{R}^2$, let $\|x\| \stackrel{\text{def}}{=} \sqrt{x_1^2 + x_2^2}$. We will occasionally use the infinity norm on \mathbb{R}^2 given by $\|x\|_\infty \stackrel{\text{def}}{=} \max\{|x_1|, |x_2|\}$ for $x = (x_1, x_2) \in \mathbb{R}^2$.

Recall the cutoff functions defined earlier. Note that for every $n \in \mathbb{N}$, there exists $K_n > 0$ such that $|\mathbf{c}_+^{(n)}|$, $|\mathbf{c}_-^{(n)}|$, $|\mathbf{c}_0^{(n)}|$, $|\mathbf{c}_\vee^{(n)}|$ are bounded above by K_n .

For $\varphi \in C^k(\mathbf{S})$, $k \in \mathbb{N}$, $F \subset \mathbf{S}$ closed, define (using multiindex notation)

$$\|\varphi\|_{C^k(F)} \stackrel{\text{def}}{=} \sup_{\substack{x \in F \\ |\alpha| \leq k}} |D^\alpha \varphi(x)|.$$

Let $\eta : \mathbb{R} \rightarrow \mathbb{R}$ be the standard mollifier, i.e.

$$\eta(x) \stackrel{\text{def}}{=} \begin{cases} C \exp\left(\frac{1}{x^2-1}\right) & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

where the constant $C > 0$ is chosen such that $\int_{\mathbb{R}} \eta = 1$. Also, for $\delta > 0$, define

$$\eta^\delta(x) \stackrel{\text{def}}{=} \frac{1}{\delta} \eta\left(\frac{x}{\delta}\right) \quad \text{for } x \in \mathbb{R}.$$

Note that there exists a constant $K > 0$ such that

$$\begin{aligned} |U'(x_1)| &\leq K|x_1| & \text{for } x_1 \in [d_L - \mathbf{u}, d_R + \mathbf{u}], \\ \|\nabla \mathbf{H}(x)\| &\leq K\|x\| & \text{for } x \in \bar{\mathbf{S}}. \end{aligned}$$

2.5. Picture Near the Origin. Let $\mathcal{U} \stackrel{\text{def}}{=} B(\mathbf{o}, \varpi_0)$, the open ball of radius ϖ_0 centered at \mathbf{o} . Then, for $x = (x_1, x_2) \in \mathcal{U}$, we have

$$\mathbf{H}(x_1, x_2) = \frac{x_2^2}{2} - \frac{x_1^2}{2}$$

Near the origin, then, it is natural to make the transformation $y = \phi(x)$ where $(y_1, y_2) = (\phi_1(x_1, x_2), \phi_2(x_1, x_2))$ is given by

$$\begin{aligned} y_1 &= \phi_1(x_1, x_2) \stackrel{\text{def}}{=} \frac{x_1 + x_2}{\sqrt{2}} \\ y_2 &= \phi_2(x_1, x_2) \stackrel{\text{def}}{=} \frac{x_2 - x_1}{\sqrt{2}} \end{aligned}$$

The map ϕ with domain \mathcal{U} is thus a clockwise rotation through an angle of $\pi/4$. Clearly, ϕ is invertible on \mathcal{U} ; let $\tilde{\phi} \stackrel{\text{def}}{=} \phi^{-1}$ on $\phi(\mathcal{U})$. Define $\tilde{\mathbf{H}} : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$\tilde{\mathbf{H}}(y_1, y_2) \stackrel{\text{def}}{=} y_1 y_2 \quad \text{for } (y_1, y_2) \in \mathbb{R}^2,$$

and for $y = (y_1, y_2) \in \mathbb{R}^2$, define $\|y\|_\infty \stackrel{\text{def}}{=} \max\{|y_1|, |y_2|\}$. Clearly, $\mathbf{H} = \tilde{\mathbf{H}} \circ \phi$ on \mathcal{U} .

Suppose now that $f \in C^2(\mathbb{R}^2)$. For $y \in \phi(\mathcal{U})$, define

$$(\tilde{\mathcal{L}}f)(y) \stackrel{\text{def}}{=} \mathcal{L}(f \circ \phi)(\tilde{\phi}(y))$$

and

$$(\tilde{\mathcal{L}}^\varepsilon f)(y) \stackrel{\text{def}}{=} \mathcal{L}^\varepsilon(f \circ \phi)(\tilde{\phi}(y)) = \frac{1}{\varepsilon^2} (\nabla^\perp \mathbf{H}, \nabla(f \circ \phi))(\tilde{\phi}(y)) + \mathcal{L}(f \circ \phi)(\tilde{\phi}(y)).$$

We can explicitly compute that

$$(\tilde{\mathcal{L}}f)(y) = \frac{1}{2}(\Delta f)(y)$$

and

$$(\nabla^\perp \mathbf{H}, \nabla(f \circ \phi))(\tilde{\phi}(y)) = (\nabla^\perp \tilde{\mathbf{H}}, \nabla f)(y)$$

for $y \in \phi(\mathcal{U})$. Hence, we have

$$(\tilde{\mathcal{L}}^\varepsilon f)(y) = \frac{1}{\varepsilon^2}(\nabla^\perp \tilde{\mathbf{H}}, \nabla f)(y) + \frac{1}{2}(\Delta f)(y)$$

for $f \in C^2(\mathbb{R}^2)$, $y \in \phi(\mathcal{U})$. It is also easily checked that

$$|\nabla \mathbf{H}|^2(x) = y_1^2 + y_2^2 = |\nabla \tilde{\mathbf{H}}|^2(y)$$

where $x = \tilde{\phi}(y)$, $y \in \phi(\mathcal{U})$.

3. CORRECTORS (I)

We now take up the issue of constructing the correctors $\{\Psi^\varepsilon : \varepsilon \in (0, \bar{\varepsilon})\}$. The main result of this section is

Theorem 3.1. *There exists $\bar{\varepsilon}_{3.1} \in (0, 1)$, a family of functions $\{\Psi^\varepsilon : \varepsilon \in (0, \bar{\varepsilon}_{3.1})\} \subset C(\bar{\mathbf{S}}) \cap C^2(\bar{\mathbf{S}} \setminus \mathbf{H}^{-1}(0))$ and a constant $K > 0$ such that for $\varepsilon \in (0, \bar{\varepsilon}_{3.1})$,*

- (1) $F + \Psi^\varepsilon$ satisfies the skewness conditions (5) at $\mathbf{H}^{-1}(0) \setminus \{\mathbf{o}\}$,
- (2)

$$|\Psi^\varepsilon(x)| \leq K\varepsilon, \quad x \in \bar{\mathbf{S}}$$

$$|(\mathcal{L}^\varepsilon \Psi^\varepsilon)(x)| \leq \frac{K}{\varepsilon^{1/4}} \mathcal{E} \left(\frac{\mathbf{H}(x)}{\varepsilon} \right) + K \left\{ \frac{1}{\varepsilon^{3/4}} + \frac{1}{\varepsilon} \exp \left[-\frac{1}{K\varepsilon^{3/4}} \right] \right\} c_0 \left(\frac{\mathbf{H}(x)}{\sqrt{2\varepsilon^{5/4}}} \right), \quad x \in \bar{\mathbf{S}} \setminus \mathbf{H}^{-1}(0).$$

3.1. Boundary Layer PDE. Let's start by noting that the skewness conditions translate to

$$(11) \quad \beta_L \left\{ \nu_L \|\nabla \mathbf{H}(x)\| + \lim_{\substack{y \rightarrow x \\ y \in \mathbf{S}_L}} (\nabla \Psi^\varepsilon(y), \nu(x)) \right\} = \beta_O \left\{ \nu_O \|\nabla \mathbf{H}(x)\| + \lim_{\substack{y \rightarrow x \\ y \in \mathbf{S}_O}} (\nabla \Psi^\varepsilon(y), \nu(x)) \right\}$$

for $x \in \mathcal{C}_L \setminus \{\mathbf{o}\}$ and

$$(12) \quad \beta_R \left\{ \nu_R \|\nabla \mathbf{H}(x)\| + \lim_{\substack{y \rightarrow x \\ y \in \mathbf{S}_R}} (\nabla \Psi^\varepsilon(y), \nu(x)) \right\} = \beta_O \left\{ \nu_O \|\nabla \mathbf{H}(x)\| + \lim_{\substack{y \rightarrow x \\ y \in \mathbf{S}_O}} (\nabla \Psi^\varepsilon(y), \nu(x)) \right\}$$

for $x \in \mathcal{C}_R \setminus \{\mathbf{o}\}$ where $\nu(x) \stackrel{\text{def}}{=} \left(\frac{\nabla \mathbf{H}}{\|\nabla \mathbf{H}\|} \right) (x)$ is the outward unit normal to $\mathbf{H}^{-1}(0)$ at $x \in \mathbf{H}^{-1}(0) \setminus \{\mathbf{o}\}$.

Essentially, we have here a *singular perturbations* problem. We can think of F as something like an outer expansion which has the requisite behavior away from $\mathbf{H}^{-1}(0)$, but fails to satisfy the boundary data at $\mathbf{H}^{-1}(0)$ (the boundary data being given by the skewness conditions). We look for correctors given in terms of a coordinate along the boundary and an expanded coordinate transversal to the boundary. The function $\frac{\mathbf{H}(x)}{\varepsilon^\alpha}$ for suitable $\alpha > 0$ is the natural choice for the coordinate transversal to the boundary. If we now apply \mathcal{L}^ε to a nonlinear function of $\frac{\mathbf{H}(x)}{\varepsilon^\alpha}$, then we will get a term of order $\varepsilon^{-2\alpha}$. This is due to the diffusion transversal to the boundary. We want this diffusion to be of the same order as the fast drift (ε^{-2}) and so we choose $\alpha = 1$. Note that we have three boundary layers, one in each of the \mathbf{S}_ℓ 's.

Let's now pin down the coordinate along the boundary. If Θ is a twice-differentiable real-valued function on some open subset \mathcal{O} of \mathbf{S} and if we set $\psi^\varepsilon(x) \stackrel{\text{def}}{=} \psi \left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon} \right)$ for $x \in \mathcal{O}$ with $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$ a C^2 function, then we have,

$$(13) \quad (\mathcal{L}^\varepsilon \psi^\varepsilon)(x) = \frac{1}{\varepsilon^2} \left\{ \frac{\partial \psi}{\partial \theta} (\nabla^\perp \mathbf{H}, \nabla \Theta)(x) + \frac{1}{2} \frac{\partial^2 \psi}{\partial h^2} \|\nabla \mathbf{H}\|^2(x) \right\} \\ + \frac{1}{\varepsilon} \left\{ \frac{\partial \psi}{\partial h} (\mathcal{L} \mathbf{H})(x) + \frac{\partial^2 \psi}{\partial h \partial \theta} (\nabla \mathbf{H}, \nabla \Theta)(x) \right\} + \left\{ \frac{\partial \psi}{\partial \theta} (\mathcal{L} \Theta)(x) + \frac{1}{2} \frac{\partial^2 \psi}{\partial \theta^2} \|\nabla \Theta\|^2(x) \right\}$$

where the various derivatives of ψ are evaluated at $\left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon} \right)$. The *Khasminskii coordinates* (see [Sow03] and [Sow05]), which we shall denote by Θ , equate the coefficients of the two dominant terms in (13), i.e. solve the PDE

$$(14) \quad (\nabla^\perp \mathbf{H}, \nabla \Theta)(x) = \|\nabla \mathbf{H}\|^2(x) \quad \text{for } x \in \mathbf{H}^{-1}(0) \setminus \{\mathbf{o}\}.$$

Before delving into the rigorous construction of the Khasminskii coordinates, let's make a few preliminary observations which provide the context for the PDE result to follow. One should think of Θ as an *angular* coordinate along the homoclinic orbit which starts at 0 near \mathbf{o} , increases to \mathbf{G}_L as one traverses \mathcal{C}_L in the direction of the flow and then increases to $\mathbf{G}_O = \mathbf{G}_L + \mathbf{G}_R$ as one traverses \mathcal{C}_R along the flow. Θ will not

exactly solve the PDE (14) away from the homoclinic orbit, but we expect to be able to control the resulting errors. Hence, if we let

$$A_L \stackrel{\text{def}}{=} 0, \quad B_L \stackrel{\text{def}}{=} G_L, \quad A_R \stackrel{\text{def}}{=} G_L, \quad B_R \stackrel{\text{def}}{=} G_O, \quad A_O \stackrel{\text{def}}{=} 0, \quad B_O \stackrel{\text{def}}{=} G_O$$

and

$$I_\ell \stackrel{\text{def}}{=} (A_\ell, B_\ell) \quad \text{for} \quad \ell \in \Lambda,$$

then Θ takes values in I_ℓ along \mathcal{C}_ℓ , $\ell \in \Lambda$.

If, as a first approximation, we take

$$\hat{\Psi}^\varepsilon(x) = \varepsilon\psi\left(\Theta(x), \frac{H(x)}{\varepsilon}\right),$$

then for $x \in \mathcal{C}_\ell \setminus \{\mathfrak{o}\}$, $\ell \in \Lambda$,

$$\lim_{\substack{y \rightarrow x \\ y \in \mathbf{S}_\ell}} (\nabla \hat{\Psi}^\varepsilon(y), \nu(x)) = \frac{(\nabla H, \nabla \Theta)(x)}{\|\nabla H(x)\|} \lim_{\substack{y \rightarrow x \\ y \in \mathbf{S}_\ell}} \varepsilon \frac{\partial \psi}{\partial \theta} \left(\Theta(y), \frac{H(y)}{\varepsilon} \right) + \|\nabla H(x)\| \lim_{\substack{y \rightarrow x \\ y \in \mathbf{S}_\ell}} \frac{\partial \psi}{\partial h} \left(\Theta(y), \frac{H(y)}{\varepsilon} \right).$$

If we choose Θ such that $(\nabla H, \nabla \Theta)(x) = 0$ for $x \in H^{-1}(0) \setminus \{\mathfrak{o}\}$, then the equation above becomes

$$\lim_{\substack{y \rightarrow x \\ y \in \mathbf{S}_\ell}} (\nabla \hat{\Psi}^\varepsilon(y), \nu(x)) = \|\nabla H(x)\| \lim_{\substack{h \rightarrow 0 \\ h \in \mathcal{R}_\ell}} \frac{\partial \psi}{\partial h} (\Theta(x), h)$$

where $\mathcal{R}_L = \mathcal{R}_R \stackrel{\text{def}}{=} (-\infty, 0)$ and $\mathcal{R}_O \stackrel{\text{def}}{=} (0, \infty)$. Using now the fact that $\|\nabla H(x)\| \neq 0$ for $x \neq \mathfrak{o}$, the skewness conditions become: for $x \in \mathcal{C}_\ell \setminus \{\mathfrak{o}\}$, $\ell \in \{L, R\}$,

$$(15) \quad \beta_\ell \left\{ \mathbf{v}_\ell + \frac{\partial \psi}{\partial h} (\Theta(x), 0-) \right\} = \beta_O \left\{ \mathbf{v}_O + \frac{\partial \psi}{\partial h} (\Theta(x), 0+) \right\}.$$

The equation (15) specifies the relation between the \mathbf{v}_ℓ 's, the β_ℓ 's and the Neumann data for the PDE $\mathcal{L}^\varepsilon \Psi^\varepsilon \approx 0$.

We now state our main result regarding the boundary layer PDEs. The proof will be given in section 5.

Proposition 3.2. *There is a triplet of functions $(\Psi_O^K, \Psi_L^K, \Psi_R^K)$ such that the following hold. For each $\ell \in \Lambda$, $\Psi_\ell^K \in C^\infty(\mathbb{R} \times \mathcal{R}_\ell)$ and*

$$(16) \quad \frac{\partial \Psi_\ell^K}{\partial \theta}(\theta, h) + \frac{1}{2} \frac{\partial^2 \Psi_\ell^K}{\partial h^2}(\theta, h) = 0$$

for $(\theta, h) \in \mathbb{R} \times \mathcal{R}_\ell$. Secondly, for each multiindex α with $|\alpha| \geq 1$, there is a constant $K > 0$ such that

$$(17) \quad |D^\alpha \Psi_\ell^K(\theta, h)| \leq K \exp \left[-\sqrt{\frac{\pi}{G_\ell}} |h| \right]$$

for all $\ell \in \Lambda$, $\theta \in \mathbb{R}$, $h \in \mathcal{R}_\ell$ with $|h| \geq 1$. Thirdly,

$$(18) \quad \Psi_\ell^K(\theta, h) = \Psi_\ell^K(\theta + G_\ell, h)$$

for $(\theta, h) \in \mathbb{R} \times \mathcal{R}_\ell$. Fourthly,

$$(19) \quad \begin{aligned} \lim_{h \searrow 0} \Psi_O^K(\cdot, h) &= \lim_{h \nearrow 0} \Psi_L^K(\cdot, h) && \text{in } L^2(I_L) \\ \lim_{h \searrow 0} \Psi_O^K(\cdot, h) &= \lim_{h \nearrow 0} \Psi_R^K(\cdot, h) && \text{in } L^2(I_R) \end{aligned}$$

and

$$(20) \quad \begin{aligned} \lim_{h \searrow 0} \beta_O \int_{\theta \in I_L} \left\{ \frac{\partial \Psi_O^K}{\partial h}(\theta, h) + \mathbf{v}_O \right\} \varphi(\theta) d\theta &= \lim_{h \nearrow 0} \beta_L \int_{\theta \in I_L} \left\{ \frac{\partial \Psi_L^K}{\partial h}(\theta, h) + \mathbf{v}_L \right\} \varphi(\theta) d\theta && \varphi \in C_c^\infty(I_L) \\ \lim_{h \searrow 0} \beta_O \int_{\theta \in I_R} \left\{ \frac{\partial \Psi_O^K}{\partial h}(\theta, h) + \mathbf{v}_O \right\} \varphi(\theta) d\theta &= \lim_{h \nearrow 0} \beta_R \int_{\theta \in I_R} \left\{ \frac{\partial \Psi_R^K}{\partial h}(\theta, h) + \mathbf{v}_R \right\} \varphi(\theta) d\theta && \varphi \in C_c^\infty(I_R) \end{aligned}$$

Lastly,

$$(21) \quad \sum_{\ell \in \Lambda} \beta_\ell \int_{h \in \mathcal{R}_\ell} \int_{\theta \in \mathfrak{l}_\ell} \left(\frac{\partial \Psi_\ell^K}{\partial h}(\theta, h) \right)^2 d\theta dh < \infty.$$

Let's now obtain bounds on the derivatives of the Ψ_ℓ^K 's near $h = 0$. For each nonnegative integer k , define

$$S_k(\theta, h) \stackrel{\text{def}}{=} \left\{ \frac{1}{|\theta|^{k/2}} \exp \left[-\frac{h^2}{4|\theta|} \right] \chi_{(0, \infty)}(\theta) + 1 \right\} \exp \left[-\sqrt{\frac{\pi}{G_O}} |h| \right]$$

for $(\theta, h) \in \mathbb{R}^2$. Note that there is a constant $K > 0$ such that for $k \in \{0, 1, 2, 3, 4\}$,

$$(22) \quad S_k(\theta, h) \leq \frac{K}{(|\theta| + h^2)^{k/2}} \exp \left[-\sqrt{\frac{\pi}{G_O}} |h| \right]$$

for all $\theta \geq 0$ and $h \in \mathbb{R}$ such that $|\theta| + |h| > 0$.

Define

$$\mathcal{S}_L \stackrel{\text{def}}{=} \mathbb{Z}G_L, \quad \mathcal{S}_R \stackrel{\text{def}}{=} G_L + \mathbb{Z}G_R \quad \text{and} \quad \mathcal{S}_O \stackrel{\text{def}}{=} \mathcal{S}_L \cup \mathcal{S}_R.$$

Also, for $\ell \in \Lambda$, define

$$d_\ell(\theta) \stackrel{\text{def}}{=} \inf_{\theta' \in \mathcal{S}_\ell} |\theta - \theta'| \quad \text{for } \theta \in \mathbb{R}.$$

The following proposition is proved using Proposition 3.2 and Lemmas 5.9 and 5.10 in a manner analogous to the proof of Proposition 8.7 in [Sow03].

Proposition 3.3. *For $\ell \in \Lambda$, the function $\Psi_\ell^K \in C^\infty(\mathbb{R} \times \overline{\mathcal{R}_\ell} \setminus (\mathcal{S}_\ell \times \{0\}))$. Also, for each pair of indices j and k in $\{0, 1, 2\}$ such that $j + k > 0$, there is a constant $K > 0$ such that*

$$(23) \quad \left| \frac{\partial^{j+k} \Psi_\ell^K}{\partial \theta^j \partial h^k}(\theta, h) \right| \leq K S_{2j+k}(d_\ell(\theta), h)$$

for all $\theta \in \mathbb{R}$ and $h \in \mathcal{R}_\ell$. Finally, for $(j, k) \in (\mathbb{Z}^+)^2$, $\ell \in \{L, R\}$, $\theta \in \mathfrak{l}_\ell$, we have

$$(24) \quad \begin{aligned} \lim_{h \searrow 0} \frac{\partial^{j+k}}{\partial \theta^j \partial h^k} (\Psi_O^K(\theta, h) + \mathbf{v}_O h) &= \lim_{h \nearrow 0} \frac{\partial^{j+k}}{\partial \theta^j \partial h^k} (\Psi_\ell^K(\theta, h) + \mathbf{v}_\ell h) && \text{if } k \text{ is even,} \\ \beta_O \lim_{h \searrow 0} \frac{\partial^{j+k}}{\partial \theta^j \partial h^k} (\Psi_O^K(\theta, h) + \mathbf{v}_O h) &= \beta_\ell \lim_{h \nearrow 0} \frac{\partial^{j+k}}{\partial \theta^j \partial h^k} (\Psi_\ell^K(\theta, h) + \mathbf{v}_\ell h) && \text{if } k \text{ is odd.} \end{aligned}$$

3.2. The Angular Coordinate Θ . Let's now start to rigorously construct the Khasminskii coordinates Θ . The ensuing is essentially lifted from [Sow05], with slight changes in notation. Our intent is to solve the PDE (14) separately on $\mathcal{C}_L \setminus \{\mathfrak{o}\}$ and $\mathcal{C}_R \setminus \{\mathfrak{o}\}$ by specifying initial data at $(d_L, 0)$ and $(d_R, 0)$ respectively, and then smoothly extending the angle to (almost) all of $\bar{\mathbf{S}}$. Let $\bar{\Theta}(d_L, 0) = G_L/2$ and let $\bar{\Theta}(d_R, 0) = G_O - G_R/2$. Then for $x \in \mathcal{C}_\ell \setminus \{\mathfrak{o}\}$, $\ell \in \{L, R\}$, define

$$\bar{\Theta}(x) = \bar{\Theta}(d_\ell, 0) + \int_0^t \|\nabla \mathbf{H}\|^2(\mathfrak{z}_s(d_\ell, 0)) ds \quad \text{where } x = \mathfrak{z}_t(d_\ell, 0), t \in \mathbb{R}.$$

Note that $\bar{\Theta} \in C^\infty(\mathbf{H}^{-1}(0) \setminus \{\mathfrak{o}\})$. Note also that we have not defined $\bar{\Theta}$ at \mathfrak{o} . This is because it is not possible to extend $\bar{\Theta}$ to a continuous function on all of $\mathbf{H}^{-1}(0)$. Indeed,

$$\lim_{t \rightarrow -\infty} \bar{\Theta}(\mathfrak{z}_t(d_L, 0)) = 0, \quad \lim_{t \rightarrow \infty} \bar{\Theta}(\mathfrak{z}_t(d_L, 0)) = \lim_{t \rightarrow -\infty} \bar{\Theta}(\mathfrak{z}_t(d_R, 0)) = G_L, \quad \lim_{t \rightarrow \infty} \bar{\Theta}(\mathfrak{z}_t(d_R, 0)) = G_O$$

which clearly precludes the possibility of extending $\bar{\Theta}$ in a continuous way.

Let's now extend the angle $\bar{\Theta}$ away from the homoclinic orbit. Let $\tilde{X} \stackrel{\text{def}}{=} \{y \in \phi(\mathcal{W}) : \tilde{\mathbf{H}}(y) = 0\}$ and consider the angle $\bar{\Theta}^\sim$ induced by $\bar{\Theta}$ on \tilde{X} by

$$\bar{\Theta}^\sim(y) \stackrel{\text{def}}{=} \begin{cases} \bar{\Theta}(\tilde{\phi}(y)) - \lim_{t \searrow 0} \bar{\Theta}(\tilde{\phi}(ty)) & \text{if } y \in \tilde{X} \setminus \{(0, 0)\} \\ 0 & \text{if } y = (0, 0). \end{cases}$$

Then, $\bar{\Theta}^\sim$ solves the PDE

$$\begin{aligned} (\nabla^\perp \tilde{H}, \nabla \bar{\Theta}^\sim)(y) &= y_1^2 + y_2^2 \quad \text{for } y \in \tilde{X} \setminus \{(0,0)\} \\ \lim_{\substack{y \rightarrow (0,0) \\ y \in \tilde{X} \setminus \{(0,0)\}}} \bar{\Theta}^\sim(y) &= 0. \end{aligned}$$

Thus,

$$\bar{\Theta}^\sim(y_1, 0) = \frac{y_1^2}{2} \quad \text{and} \quad \bar{\Theta}^\sim(0, y_2) = -\frac{y_2^2}{2}.$$

We now define

$$\tilde{\Theta}(y_1, y_2) \stackrel{\text{def}}{=} \frac{y_1^2}{2} - \frac{y_2^2}{2} \quad \text{for } (y_1, y_2) \in \phi(\mathcal{U}).$$

Let $\tilde{\square} \stackrel{\text{def}}{=} \{y \in \mathbb{R}^2 : \|y\|_\infty < \varpi_0/\sqrt{2}\} \subset \phi(\mathcal{U})$. We now extend the angle $\bar{\Theta}$ away from $H^{-1}(0)$ by means of a retract. Consider the flow on $\bar{\mathbf{S}}$ defined by

$$\begin{aligned} \dot{\varphi}_t^*(x) &= -H(x) \left(\frac{\nabla H}{\|\nabla H\|^2} \right) (\varphi_t^*(x)) \\ \varphi_0^*(x) &= x. \end{aligned}$$

Define $\varphi : \bar{\mathbf{S}} \rightarrow H^{-1}(0)$ by

$$\varphi(x) \stackrel{\text{def}}{=} \lim_{t \nearrow 1} \varphi_t^*(x).$$

Note that

$$\mathbf{X} \stackrel{\text{def}}{=} \varphi^{-1}(\{0\}) = \{(x_1, x_2) \in \bar{\mathbf{S}} : x_1 x_2 = 0\}.$$

The following two lemmas are proved in [Sow05].

Lemma 3.4 (Retract). *The retract $\varphi : \bar{\mathbf{S}} \rightarrow H^{-1}(0)$ is continuous and satisfies the following:*

- (1) φ is C^∞ on $\bar{\mathbf{S}} \setminus \mathbf{X}$,
- (2) $\bar{\Theta}^\sim \circ \phi \circ \varphi$ is C^∞ on $\varphi^{-1}(\tilde{\phi}(\tilde{\square}))$, and
- (3) there is a neighborhood \mathcal{V} of \mathfrak{o} such that $\mathcal{V} \subset \tilde{\phi}(\tilde{\square})$ and $\overline{\varphi(\mathcal{V})} \subset \tilde{\phi}(\tilde{\square})$ and such that $\bar{\Theta}^\sim \circ \phi \circ \varphi = \tilde{\Theta} \circ \phi$ on \mathcal{V} .

Let $\varpi \in (0, h)$ be small enough that the set $\square \stackrel{\text{def}}{=} \{y \in \mathbb{R}^2 : \|y\|_\infty < \varpi\}$ is contained in $\phi(\mathcal{V})$. We can now define Θ on $\bar{\mathbf{S}} \setminus \mathbf{X}$ by $\Theta \stackrel{\text{def}}{=} \bar{\Theta} \circ \varphi$.

Lemma 3.5. *The function Θ is C^∞ on $\bar{\mathbf{S}} \setminus \mathbf{X}$. There is a constant $K > 0$ such that for all $x \in \bar{\mathbf{S}} \setminus \mathbf{X}$,*

$$(25) \quad |\Theta(x)| \leq K, \quad \|\nabla \Theta(x)\| \leq K, \quad |\mathcal{L}\Theta(x)| \leq K,$$

$$(26) \quad |(\nabla^\perp H, \nabla \Theta)(x) - \|\nabla H\|^2(x)| \leq K|H(x)|,$$

and for all $y \in \square$,

$$|\tilde{\Theta}(y)| \leq K\|y\|_\infty^2 \quad \text{and} \quad \|\nabla \tilde{\Theta}(y)\| \leq K\|y\|_\infty.$$

At this point, we have an angle Θ which is *globally* defined and an angle $\tilde{\Theta}$ which is only *locally* defined near the origin $y = (0, 0)$. Let's understand how the two are related. To start, let's note that images (via the map ϕ) of points $x \in \mathbf{X}$ sufficiently close to \mathfrak{o} have $\tilde{\Theta}$ -angle equal to zero. Indeed, if $x \in \mathbf{X} \cap \mathcal{V}$, then by Lemma 3.4,

$$\tilde{\Theta}(\phi(x)) = \tilde{\Theta} \circ \phi(x) = \bar{\Theta}^\sim \circ \phi \circ \varphi(x) = \bar{\Theta}^\sim(0, 0) = 0.$$

Thus, the image (via ϕ) of the set \mathbf{X} is contained in $\{y \in \square : \tilde{\Theta}(y) = 0\}$. (Note that $\tilde{\Theta}(y_1, y_2) = 0$ if and only if $|y_1| = |y_2|$.) The set $\phi(\mathbf{X})$ partitions \square into the following:

$$\begin{aligned}\mathcal{C}_U &\stackrel{\text{def}}{=} \{(y_1, y_2) \in \square : y_2 > |y_1|\}, \\ \mathcal{C}_D &\stackrel{\text{def}}{=} \{(y_1, y_2) \in \square : y_2 < -|y_1|\}, \\ \mathcal{C}_L &\stackrel{\text{def}}{=} \{(y_1, y_2) \in \square : y_1 < -|y_2|\}, \\ \mathcal{C}_R &\stackrel{\text{def}}{=} \{(y_1, y_2) \in \square : y_1 > |y_2|\}.\end{aligned}$$

These sets correspond, respectively, to the upper, lower, left and right sectors of \square as determined by $\phi(\mathbf{X}) = \{(y_1, y_2) \in \square : |y_1| = |y_2|\}$. We have changed font here to distinguish between the index sets $\Lambda = \{O, L, R\}$ and $\{\mathbf{U}, \mathbf{D}, \mathbf{L}, \mathbf{R}\}$. Thus, “ L ” refers to the left loop of the figure eight, while “ \mathbf{L} ” refers to the left sector of \square .

It is now easily seen, using Lemma 3.4, that for $y \in \square \setminus \phi(\mathbf{X})$, $x = \tilde{\phi}(y)$,

$$\tilde{\Theta}(y) = \begin{cases} \Theta(x) - \mathbf{G}_L & \text{if } y \in \mathcal{C}_U, \\ \Theta(x) - \mathbf{G}_O & \text{if } y \in \mathcal{C}_D, \\ \Theta(x) & \text{if } y \in \mathcal{C}_L, \\ \Theta(x) - \mathbf{G}_L & \text{if } y \in \mathcal{C}_R. \end{cases}$$

3.3. The Principal Part of the Corrector (Ψ_A^ε). Let $\underline{\mathbf{G}} \stackrel{\text{def}}{=} \min\{\mathbf{G}_L, \mathbf{G}_R\}$ and define the four functions u_U, u_D, u_L, u_R by

$$\begin{aligned}u_U(t, h) &\stackrel{\text{def}}{=} \begin{cases} \Psi_O^K(\mathbf{G}_L - t, h) + \mathbf{v}_O h & \text{if } h > 0, t \in (0, \underline{\mathbf{G}}) \\ \Psi_L^K(\mathbf{G}_L - t, h) + \mathbf{v}_L h & \text{if } h < 0, t \in (0, \underline{\mathbf{G}}) \end{cases} \\ u_D(t, h) &\stackrel{\text{def}}{=} \begin{cases} \Psi_O^K(\mathbf{G}_O - t, h) + \mathbf{v}_O h & \text{if } h > 0, t \in (0, \underline{\mathbf{G}}) \\ \Psi_R^K(\mathbf{G}_O - t, h) + \mathbf{v}_R h & \text{if } h < 0, t \in (0, \underline{\mathbf{G}}) \end{cases} \\ u_L(t, h) &\stackrel{\text{def}}{=} \begin{cases} \Psi_O^K(-t, h) + \mathbf{v}_O h & \text{if } h > 0, t \in (-\underline{\mathbf{G}}, 0) \\ \Psi_L^K(-t, h) + \mathbf{v}_L h & \text{if } h < 0, t \in (-\underline{\mathbf{G}}, 0) \end{cases} \\ u_R(t, h) &\stackrel{\text{def}}{=} \begin{cases} \Psi_O^K(\mathbf{G}_L - t, h) + \mathbf{v}_O h & \text{if } h > 0, t \in (-\underline{\mathbf{G}}, 0) \\ \Psi_R^K(\mathbf{G}_L - t, h) + \mathbf{v}_R h & \text{if } h < 0, t \in (-\underline{\mathbf{G}}, 0) \end{cases}\end{aligned}$$

The principal part of the corrector Ψ^ε is given by

$$\Psi_A^\varepsilon(x) \stackrel{\text{def}}{=} \varepsilon \sum_{\ell \in \Lambda} \Psi_\ell^K \left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon} \right) \chi_{\mathbf{S}_\ell}(x) \quad \text{for } x \in \bar{\mathbf{S}} \setminus (\mathbf{H}^{-1}(0) \cup \mathbf{X}).$$

Let

$$\tilde{F} \stackrel{\text{def}}{=} F \circ \tilde{\phi} \quad \text{on } \square.$$

Lemma 3.6. *The function $\Psi_A^\varepsilon + F$ has a continuous extension to $\bar{\mathbf{S}} \setminus \{\mathbf{o}\}$. Also, $\Psi_A^\varepsilon + F$ satisfies the skewness conditions at $\mathbf{H}^{-1}(0) \setminus \{\mathbf{o}\}$ and is smooth at $\mathbf{X} \setminus \{\mathbf{o}\}$.*

Proof. The proof is similar to that of Lemma 3.3 in [Sow05]. □

Define

$$\Psi_A^\varepsilon(x) \stackrel{\text{def}}{=} \lim_{\substack{x' \rightarrow x \\ x' \in \bar{\mathbf{S}} \setminus (\mathbf{H}^{-1}(0) \cup \mathbf{X})}} \Psi_A^\varepsilon(x') \quad \text{for } x \in \bar{\mathbf{S}} \setminus \{\mathbf{o}\}$$

and

$$\tilde{\Psi}_A^\varepsilon \stackrel{\text{def}}{=} \Psi_A^\varepsilon \circ \tilde{\phi} \quad \text{on } \square \setminus \{(0, 0)\}.$$

Let $\tilde{\mathcal{N}} \stackrel{\text{def}}{=} \square \setminus (\tilde{X} \cup \phi(\mathbf{X}))$. For $y \in \tilde{\mathcal{N}}$, we have

$$\tilde{\Psi}_A^\varepsilon(y) + \tilde{F}(y) = \varepsilon \sum_{s \in \{\mathbf{U}, \mathbf{D}, \mathbf{L}, \mathbf{R}\}} u_s \left(-\tilde{\Theta}(y), \frac{\tilde{H}(y)}{\varepsilon} \right) \chi_{\mathcal{C}_s}(y)$$

Using Propositions 3.2 and 3.3, it follows that for $s \in \{\mathbf{U}, \mathbf{D}, \mathbf{L}, \mathbf{R}\}$, the functions u_s solve:

$s = \mathbf{U}$

$$(27) \quad \begin{aligned} \frac{\partial u_{\mathbf{U}}}{\partial t}(t, h) &= \frac{1}{2} \frac{\partial^2 u_{\mathbf{U}}}{\partial h^2}(t, h) & (t, h) &\in (0, \underline{\mathbf{G}}) \times \mathbb{R} \setminus \{0\} \\ u_{\mathbf{U}}(t, 0+) &= u_{\mathbf{U}}(t, 0-) & t &\in (0, \underline{\mathbf{G}}] \\ \beta_{\mathbf{O}} \frac{\partial u_{\mathbf{U}}}{\partial h}(t, 0+) &= \beta_{\mathbf{L}} \frac{\partial u_{\mathbf{U}}}{\partial h}(t, 0-) & t &\in (0, \underline{\mathbf{G}}] \end{aligned}$$

$s = \mathbf{D}$

$$(28) \quad \begin{aligned} \frac{\partial u_{\mathbf{D}}}{\partial t}(t, h) &= \frac{1}{2} \frac{\partial^2 u_{\mathbf{D}}}{\partial h^2}(t, h) & (t, h) &\in (0, \underline{\mathbf{G}}) \times \mathbb{R} \setminus \{0\} \\ u_{\mathbf{D}}(t, 0+) &= u_{\mathbf{D}}(t, 0-) & t &\in (0, \underline{\mathbf{G}}] \\ \beta_{\mathbf{O}} \frac{\partial u_{\mathbf{D}}}{\partial h}(t, 0+) &= \beta_{\mathbf{R}} \frac{\partial u_{\mathbf{D}}}{\partial h}(t, 0-) & t &\in (0, \underline{\mathbf{G}}] \end{aligned}$$

$s = \mathbf{L}$

$$(29) \quad \begin{aligned} \frac{\partial u_{\mathbf{L}}}{\partial t}(t, h) &= \frac{1}{2} \frac{\partial^2 u_{\mathbf{L}}}{\partial h^2}(t, h) & (t, h) &\in (-\underline{\mathbf{G}}, 0) \times \mathbb{R} \setminus \{0\} \\ u_{\mathbf{L}}(t, 0+) &= u_{\mathbf{L}}(t, 0-) & t &\in (-\underline{\mathbf{G}}, 0] \\ \beta_{\mathbf{O}} \frac{\partial u_{\mathbf{L}}}{\partial h}(t, 0+) &= \beta_{\mathbf{L}} \frac{\partial u_{\mathbf{L}}}{\partial h}(t, 0-) & t &\in (-\underline{\mathbf{G}}, 0] \end{aligned}$$

$s = \mathbf{R}$

$$(30) \quad \begin{aligned} \frac{\partial u_{\mathbf{R}}}{\partial t}(t, h) &= \frac{1}{2} \frac{\partial^2 u_{\mathbf{R}}}{\partial h^2}(t, h) & (t, h) &\in (-\underline{\mathbf{G}}, 0) \times \mathbb{R} \setminus \{0\} \\ u_{\mathbf{R}}(t, 0+) &= u_{\mathbf{R}}(t, 0-) & t &\in (-\underline{\mathbf{G}}, 0] \\ \beta_{\mathbf{O}} \frac{\partial u_{\mathbf{R}}}{\partial h}(t, 0+) &= \beta_{\mathbf{R}} \frac{\partial u_{\mathbf{R}}}{\partial h}(t, 0-) & t &\in (-\underline{\mathbf{G}}, 0] \end{aligned}$$

Now extend u_s to $h = 0$ by continuity, i.e. for $s \in \{\mathbf{U}, \mathbf{D}, \mathbf{L}, \mathbf{R}\}$, define

$$u_s(t, 0) \stackrel{\text{def}}{=} \lim_{h \rightarrow 0} u_s(t, h)$$

for $t \in (0, \underline{\mathbf{G}})$ if $s \in \{\mathbf{U}, \mathbf{D}\}$ and $t \in (-\underline{\mathbf{G}}, 0)$ if $s \in \{\mathbf{L}, \mathbf{R}\}$. The above system of PDE's assures us that the limits exist. For $h \in \mathbb{R}$, define

$$u_{\mathbf{L}}^\circ(h) \stackrel{\text{def}}{=} \lim_{t \nearrow 0} u_{\mathbf{L}}(t, h), \quad u_{\mathbf{R}}^\circ(h) \stackrel{\text{def}}{=} \lim_{t \nearrow 0} u_{\mathbf{R}}(t, h).$$

Using the above system of PDE's together with Lemma 5.9, it follows that for $s \in \{\mathbf{L}, \mathbf{R}\}$, u_s° belongs to $C(\mathbb{R})$, $u_s^\circ|_{\mathbb{R}^+} \in C^\infty(\mathbb{R}^+)$, $u_s^\circ|_{\mathbb{R}^-} \in C^\infty(\mathbb{R}^-)$ and

$$(31) \quad \beta_{\mathbf{O}} \frac{du_{\mathbf{L}}^\circ}{dh}(0+) = \beta_{\mathbf{L}} \frac{du_{\mathbf{L}}^\circ}{dh}(0-), \quad \beta_{\mathbf{O}} \frac{du_{\mathbf{R}}^\circ}{dh}(0+) = \beta_{\mathbf{R}} \frac{du_{\mathbf{R}}^\circ}{dh}(0-).$$

In light of the G_ℓ -periodicity of the Ψ_ℓ^K 's together with smoothness of the latter away from $h = 0$, it is natural to define

$$(32) \quad \begin{aligned} u_{\mathbf{U}}^\circ(h) &\stackrel{\text{def}}{=} \begin{cases} u_{\mathbf{R}}^\circ(h), & h > 0 \\ u_{\mathbf{L}}^\circ(h), & h < 0 \end{cases} \\ u_{\mathbf{D}}^\circ(h) &\stackrel{\text{def}}{=} \begin{cases} u_{\mathbf{L}}^\circ(h), & h > 0 \\ u_{\mathbf{R}}^\circ(h), & h < 0 \end{cases} \end{aligned}$$

Then, for $s \in \{\mathbf{U}, \mathbf{D}\}$, $u_s^\circ|_{\mathbb{R}^+} \in C^\infty(\mathbb{R}^+)$, $u_s^\circ|_{\mathbb{R}^-} \in C^\infty(\mathbb{R}^-)$. However, there is *no* guarantee that

$$\beta_O \frac{du_{\mathbf{U}}^\circ}{dh}(0+) = \beta_L \frac{du_{\mathbf{U}}^\circ}{dh}(0-) \quad \text{or that} \quad \beta_O \frac{du_{\mathbf{D}}^\circ}{dh}(0+) = \beta_R \frac{du_{\mathbf{D}}^\circ}{dh}(0-)$$

The upshot is that we must develop a suitable approximation to Ψ_A^ε near the origin.

3.4. Approximation Near the Origin (Ψ_B^ε). For $(y_1, y_2) \in \square$, define

$$\tilde{\vartheta}_1(y_1, y_2) \stackrel{\text{def}}{=} \sqrt{2}y_1 \quad \text{and} \quad \tilde{\vartheta}_2(y_1, y_2) \stackrel{\text{def}}{=} \frac{y_2}{\sqrt{2}}.$$

For $z \in \mathbb{R}$, let $\mathfrak{s}(z) \stackrel{\text{def}}{=} z/|z|$ if $z \neq 0$ and let $\mathfrak{s}(0) \stackrel{\text{def}}{=} 0$. Then, for $y_2 \neq 0$,

$$\sqrt{-\tilde{\Theta}(0, y_2)} = \mathfrak{s}(y_2)\tilde{\vartheta}_2(y_1, y_2) \quad \text{and} \quad \frac{\tilde{\mathbf{H}}(y_1, y_2)}{\sqrt{-\tilde{\Theta}(0, y_2)}} = \mathfrak{s}(y_2)\tilde{\vartheta}_1(y_1, y_2).$$

Also, let \mathfrak{G} be the Gaussian kernel:

$$\mathfrak{G}(z) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad \text{for } z \in \mathbb{R}.$$

For $(y_1, y_2) \in \square$ such that $y_2 \neq 0$, define

$$\tilde{\Psi}_B^\varepsilon(y_1, y_2) \stackrel{\text{def}}{=} \begin{cases} \varepsilon u_{\mathbf{U}} \left(-\tilde{\Theta}(0, y_2), \frac{\tilde{\mathbf{H}}(y_1, y_2)}{\varepsilon} \right) - \tilde{F}(y_1, y_2), & y_2 > 0 \\ \varepsilon u_{\mathbf{D}} \left(-\tilde{\Theta}(0, y_2), \frac{\tilde{\mathbf{H}}(y_1, y_2)}{\varepsilon} \right) - \tilde{F}(y_1, y_2), & y_2 < 0 \end{cases}$$

Thus, for $y = (y_1, y_2) \in \square$ such that $y_2 \neq 0$, $\tilde{\Psi}_B^\varepsilon(y)$ is obtained by evaluating $\tilde{\Psi}_A^\varepsilon$ at the point \tilde{y} , where \tilde{y} satisfies $\tilde{\Theta}(\tilde{y}) = \tilde{\Theta}(0, y_2)$ and $\tilde{\mathbf{H}}(\tilde{y}) = \tilde{\mathbf{H}}(y_1, y_2)$, i.e. one pushes back along the constant energy set to the point where the angle matches that at $(0, y_2)$ and evaluates $\tilde{\Psi}_A^\varepsilon$ there.

Lemma 3.7. For $y = (y_1, y_2) \in \square$ such that $y_2 > 0$, we have

$$\begin{aligned}
\tilde{\Psi}_B^\varepsilon(y) + \tilde{F}(y) &= \chi_{\{y_1 > 0\}} \left[\varepsilon \int_{z=0}^{\infty} u_R^\circ(\tilde{\vartheta}_2(y)z) \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) dz \right. \\
&\quad + \frac{\beta_O - \beta_L}{\beta_O + \beta_L} \varepsilon \int_{z=0}^{\infty} u_R^\circ(\tilde{\vartheta}_2(y)z) \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} + z \right) dz \\
&\quad \left. + 2 \frac{\beta_L}{\beta_O + \beta_L} \varepsilon \int_{z=-\infty}^0 u_L^\circ(\tilde{\vartheta}_2(y)z) \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) dz \right] \\
(33) \quad &+ \chi_{\{y_1 = 0\}} \left[2 \frac{\beta_O}{\beta_O + \beta_L} \varepsilon \int_{z=0}^{\infty} u_R^\circ(\tilde{\vartheta}_2(y)z) \mathfrak{G}(z) dz + 2 \frac{\beta_L}{\beta_O + \beta_L} \varepsilon \int_{z=-\infty}^0 u_L^\circ(\tilde{\vartheta}_2(y)z) \mathfrak{G}(z) dz \right] \\
&+ \chi_{\{y_1 < 0\}} \left[2 \frac{\beta_O}{\beta_O + \beta_L} \varepsilon \int_{z=0}^{\infty} u_R^\circ(\tilde{\vartheta}_2(y)z) \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) dz \right. \\
&\quad + \varepsilon \int_{z=-\infty}^0 u_L^\circ(\tilde{\vartheta}_2(y)z) \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) dz \\
&\quad \left. + \frac{\beta_L - \beta_O}{\beta_O + \beta_L} \varepsilon \int_{z=-\infty}^0 u_L^\circ(\tilde{\vartheta}_2(y)z) \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} + z \right) dz \right]
\end{aligned}$$

and for $y = (y_1, y_2) \in \square$ such that $y_2 < 0$,

$$\begin{aligned}
(34) \quad \tilde{\Psi}_B^\varepsilon(y) + \tilde{F}(y) &= \chi_{\{y_1 > 0\}} \left[\varepsilon \int_{z=0}^{\infty} u_R^\circ(\tilde{\vartheta}_2(y)z) \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) dz \right. \\
&\quad + \frac{\beta_R - \beta_O}{\beta_O + \beta_R} \varepsilon \int_{z=0}^{\infty} u_R^\circ(\tilde{\vartheta}_2(y)z) \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} + z \right) dz \\
&\quad \left. + 2 \frac{\beta_O}{\beta_O + \beta_R} \varepsilon \int_{z=-\infty}^0 u_L^\circ(\tilde{\vartheta}_2(y)z) \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) dz \right] \\
&+ \chi_{\{y_1 = 0\}} \left[2 \frac{\beta_R}{\beta_O + \beta_R} \varepsilon \int_{z=0}^{\infty} u_R^\circ(\tilde{\vartheta}_2(y)z) \mathfrak{G}(z) dz + 2 \frac{\beta_O}{\beta_O + \beta_R} \varepsilon \int_{z=-\infty}^0 u_L^\circ(\tilde{\vartheta}_2(y)z) \mathfrak{G}(z) dz \right] \\
&+ \chi_{\{y_1 < 0\}} \left[2 \frac{\beta_R}{\beta_O + \beta_R} \varepsilon \int_{z=0}^{\infty} u_R^\circ(\tilde{\vartheta}_2(y)z) \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) dz \right. \\
&\quad + \varepsilon \int_{z=-\infty}^0 u_L^\circ(\tilde{\vartheta}_2(y)z) \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) dz \\
&\quad \left. + \frac{\beta_O - \beta_R}{\beta_O + \beta_R} \varepsilon \int_{z=-\infty}^0 u_L^\circ(\tilde{\vartheta}_2(y)z) \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} + z \right) dz \right]
\end{aligned}$$

The functions $\tilde{\Psi}_B^\varepsilon + \tilde{F}$ and $\tilde{\Psi}_B^\varepsilon$ have continuous extensions to \square . Also, $\tilde{\Psi}_B^\varepsilon + \tilde{F}$ satisfies the skewness conditions at $\tilde{X} \setminus \{(0, 0)\}$.

Proof. Consider the case $y_2 > 0$. Since u_U solves the heat equation (27) with initial data given by (32), we can use Lemma 5.9 with $\gamma = \frac{\beta_O}{\beta_O + \beta_L}$ to conclude that

$$u_U(t, h) = \int_{z=0}^{\infty} u_R^\circ(z) p(t, h, z) dz + \int_{z=-\infty}^0 u_L^\circ(z) p(t, h, z) dz \quad \text{for } (t, h) \in (0, \underline{\mathbf{G}}) \times \mathbb{R}.$$

Using the expression for $p(t, h, z)$ (note that this expression varies depending on whether $h > 0$, $h = 0$ or $h < 0$), replacing t and h by $-\tilde{\Theta}(0, y_2)$ and $\frac{\tilde{H}(y_1, y_2)}{\varepsilon}$ respectively, and rescaling the variable of integration $z \mapsto \tilde{\vartheta}_2(y)z$, we get (33).

For the case $y_2 < 0$, the function u_D solves the heat equation (28) with initial data given by (32). Using Lemma 5.9 with $\gamma = \frac{\beta_O}{\beta_O + \beta_R}$, we get

$$u_D(t, h) = \int_{z=0}^{\infty} u_L^\circ(z)p(t, h, z)dz + \int_{z=-\infty}^0 u_R^\circ(z)p(t, h, z)dz \quad \text{for } (t, h) \in (0, \underline{\mathbb{G}}) \times \mathbb{R}$$

Proceeding as above and noting that $\tilde{\vartheta}_2(y)$ is negative for $y_2 < 0$, we get (34).

Let's now address the issue of continuity. It is easily seen from the expressions for $\tilde{\Psi}_B^\varepsilon + \tilde{F}$ that for $y_2 \neq 0$,

$$\lim_{y_1 \searrow 0} [\tilde{\Psi}_B^\varepsilon(y_1, y_2) + \tilde{F}(y_1, y_2)] = \lim_{y_1 \nearrow 0} [\tilde{\Psi}_B^\varepsilon(y_1, y_2) + \tilde{F}(y_1, y_2)] = \tilde{\Psi}_B^\varepsilon(0, y_2) + \tilde{F}(0, y_2).$$

For continuity (of an extension) at $y_2 = 0$, let's first consider the case when $y_1 > 0$. Then, from (33) and (34), we have, respectively,

$$\begin{aligned} \lim_{y_2 \searrow 0} [\tilde{\Psi}_B^\varepsilon(y) + \tilde{F}(y)] &= \varepsilon u_R^\circ(0) \int_{z=0}^{\infty} \mathfrak{G} \left(\frac{\sqrt{2}y_1}{\varepsilon} - z \right) dz + \varepsilon \frac{\beta_O - \beta_L}{\beta_O + \beta_L} u_R^\circ(0) \int_{z=0}^{\infty} \mathfrak{G} \left(\frac{\sqrt{2}y_1}{\varepsilon} + z \right) dz \\ &\quad + 2\varepsilon \frac{\beta_L}{\beta_O + \beta_L} u_L^\circ(0) \int_{z=0}^{\infty} \mathfrak{G} \left(\frac{\sqrt{2}y_1}{\varepsilon} + z \right) dz \end{aligned}$$

and

$$\begin{aligned} \lim_{y_2 \nearrow 0} [\tilde{\Psi}_B^\varepsilon(y) + \tilde{F}(y)] &= \varepsilon u_R^\circ(0) \int_{z=0}^{\infty} \mathfrak{G} \left(\frac{\sqrt{2}y_1}{\varepsilon} - z \right) dz + \varepsilon \frac{\beta_R - \beta_O}{\beta_O + \beta_R} u_R^\circ(0) \int_{z=0}^{\infty} \mathfrak{G} \left(\frac{\sqrt{2}y_1}{\varepsilon} + z \right) dz \\ &\quad + 2\varepsilon \frac{\beta_O}{\beta_O + \beta_R} u_L^\circ(0) \int_{z=0}^{\infty} \mathfrak{G} \left(\frac{\sqrt{2}y_1}{\varepsilon} + z \right) dz. \end{aligned}$$

Since u_R° and u_L° are continuous, we can use $u_R^\circ(0)$ and $u_L^\circ(0)$ instead of the appropriate right and left hand limits. Note that we do *not* assume any relationship between $u_R^\circ(0)$ and $u_L^\circ(0)$. Hence, the limits above agree if and only if

$$\frac{\beta_O - \beta_L}{\beta_O + \beta_L} u_R^\circ(0) + 2 \frac{\beta_L}{\beta_O + \beta_L} u_L^\circ(0) = \frac{\beta_R - \beta_O}{\beta_O + \beta_R} u_R^\circ(0) + 2 \frac{\beta_O}{\beta_O + \beta_R} u_L^\circ(0)$$

which is easily seen to yield

$$2(u_R^\circ(0) - u_L^\circ(0))(\beta_O^2 - \beta_L\beta_R) = 0.$$

This is the origin of the requirement $\beta_O^2 = \beta_L\beta_R$ between the skewness coefficients. The case $y_1 < 0$ is similar and also yields the last equation as a necessary and sufficient condition for continuity at $y_2 = 0$.

To establish continuity at $(0, 0)$, note that

$$\lim_{\substack{(y_1, y_2) \rightarrow (0, 0) \\ y_2 > 0}} [\tilde{\Psi}_B^\varepsilon(y_1, y_2) + \tilde{F}(y_1, y_2)] = \varepsilon \frac{\beta_O}{\beta_O + \beta_L} u_R^\circ(0) + \varepsilon \frac{\beta_L}{\beta_O + \beta_L} u_L^\circ(0)$$

and

$$\lim_{\substack{(y_1, y_2) \rightarrow (0, 0) \\ y_2 < 0}} [\tilde{\Psi}_B^\varepsilon(y_1, y_2) + \tilde{F}(y_1, y_2)] = \varepsilon \frac{\beta_R}{\beta_O + \beta_R} u_R^\circ(0) + \varepsilon \frac{\beta_O}{\beta_O + \beta_R} u_L^\circ(0)$$

The limits above agree if and only if $\beta_O^2 = \beta_L\beta_R$.

Since \tilde{F} vanishes when either y_1 or y_2 (or both) equals zero, $\tilde{\Psi}_B^\varepsilon$ also has a continuous extension to \square .

To establish the skewness conditions, we need to show

$$(35) \quad \beta_O \lim_{y_1 \searrow 0} \frac{\partial}{\partial y_1} [\tilde{\Psi}_B^\varepsilon(y_1, y_2) + \tilde{F}(y_1, y_2)] = \beta_L \lim_{y_1 \nearrow 0} \frac{\partial}{\partial y_1} [\tilde{\Psi}_B^\varepsilon(y_1, y_2) + \tilde{F}(y_1, y_2)] \quad \text{for } y_2 > 0$$

$$(36) \quad \beta_O \lim_{y_2 \searrow 0} \frac{\partial}{\partial y_2} [\tilde{\Psi}_B^\varepsilon(y_1, y_2) + \tilde{F}(y_1, y_2)] = \beta_R \lim_{y_2 \nearrow 0} \frac{\partial}{\partial y_2} [\tilde{\Psi}_B^\varepsilon(y_1, y_2) + \tilde{F}(y_1, y_2)] \quad \text{for } y_1 > 0$$

$$(37) \quad \beta_R \lim_{y_1 \searrow 0} \frac{\partial}{\partial y_1} [\tilde{\Psi}_B^\varepsilon(y_1, y_2) + \tilde{F}(y_1, y_2)] = \beta_O \lim_{y_1 \nearrow 0} \frac{\partial}{\partial y_1} [\tilde{\Psi}_B^\varepsilon(y_1, y_2) + \tilde{F}(y_1, y_2)] \quad \text{for } y_2 < 0$$

$$(38) \quad \beta_L \lim_{y_2 \searrow 0} \frac{\partial}{\partial y_2} [\tilde{\Psi}_B^\varepsilon(y_1, y_2) + \tilde{F}(y_1, y_2)] = \beta_O \lim_{y_2 \nearrow 0} \frac{\partial}{\partial y_2} [\tilde{\Psi}_B^\varepsilon(y_1, y_2) + \tilde{F}(y_1, y_2)] \quad \text{for } y_1 < 0$$

Noting that $\mathfrak{G}'(z) = -z\mathfrak{G}(z)$ for all $z \in \mathbb{R}$, we directly compute, for $y_2 > 0$,

$$(39) \quad \lim_{y_1 \searrow 0} \frac{\partial}{\partial y_1} [\tilde{\Psi}_B^\varepsilon(y) + \tilde{F}(y)] = 2\sqrt{2} \frac{\beta_L}{\beta_O + \beta_L} \int_{z=0}^{\infty} u_R^\circ(\tilde{\vartheta}_2(y)z) z \mathfrak{G}(z) dz \\ + 2\sqrt{2} \frac{\beta_L}{\beta_O + \beta_L} \int_{z=-\infty}^0 u_L^\circ(\tilde{\vartheta}_2(y)z) z \mathfrak{G}(z) dz$$

and

$$(40) \quad \lim_{y_1 \nearrow 0} \frac{\partial}{\partial y_1} [\tilde{\Psi}_B^\varepsilon(y) + \tilde{F}(y)] = 2\sqrt{2} \frac{\beta_O}{\beta_O + \beta_L} \int_{z=0}^{\infty} u_R^\circ(\tilde{\vartheta}_2(y)z) z \mathfrak{G}(z) dz \\ + 2\sqrt{2} \frac{\beta_O}{\beta_O + \beta_L} \int_{z=-\infty}^0 u_L^\circ(\tilde{\vartheta}_2(y)z) z \mathfrak{G}(z) dz$$

The equations (39) and (40) readily yield (35). The proof of (37) is similar.

Let's now prove (36). For $(y_1, y_2) \in \square$, we have

$$\frac{\partial \tilde{\vartheta}_1}{\partial y_2}(y_1, y_2) \equiv 0 \quad \text{and} \quad \frac{\partial \tilde{\vartheta}_2}{\partial y_2}(y_1, y_2) \equiv \frac{1}{\sqrt{2}}$$

For $y_1 > 0$, we calculate

$$\lim_{y_2 \searrow 0} \frac{\partial}{\partial y_2} [\tilde{\Psi}_B^\varepsilon(y) + \tilde{F}(y)] = \frac{\varepsilon}{\sqrt{2}} \frac{du_R^\circ}{dh}(0+) \int_{z=0}^{\infty} z \mathfrak{G} \left(\frac{\sqrt{2}y_1}{\varepsilon} - z \right) dz \\ + \frac{\varepsilon}{\sqrt{2}} \frac{\beta_O - \beta_L}{\beta_O + \beta_L} \frac{du_R^\circ}{dh}(0+) \int_{z=0}^{\infty} z \mathfrak{G} \left(\frac{\sqrt{2}y_1}{\varepsilon} + z \right) dz \\ + 2 \frac{\varepsilon}{\sqrt{2}} \frac{\beta_L}{\beta_O + \beta_L} \frac{du_L^\circ}{dh}(0-) \int_{z=-\infty}^0 z \mathfrak{G} \left(\frac{\sqrt{2}y_1}{\varepsilon} - z \right) dz$$

Making the change of variables $z \mapsto -z$ in the last integral, we get

$$(41) \quad \lim_{y_2 \searrow 0} \frac{\partial}{\partial y_2} [\tilde{\Psi}_B^\varepsilon(y) + \tilde{F}(y)] = \frac{\varepsilon}{\sqrt{2}} \frac{du_R^\circ}{dh}(0+) \int_{z=0}^{\infty} z \mathfrak{G} \left(\frac{\sqrt{2}y_1}{\varepsilon} - z \right) dz \\ + \frac{\varepsilon}{\sqrt{2}} \left[\frac{\beta_O - \beta_L}{\beta_O + \beta_L} \frac{du_R^\circ}{dh}(0+) - 2 \frac{\beta_L}{\beta_O + \beta_L} \frac{du_L^\circ}{dh}(0-) \right] \int_{z=0}^{\infty} z \mathfrak{G} \left(\frac{\sqrt{2}y_1}{\varepsilon} + z \right) dz$$

Also, for $y_1 > 0$,

$$\lim_{y_2 \nearrow 0} \frac{\partial}{\partial y_2} [\tilde{\Psi}_B^\varepsilon(y) + \tilde{F}(y)] = \frac{\varepsilon}{\sqrt{2}} \frac{du_R^\circ}{dh}(0-) \int_{z=0}^{\infty} z \mathfrak{G} \left(\frac{\sqrt{2}y_1}{\varepsilon} - z \right) dz \\ + \frac{\varepsilon}{\sqrt{2}} \frac{\beta_R - \beta_O}{\beta_O + \beta_R} \frac{du_R^\circ}{dh}(0-) \int_{z=0}^{\infty} z \mathfrak{G} \left(\frac{\sqrt{2}y_1}{\varepsilon} + z \right) dz \\ + 2 \frac{\varepsilon}{\sqrt{2}} \frac{\beta_O}{\beta_O + \beta_R} \frac{du_L^\circ}{dh}(0+) \int_{z=-\infty}^0 z \mathfrak{G} \left(\frac{\sqrt{2}y_1}{\varepsilon} - z \right) dz$$

Again, making the change of variables $z \mapsto -z$ in the last integral, we get

$$(42) \quad \lim_{y_2 \nearrow 0} \frac{\partial}{\partial y_2} [\tilde{\Psi}_B^\varepsilon(y) + \tilde{F}(y)] = \frac{\varepsilon}{\sqrt{2}} \frac{du_R^\circ(0-)}{dh} \int_{z=0}^{\infty} z \mathfrak{G} \left(\frac{\sqrt{2}y_1}{\varepsilon} - z \right) dz \\ + \frac{\varepsilon}{\sqrt{2}} \left[\frac{\beta_R - \beta_O}{\beta_O + \beta_R} \frac{du_R^\circ(0-)}{dh} - 2 \frac{\beta_O}{\beta_O + \beta_R} \frac{du_L^\circ(0+)}{dh} \right] \int_{z=0}^{\infty} z \mathfrak{G} \left(\frac{\sqrt{2}y_1}{\varepsilon} + z \right) dz$$

From (41) and (42), we see that (36) holds iff

$$(43) \quad \beta_O \frac{du_R^\circ(0+)}{dh} = \beta_R \frac{du_R^\circ(0-)}{dh}$$

and

$$(44) \quad \frac{\beta_O - \beta_L}{\beta_O + \beta_L} \beta_O \frac{du_R^\circ(0+)}{dh} - 2 \frac{\beta_O \beta_L}{\beta_O + \beta_L} \frac{du_L^\circ(0-)}{dh} = \frac{\beta_R - \beta_O}{\beta_O + \beta_R} \beta_R \frac{du_R^\circ(0-)}{dh} - 2 \frac{\beta_R \beta_O}{\beta_O + \beta_R} \frac{du_L^\circ(0+)}{dh}$$

hold. Note that (43) is precisely the second equality in (31). Since the first (one-sided) derivatives of u_R° are *independent* of the first (one-sided) derivatives of u_L° , the equation (44) holds iff

$$(45) \quad \frac{\beta_O - \beta_L}{\beta_O + \beta_L} = \frac{\beta_R - \beta_O}{\beta_O + \beta_R} \quad \text{and} \quad \frac{\beta_O}{\beta_O + \beta_L} = \frac{\beta_R}{\beta_O + \beta_R}$$

where we have once again used (31). It is now easily seen that both the conditions in (45) reduce to

$$\beta_O^2 = \beta_L \beta_R$$

The proof of (38) is similar. □

Using Lemma 3.7, we can now define, for $y \in \square$,

$$\tilde{\Psi}_B^\varepsilon(y) \stackrel{\text{def}}{=} \lim_{\substack{y' \rightarrow y \\ y' \in \square \setminus \{y_2=0\}}} \tilde{\Psi}_B^\varepsilon(y').$$

If we now define Ψ_B^ε on $\tilde{\phi}(\square)$ by

$$\Psi_B^\varepsilon \stackrel{\text{def}}{=} \tilde{\Psi}_B^\varepsilon \circ \tilde{\phi},$$

then Ψ_B^ε is continuous on $\tilde{\phi}(\square)$ and $F + \Psi_B^\varepsilon$ satisfies the skewness conditions at $\mathbf{H}^{-1}(0) \cap \tilde{\phi}(\square)$.

3.5. Calculations and Estimates. The idea is now to obtain Ψ^ε by using Ψ_A^ε away from \mathfrak{o} and Ψ_B^ε near \mathfrak{o} . Define

$$\tilde{\mathfrak{D}}^\varepsilon(y) \stackrel{\text{def}}{=} \left\{ |\tilde{\Theta}(y)|^2 + \left| \frac{\tilde{\mathbf{H}}(y)}{\varepsilon} \right|^4 \right\}^{\frac{1}{2}} \quad \text{for } y \in \square$$

and

$$\mathfrak{D}^\varepsilon(x) \stackrel{\text{def}}{=} \tilde{\mathfrak{D}}^\varepsilon(\tilde{\phi}(x)) \quad \text{for } x \in \tilde{\phi}(\square).$$

The following Lemma is taken from [Sow05].

Lemma 3.8. *There is a constant $K_{3.8} > 0$ such that for all $\varepsilon \in (0, 1)$, $y \in \square$,*

$$\|y\|_\infty \leq K_{3.8} \{ (\tilde{\mathfrak{D}}^\varepsilon(y))^{\frac{1}{2}} + (\tilde{\mathfrak{D}}^\varepsilon(y))^{\frac{1}{4}} \} \\ \|y\|_\infty \leq K_{3.8} \{ (\tilde{\mathfrak{D}}^\varepsilon(y))^{\frac{1}{2}} + \varepsilon \} \\ |\tilde{\mathbf{H}}(y)| \leq \varepsilon (\tilde{\mathfrak{D}}^\varepsilon(y))^{\frac{1}{2}}$$

For each $\delta > 0$, define

$$\tilde{\mathcal{B}}^\varepsilon(\delta) \stackrel{\text{def}}{=} \{y \in \square : \tilde{\mathcal{D}}^\varepsilon(y) \leq \delta\} \quad \text{and} \quad \mathcal{B}^\varepsilon(\delta) \stackrel{\text{def}}{=} \tilde{\phi}(\tilde{\mathcal{B}}^\varepsilon(\delta)).$$

Obviously,

$$\mathcal{B}^\varepsilon(\delta) = \{x \in \tilde{\phi}(\square) : \mathcal{D}^\varepsilon(x) \leq \delta\}.$$

If we now fix $\bar{\delta} \in (0, 1)$ such that

$$\bar{\delta} < \left(\frac{\varpi}{4K_{3.8}} \right)^4,$$

then for $\delta \in (0, \bar{\delta})$, $\varepsilon \in (0, 1)$, $y \in \tilde{\mathcal{B}}^\varepsilon(2\delta)$, we have

$$\|y\|_\infty \leq K_{3.8} \{2^{\frac{1}{2}} \delta^{\frac{1}{2}} + 2^{\frac{1}{4}} \delta^{\frac{1}{4}}\} \leq 2K_{3.8} \delta^{\frac{1}{4}} (1 + \delta^{\frac{1}{4}}) \leq 4K_{3.8} \delta^{\frac{1}{4}} < \varpi.$$

Hence, $\tilde{\mathcal{B}}^\varepsilon(2\delta) \subset \subset \square$.

For $\delta \in (0, \bar{\delta})$, $\varepsilon \in (0, 1)$, let $\eta^{\delta, \varepsilon}(x) \stackrel{\text{def}}{=} \mathbf{c}_0 \left(\frac{\mathcal{D}^\varepsilon(x)}{\delta} \right) \chi_{\tilde{\phi}(\square)}(x)$ for $x \in \bar{\mathbf{S}}$. Define

$$\Psi^{\delta, \varepsilon}(x) \stackrel{\text{def}}{=} \Psi_A^\varepsilon(x) (1 - \eta^{\delta, \varepsilon}(x)) + \Psi_B^\varepsilon(x) \eta^{\delta, \varepsilon}(x) \quad \text{for } x \in \bar{\mathbf{S}}$$

and

$$\tilde{\Psi}^{\delta, \varepsilon}(y) \stackrel{\text{def}}{=} \Psi^{\delta, \varepsilon}(\tilde{\phi}(y)) \quad \text{for } y \in \square.$$

Note that

$$\text{supp}(1 - \eta^{\delta, \varepsilon}) \subset \bar{\mathbf{S}} \setminus \mathcal{B}_0^\varepsilon(\delta) \quad \text{and} \quad \text{supp}(\eta^{\delta, \varepsilon}) \subset \mathcal{B}^\varepsilon(2\delta)$$

where $\mathcal{B}_0^\varepsilon(\delta) \stackrel{\text{def}}{=} \{x \in \tilde{\phi}(\square) : \mathcal{D}^\varepsilon(x) < \delta\}$.

Lemma 3.9. For $\delta \in (0, \bar{\delta})$, $\varepsilon \in (0, 1)$ and $x \in \bar{\mathbf{S}} \setminus \mathbf{H}^{-1}(0)$,

$$\mathcal{L}^\varepsilon \Psi^{\delta, \varepsilon}(x) = \sum_{i=1}^4 I_i^{\delta, \varepsilon}(x)$$

where

$$\begin{aligned} I_1^{\delta, \varepsilon}(x) &= (\mathcal{L}^\varepsilon \Psi_A^\varepsilon)(x) (1 - \eta^{\delta, \varepsilon}(x)), \\ I_2^{\delta, \varepsilon}(x) &= (\mathcal{L}^\varepsilon \Psi_B^\varepsilon)(x) \eta^{\delta, \varepsilon}(x), \\ I_3^{\delta, \varepsilon}(x) &= (\Psi_B^\varepsilon(x) - \Psi_A^\varepsilon(x)) (\mathcal{L}^\varepsilon \eta^{\delta, \varepsilon})(x), \\ I_4^{\delta, \varepsilon}(x) &= (\nabla(\Psi_B^\varepsilon - \Psi_A^\varepsilon))(x), \nabla \eta^{\delta, \varepsilon}(x) \end{aligned}$$

Proof. Straightforward. □

Recall the function $\mathcal{E}(z)$ in the statement of the Main Theorem. \mathcal{E} is smooth and there is a constant $K > 0$ such that

$$\frac{1}{K} \exp \left[-\frac{1}{2} \sqrt{\frac{\pi}{\mathbf{G}_O}} |z| \right] \leq \mathcal{E}(z) \leq K \exp \left[-\frac{1}{2} \sqrt{\frac{\pi}{\mathbf{G}_O}} |z| \right]$$

for all $z \in \mathbb{R}$. Also, there exists a constant $K > 0$ such that

$$|z| \exp \left[-\sqrt{\frac{\pi}{\mathbf{G}_O}} |z| \right] \leq K \exp \left[-\frac{1}{2} \sqrt{\frac{\pi}{\mathbf{G}_O}} |z| \right]$$

for all $z \in \mathbb{R}$.

Lemma 3.10. There is a constant $K > 0$ such that for all $x \in \bar{\mathbf{S}} \setminus (\mathcal{B}_0^\varepsilon(\delta) \cup \mathbf{H}^{-1}(0))$,

$$(46) \quad |(\mathcal{L}^\varepsilon \Psi_A^\varepsilon)(x)| \leq \frac{K}{\sqrt{\delta}} \left\{ 1 + \left(\frac{\varepsilon}{\sqrt{\delta}} \right)^3 \right\} \mathcal{E} \left(\frac{\mathbf{H}(x)}{\varepsilon} \right).$$

Proof. The proof is similar to that of Lemma 4.1 in [Sow05]. □

Lemma 3.11. *There is a constant $K > 0$ such that for $\varepsilon \in (0, 1)$ and $y \in \mathcal{C}_L \cup \mathcal{C}_R$,*

$$\begin{aligned} |\tilde{\Psi}_B^\varepsilon(y) - \tilde{\Psi}_A^\varepsilon(y)| &\leq K\varepsilon(\varepsilon + \|y\|_\infty) + K\varepsilon \exp\left[-\frac{\|y\|_\infty^2}{K\varepsilon^2}\right], \\ \|\nabla(\tilde{\Psi}_B^\varepsilon - \tilde{\Psi}_A^\varepsilon)(y)\| &\leq K(\varepsilon\|y\|_\infty + \|y\|_\infty^2 + \varepsilon) + K \exp\left[-\frac{\|y\|_\infty^2}{K\varepsilon^2}\right]. \end{aligned}$$

Proof. Let's start by noting that for $y \in \mathcal{C}_L \cup \mathcal{C}_R$, $|y_1| \geq |y_2|$ and hence $\|y\|^2 \leq 2y_1^2$. This implies, in particular, that $e^{-|\tilde{\vartheta}_1(y)|^2/2\varepsilon^2} \leq e^{-\|y\|^2/2\varepsilon^2}$. We will also repeatedly use the fact that for $k \geq 0$, the map $z \mapsto |z|^k \mathfrak{G}(z)$ is bounded on \mathbb{R} (typically, we will have $z = \frac{\|y\|}{\varepsilon}$).

Define

$$\alpha(y) \stackrel{\text{def}}{=} \begin{cases} \frac{\beta_O}{\beta_O + \beta_L} & \text{if } y_2 > 0, \\ \frac{\beta_R}{\beta_O + \beta_R} & \text{if } y_2 < 0. \end{cases}$$

For $y \in \mathcal{C}_s$, $s \in \{L, R\}$ such that $y_2 \neq 0$, we have

$$\tilde{\Psi}_B^\varepsilon(y) - \tilde{\Psi}_A^\varepsilon(y) = \sum_{i=1}^3 \Upsilon_{i,s}^\varepsilon(y)$$

where

$$\begin{aligned} \Upsilon_{1,R}^\varepsilon(y) &\stackrel{\text{def}}{=} \varepsilon \int_{z=0}^{\infty} \left\{ u_R^\circ(\tilde{\vartheta}_2(y)z) - u_R^\circ\left(\frac{\tilde{H}(y)}{\varepsilon}\right) \right\} \mathfrak{G}\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z\right) dz \\ &\quad + (2\alpha(y) - 1)\varepsilon \int_{z=0}^{\infty} \left\{ u_R^\circ(\tilde{\vartheta}_2(y)z) - u_R^\circ\left(\frac{\tilde{H}(y)}{\varepsilon}\right) \right\} \mathfrak{G}\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} + z\right) dz \\ &\quad + 2(1 - \alpha(y))\varepsilon \int_{z=-\infty}^0 \left\{ u_R^\circ(\tilde{\vartheta}_2(y)z) - u_R^\circ\left(\frac{\tilde{H}(y)}{\varepsilon}\right) \right\} \mathfrak{G}\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z\right) dz, \\ \Upsilon_{1,L}^\varepsilon(y) &\stackrel{\text{def}}{=} 2\alpha(y)\varepsilon \int_{z=0}^{\infty} \left\{ u_L^\circ(\tilde{\vartheta}_2(y)z) - u_L^\circ\left(\frac{\tilde{H}(y)}{\varepsilon}\right) \right\} \mathfrak{G}\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z\right) dz \\ &\quad + \varepsilon \int_{z=-\infty}^0 \left\{ u_L^\circ(\tilde{\vartheta}_2(y)z) - u_L^\circ\left(\frac{\tilde{H}(y)}{\varepsilon}\right) \right\} \mathfrak{G}\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z\right) dz \\ &\quad + (1 - 2\alpha(y))\varepsilon \int_{z=-\infty}^0 \left\{ u_L^\circ(\tilde{\vartheta}_2(y)z) - u_L^\circ\left(\frac{\tilde{H}(y)}{\varepsilon}\right) \right\} \mathfrak{G}\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} + z\right) dz, \\ \Upsilon_{2,R}^\varepsilon(y) &\stackrel{\text{def}}{=} 2(1 - \alpha(y))\varepsilon \int_{z=-\infty}^0 \{u_L^\circ(\tilde{\vartheta}_2(y)z) - u_R^\circ(\tilde{\vartheta}_2(y)z)\} \mathfrak{G}\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z\right) dz, \\ \Upsilon_{2,L}^\varepsilon(y) &\stackrel{\text{def}}{=} 2\alpha(y)\varepsilon \int_{z=0}^{\infty} \{u_R^\circ(\tilde{\vartheta}_2(y)z) - u_L^\circ(\tilde{\vartheta}_2(y)z)\} \mathfrak{G}\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z\right) dz, \\ \Upsilon_{3,s}^\varepsilon(y) &\stackrel{\text{def}}{=} \varepsilon \left\{ u_s^\circ\left(\frac{\tilde{H}(y)}{\varepsilon}\right) - u_s\left(-\tilde{\Theta}(y), \frac{\tilde{H}(y)}{\varepsilon}\right) \right\} \quad \text{for } s \in \{L, R\}. \end{aligned}$$

Let's start with $\Upsilon_{3,s}^\varepsilon$, $s \in \{L, R\}$. Since u_s solves a heat equation, it is easily seen that there is a constant $K > 0$ such that for all $y \in \mathcal{C}_s$,

$$\left| u_s^\circ\left(\frac{\tilde{H}(y)}{\varepsilon}\right) - u_s\left(-\tilde{\Theta}(y), \frac{\tilde{H}(y)}{\varepsilon}\right) \right| \leq K|\tilde{\Theta}(y)|$$

and for all $y \in \mathcal{C}_s$ such that $y_2 \neq 0$,

$$\left| \dot{u}_s^\circ \left(\frac{\tilde{H}(y)}{\varepsilon} \right) - \frac{\partial u_s}{\partial h} \left(-\tilde{\Theta}(y), \frac{\tilde{H}(y)}{\varepsilon} \right) \right| \leq K |\tilde{\Theta}(y)|.$$

By direct calculation we have that

$$\nabla \Upsilon_{3,s}^\varepsilon(y) = \varepsilon \frac{\partial u_s}{\partial t} \left(-\tilde{\Theta}(y), \frac{\tilde{H}(y)}{\varepsilon} \right) \nabla \tilde{\Theta}(y) + \left\{ \dot{u}_s^\circ \left(\frac{\tilde{H}(y)}{\varepsilon} \right) - \frac{\partial u_s}{\partial h} \left(-\tilde{\Theta}(y), \frac{\tilde{H}(y)}{\varepsilon} \right) \right\} \nabla \tilde{H}(y).$$

Using Lemma 3.5 and the observations above, we see that there exists $K > 0$ such that

$$|\Upsilon_{3,s}^\varepsilon(y)| \leq K \varepsilon \|y\|_\infty^2 \quad \text{for } y \in \mathcal{C}_s$$

and

$$\begin{aligned} \|\nabla \Upsilon_{3,s}^\varepsilon(y)\| &\leq K \varepsilon \|y\|_\infty + K |\tilde{\Theta}(y)| \|y\|_\infty \\ &\leq K \{\varepsilon \|y\|_\infty + \|y\|_\infty^3\} \quad \text{for } y \in \mathcal{C}_s, y_2 \neq 0. \end{aligned}$$

Let's now tackle $\Upsilon_{1,s}^\varepsilon$, $s \in \{\mathbf{L}, \mathbf{R}\}$. Start by defining for $s \in \{\mathbf{L}, \mathbf{R}\}$,

$$\mathbf{E}_s(z, h) \stackrel{\text{def}}{=} \begin{cases} u_s^\circ(h+z) - u_s^\circ(h) - \dot{u}_s^\circ(h+)z & \text{for } h \in \mathbb{R}, z \geq 0 \\ u_s^\circ(h+z) - u_s^\circ(h) - \dot{u}_s^\circ(h-)z & \text{for } h \in \mathbb{R}, z < 0. \end{cases}$$

Then

$$\mathbf{E}_s(z, h) = z^2 \int_{t=0}^1 (1-t) \ddot{u}_s^\circ(h+tz) dt.$$

It is easily seen that for $(z, h) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, $\mathbf{E}_s(z, h)$ is differentiable in z and h . Moreover, there is a $K > 0$ such that

$$|\mathbf{E}_s(z, h)| \leq K z^2, \quad \left| \frac{\partial \mathbf{E}_s}{\partial z}(z, h) \right| \leq K(|z| + z^2), \quad \left| \frac{\partial \mathbf{E}_s}{\partial h}(z, h) \right| \leq K z^2$$

for $(z, h) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. We will establish the desired bounds for $\Upsilon_{1,\mathbf{R}}^\varepsilon$ for the case $y_2 > 0$. The calculations for the case $y_2 < 0$ and for $\Upsilon_{1,\mathbf{L}}^\varepsilon(y)$, $y_2 \neq 0$ are similar. Now

$$\begin{aligned} I_1(y) &\stackrel{\text{def}}{=} \varepsilon \int_{z=0}^\infty \left\{ u_{\mathbf{R}}^\circ(\tilde{\vartheta}_2(y)z) - u_{\mathbf{R}}^\circ \left(\frac{\tilde{H}(y)}{\varepsilon} \right) \right\} \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) dz \\ &= \varepsilon \int_{w=-\infty}^{\frac{\tilde{\vartheta}_1(y)}{\varepsilon}} \left\{ u_{\mathbf{R}}^\circ \left(\frac{\tilde{H}(y)}{\varepsilon} - \tilde{\vartheta}_2(y)w \right) - u_{\mathbf{R}}^\circ \left(\frac{\tilde{H}(y)}{\varepsilon} \right) \right\} \mathfrak{G}(w) dw \\ &= \varepsilon \int_{w=-\infty}^{\frac{\tilde{\vartheta}_1(y)}{\varepsilon}} \left\{ \mathbf{E}_{\mathbf{R}} \left(-\tilde{\vartheta}_2(y)w, \frac{\tilde{H}(y)}{\varepsilon} \right) - \dot{u}_{\mathbf{R}}^\circ \left(\frac{\tilde{H}(y)}{\varepsilon} \right) \tilde{\vartheta}_2(y)w \right\} \mathfrak{G}(w) dw. \end{aligned}$$

Hence, there exists $K > 0$ such that

$$\begin{aligned} |I_1(y)| &\leq \varepsilon \int_{w \in \mathbb{R}} K |\tilde{\vartheta}_2(y)|^2 w^2 \mathfrak{G}(w) dw + \varepsilon \int_{w \in \mathbb{R}} K |\tilde{\vartheta}_2(y)| |w| \mathfrak{G}(w) dw \\ &\leq K \varepsilon \left(|\tilde{\vartheta}_2(y)|^2 + |\tilde{\vartheta}_2(y)| \right) \\ &\leq K \varepsilon (\|y\|_\infty^2 + \|y\|_\infty) \\ &\leq K \varepsilon \|y\|_\infty \end{aligned}$$

where the last inequality follows from the fact that $\|y\|_\infty < \varpi$ for $y \in \square$.

Also, by differentiation, we have,

$$\begin{aligned} \nabla I_1(y) &= \nabla \tilde{\vartheta}_1(y) \left[\mathbf{E}_R \left(-\frac{\tilde{H}(y)}{\varepsilon}, \frac{\tilde{H}(y)}{\varepsilon} \right) - \frac{\tilde{H}(y)}{\varepsilon} \dot{u}_R^\circ \left(\frac{\tilde{H}(y)}{\varepsilon} \right) \right] \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} \right) \\ &\quad + \int_{w=-\infty}^{\frac{\tilde{\vartheta}_1(y)}{\varepsilon}} \left\{ -\varepsilon w \frac{\partial \mathbf{E}_R}{\partial z} \left(-\tilde{\vartheta}_2(y)w, \frac{\tilde{H}(y)}{\varepsilon} \right) \nabla \tilde{\vartheta}_2(y) + \frac{\partial \mathbf{E}_R}{\partial h} \left(-\tilde{\vartheta}_2(y)w, \frac{\tilde{H}(y)}{\varepsilon} \right) \nabla \tilde{H}(y) \right\} \mathfrak{G}(w) dw \\ &\quad + \int_{w=-\infty}^{\frac{\tilde{\vartheta}_1(y)}{\varepsilon}} \left\{ -\dot{u}_R^\circ \left(\frac{\tilde{H}(y)}{\varepsilon} \right) \tilde{\vartheta}_2(y)w \nabla \tilde{H}(y) - \varepsilon w \dot{u}_R^\circ \left(\frac{\tilde{H}(y)}{\varepsilon} \right) \nabla \tilde{\vartheta}_2(y) \right\} \mathfrak{G}(w) dw. \end{aligned}$$

It now follows, using Lemma 3.5 and the estimates on \mathbf{E}_R and its derivatives established above that

$$\begin{aligned} \|\nabla I_1(y)\| &\leq K \|\nabla \tilde{\vartheta}_1(y)\| \left[\left| \frac{\tilde{H}(y)}{\varepsilon} \right|^2 + \left| \frac{\tilde{H}(y)}{\varepsilon} \right| \right] e^{-|\tilde{\vartheta}_1(y)|^2/2\varepsilon^2} \\ &\quad + K \int_{w=-\infty}^{\frac{\tilde{\vartheta}_1(y)}{\varepsilon}} \left\{ \varepsilon |w| (\|y\|^2 w^2 + \|y\| |w|) \|\nabla \tilde{\vartheta}_2(y)\| + \|y\|^3 w^2 + |w| \|y\|^2 + \varepsilon |w| \|\nabla \tilde{\vartheta}_2(y)\| \right\} \mathfrak{G}(w) dw. \end{aligned}$$

By simplification, we now get

$$\begin{aligned} \|\nabla I_1(y)\| &\leq K \left(\frac{\|y\|^4}{\varepsilon^2} + \frac{\|y\|^2}{\varepsilon} \right) e^{-\|y\|^2/2\varepsilon^2} + K(\varepsilon \|y\|_\infty + \|y\|_\infty^2 + \varepsilon) \\ &\leq K(\varepsilon \|y\|_\infty + \|y\|_\infty^2 + \varepsilon). \end{aligned}$$

Note that there is a constant $K > 0$ such that for all $t \geq 0$,

$$(47) \quad \begin{aligned} \int_t^\infty \mathfrak{G}(z) dz &\leq K e^{-t^2/2}, & \int_t^\infty z \mathfrak{G}(z) dz &= \mathfrak{G}(t), \\ \int_t^\infty z^2 \mathfrak{G}(z) dz &\leq K(1+t) e^{-t^2/2}, & \int_t^\infty z^3 \mathfrak{G}(z) dz &= (2+t^2) \mathfrak{G}(t). \end{aligned}$$

Now

$$\begin{aligned} I_2(y) &\stackrel{\text{def}}{=} \frac{\beta_O - \beta_L}{\beta_O + \beta_L} \varepsilon \int_{z=0}^\infty \left\{ u_R^\circ(\tilde{\vartheta}_2(y)z) - u_R^\circ \left(\frac{\tilde{H}(y)}{\varepsilon} \right) \right\} \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} + z \right) dz \\ &= \frac{\beta_O - \beta_L}{\beta_O + \beta_L} \varepsilon \int_{w=\frac{\tilde{\vartheta}_1(y)}{\varepsilon}}^\infty \left\{ u_R^\circ \left(-\frac{\tilde{H}(y)}{\varepsilon} + \tilde{\vartheta}_2(y)w \right) - u_R^\circ \left(\frac{\tilde{H}(y)}{\varepsilon} \right) \right\} \mathfrak{G}(w) dw \\ &= \frac{\beta_O - \beta_L}{\beta_O + \beta_L} \varepsilon \int_{w=\frac{\tilde{\vartheta}_1(y)}{\varepsilon}}^\infty \left\{ \mathbf{E}_R \left(\tilde{\vartheta}_2(y)w - 2\frac{\tilde{H}(y)}{\varepsilon}, \frac{\tilde{H}(y)}{\varepsilon} \right) + \dot{u}_R^\circ \left(\frac{\tilde{H}(y)}{\varepsilon} \right) \left(\tilde{\vartheta}_2(y)w - 2\frac{\tilde{H}(y)}{\varepsilon} \right) \right\} \mathfrak{G}(w) dw. \end{aligned}$$

Elementary computations using equation (47) show that there is a constant $K > 0$ such that

$$\begin{aligned} |I_2(y)| &\leq K \varepsilon \int_{w=\frac{\tilde{\vartheta}_1(y)}{\varepsilon}}^\infty \left[|\tilde{\vartheta}_2(y)| |w| + \left| \frac{\tilde{H}(y)}{\varepsilon} \right| + |\tilde{\vartheta}_2(y)|^2 |w|^2 + \left| \frac{\tilde{H}(y)}{\varepsilon} \right|^2 \right] \mathfrak{G}(w) dw \\ &\leq K \left\{ \varepsilon |\tilde{\vartheta}_2(y)| + |\tilde{H}(y)| + \varepsilon |\tilde{\vartheta}_2(y)|^2 \left(1 + \frac{|\tilde{\vartheta}_1(y)|}{\varepsilon} \right) + \frac{|\tilde{H}(y)|^2}{\varepsilon} \right\} e^{-\frac{|\tilde{\vartheta}_1(y)|^2}{2\varepsilon^2}} \\ &\leq K \left\{ \varepsilon \|y\| + \|y\|^2 + \frac{\|y\|^4}{\varepsilon} \right\} e^{-\frac{\|y\|^2}{2\varepsilon^2}} \\ &\leq K \varepsilon^2. \end{aligned}$$

Differentiating, we get

$$\nabla I_2(y) = \sum_{i=1}^5 J_i(y)$$

where

$$\begin{aligned} J_1(y) &\stackrel{\text{def}}{=} \frac{\beta_O - \beta_L}{\beta_O + \beta_L} (-1) \nabla \tilde{\vartheta}_1(y) \left[\mathbf{E}_R \left(-\frac{\tilde{H}(y)}{\varepsilon}, \frac{\tilde{H}(y)}{\varepsilon} \right) - \frac{\tilde{H}(y)}{\varepsilon} \dot{u}_R^\circ \left(\frac{\tilde{H}(y)}{\varepsilon} \right) \right] \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} \right), \\ J_2(y) &\stackrel{\text{def}}{=} \frac{\beta_O - \beta_L}{\beta_O + \beta_L} \varepsilon \int_{w=\frac{\tilde{\vartheta}_1(y)}{\varepsilon}}^{\infty} \left\{ \frac{\partial \mathbf{E}_R}{\partial z} \left(-2\frac{\tilde{H}(y)}{\varepsilon} + \tilde{\vartheta}_2(y)w, \frac{\tilde{H}(y)}{\varepsilon} \right) \left(-\frac{2}{\varepsilon} \nabla \tilde{H}(y) + w \nabla \tilde{\vartheta}_2(y) \right) \right\} \mathfrak{G}(w) dw, \\ J_3(y) &\stackrel{\text{def}}{=} \frac{\beta_O - \beta_L}{\beta_O + \beta_L} \varepsilon \int_{w=\frac{\tilde{\vartheta}_1(y)}{\varepsilon}}^{\infty} \left\{ \frac{\partial \mathbf{E}_R}{\partial h} \left(-2\frac{\tilde{H}(y)}{\varepsilon} + \tilde{\vartheta}_2(y)w, \frac{\tilde{H}(y)}{\varepsilon} \right) \frac{1}{\varepsilon} \nabla \tilde{H}(y) \right\} \mathfrak{G}(w) dw, \\ J_4(y) &\stackrel{\text{def}}{=} \frac{\beta_O - \beta_L}{\beta_O + \beta_L} \varepsilon \int_{w=\frac{\tilde{\vartheta}_1(y)}{\varepsilon}}^{\infty} \left\{ \dot{u}_R^\circ \left(\frac{\tilde{H}(y)}{\varepsilon} \right) \frac{1}{\varepsilon} \nabla \tilde{H}(y) \left(\tilde{\vartheta}_2(y)w - 2\frac{\tilde{H}(y)}{\varepsilon} \right) \right\} \mathfrak{G}(w) dw, \\ J_5(y) &\stackrel{\text{def}}{=} \frac{\beta_O - \beta_L}{\beta_O + \beta_L} \varepsilon \int_{w=\frac{\tilde{\vartheta}_1(y)}{\varepsilon}}^{\infty} \left\{ \dot{u}_R^\circ \left(\frac{\tilde{H}(y)}{\varepsilon} \right) \left(w \nabla \tilde{\vartheta}_2(y) - \frac{2}{\varepsilon} \nabla \tilde{H}(y) \right) \right\} \mathfrak{G}(w) dw. \end{aligned}$$

A good bit of tedious calculation using (47), together with the estimates on \mathbf{E}_R and its derivatives, yields

$$\begin{aligned} \|J_1(y)\| &\leq K \left[\left| \frac{\tilde{H}(y)}{\varepsilon} \right|^2 + \left| \frac{\tilde{H}(y)}{\varepsilon} \right| \right] e^{-|\tilde{\vartheta}_1(y)|^2/2\varepsilon^2} \\ &\leq K \left(\frac{\|y\|^4}{\varepsilon^2} + \frac{\|y\|^2}{\varepsilon} \right) e^{-\|y\|^2/2\varepsilon^2} \\ &\leq K\varepsilon, \\ \|J_2(y)\| &\leq K\varepsilon \int_{w=\frac{\tilde{\vartheta}_1(y)}{\varepsilon}}^{\infty} \left(\left| \frac{\tilde{H}(y)}{\varepsilon} \right|^2 + y_2^2 w^2 + \left| \frac{\tilde{H}(y)}{\varepsilon} \right| + |y_2| |w| \right) \left(\frac{\|y\|}{\varepsilon} + |w| \right) \mathfrak{G}(w) dw \\ &\leq K \left(\frac{\|y\|^5}{\varepsilon^2} + \frac{\|y\|^4}{\varepsilon} + \|y\|^3 + \varepsilon \|y\|^2 + \frac{\|y\|^3}{\varepsilon} + \|y\|^2 + \varepsilon \|y\| \right) e^{-\|y\|^2/2\varepsilon^2} \\ &\leq K\varepsilon^2, \\ \|J_3(y)\| &\leq K\varepsilon \int_{w=\frac{\tilde{\vartheta}_1(y)}{\varepsilon}}^{\infty} \left(\left| \frac{\tilde{H}(y)}{\varepsilon} \right|^2 + y_2^2 w^2 \right) \frac{\|y\|}{\varepsilon} \mathfrak{G}(w) dw \\ &\leq K \left(\frac{\|y\|^5}{\varepsilon^2} + \|y\|^3 + \frac{\|y\|^4}{\varepsilon} \right) e^{-\|y\|^2/2\varepsilon^2} \\ &\leq K\varepsilon^3, \\ \|J_4(y)\| &\leq K\varepsilon \int_{w=\frac{\tilde{\vartheta}_1(y)}{\varepsilon}}^{\infty} \frac{\|y\|}{\varepsilon} \left(|y_2| |w| + \left| \frac{\tilde{H}(y)}{\varepsilon} \right| \right) \mathfrak{G}(w) dw \\ &\leq K \left(\|y\|^2 + \frac{\|y\|^3}{\varepsilon} \right) e^{-\|y\|^2/2\varepsilon^2} \\ &\leq K\varepsilon^2, \end{aligned}$$

$$\begin{aligned}
\|J_5(y)\| &\leq K\varepsilon \int_{w=\frac{\tilde{\vartheta}_1(y)}{\varepsilon}}^{\infty} \left(|w| + \frac{\|y\|}{\varepsilon} \right) \mathfrak{G}(w) dw \\
&\leq K(\varepsilon + \|y\|) e^{-\|y\|^2/2\varepsilon^2} \\
&\leq K\varepsilon.
\end{aligned}$$

Putting things together, we get

$$\|\nabla I_2(y)\| \leq K\varepsilon.$$

We have

$$\begin{aligned}
I_3(y) &\stackrel{\text{def}}{=} 2 \frac{\beta_L}{\beta_O + \beta_L} \varepsilon \int_{z=-\infty}^0 \left\{ u_{\mathbb{R}}^{\circ}(\tilde{\vartheta}_2(y)z) - u_{\mathbb{R}}^{\circ}\left(\frac{\tilde{\mathbb{H}}(y)}{\varepsilon}\right) \right\} \mathfrak{G}\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z\right) dz \\
&= 2 \frac{\beta_L}{\beta_O + \beta_L} \varepsilon \int_{w=\frac{\tilde{\vartheta}_1(y)}{\varepsilon}}^{\infty} \left\{ u_{\mathbb{R}}^{\circ}\left(\frac{\tilde{\mathbb{H}}(y)}{\varepsilon} - \tilde{\vartheta}_2(y)w\right) - u_{\mathbb{R}}^{\circ}\left(\frac{\tilde{\mathbb{H}}(y)}{\varepsilon}\right) \right\} \mathfrak{G}(w) dw \\
&= 2 \frac{\beta_L}{\beta_O + \beta_L} \varepsilon \int_{w=\frac{\tilde{\vartheta}_1(y)}{\varepsilon}}^{\infty} \left\{ \mathbb{E}_{\mathbb{R}}\left(-\tilde{\vartheta}_2(y)w, \frac{\tilde{\mathbb{H}}(y)}{\varepsilon}\right) - \dot{u}_{\mathbb{R}}^{\circ}\left(\frac{\tilde{\mathbb{H}}(y)}{\varepsilon}\right) \tilde{\vartheta}_2(y)w \right\} \mathfrak{G}(w) dw.
\end{aligned}$$

Arguing as in the case of I_1 , we see that there exists $K > 0$ such that

$$|I_3(y)| \leq K\varepsilon\|y\|_{\infty}.$$

By differentiation, we get

$$\begin{aligned}
\nabla I_3(y) &= -2 \frac{\beta_L}{\beta_O + \beta_L} \nabla \tilde{\vartheta}_1(y) \left[\mathbb{E}_{\mathbb{R}}\left(-\frac{\tilde{\mathbb{H}}(y)}{\varepsilon}, \frac{\tilde{\mathbb{H}}(y)}{\varepsilon}\right) - \frac{\tilde{\mathbb{H}}(y)}{\varepsilon} \dot{u}_{\mathbb{R}}^{\circ}\left(\frac{\tilde{\mathbb{H}}(y)}{\varepsilon}\right) \right] \mathfrak{G}\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon}\right) \\
&\quad + 2 \frac{\beta_L}{\beta_O + \beta_L} \int_{w=\frac{\tilde{\vartheta}_1(y)}{\varepsilon}}^{\infty} \left\{ -\varepsilon w \frac{\partial \mathbb{E}_{\mathbb{R}}}{\partial z}\left(-\tilde{\vartheta}_2(y)w, \frac{\tilde{\mathbb{H}}(y)}{\varepsilon}\right) \nabla \tilde{\vartheta}_2(y) \right\} \mathfrak{G}(w) dw \\
&\quad + 2 \frac{\beta_L}{\beta_O + \beta_L} \int_{w=\frac{\tilde{\vartheta}_1(y)}{\varepsilon}}^{\infty} \left\{ \frac{\partial \mathbb{E}_{\mathbb{R}}}{\partial h}\left(-\tilde{\vartheta}_2(y)w, \frac{\tilde{\mathbb{H}}(y)}{\varepsilon}\right) \nabla \tilde{\mathbb{H}}(y) \right\} \mathfrak{G}(w) dw \\
&\quad + 2 \frac{\beta_L}{\beta_O + \beta_L} \int_{w=\frac{\tilde{\vartheta}_1(y)}{\varepsilon}}^{\infty} \left\{ -\ddot{u}_{\mathbb{R}}^{\circ}\left(\frac{\tilde{\mathbb{H}}(y)}{\varepsilon}\right) \tilde{\vartheta}_2(y)w \nabla \tilde{\mathbb{H}}(y) - \varepsilon w \dot{u}_{\mathbb{R}}^{\circ}\left(\frac{\tilde{\mathbb{H}}(y)}{\varepsilon}\right) \nabla \tilde{\vartheta}_2(y) \right\} \mathfrak{G}(w) dw.
\end{aligned}$$

Again, reasoning as in the case of $\|\nabla I_1\|$, we get

$$\|\nabla I_3(y)\| \leq K(\varepsilon\|y\|_{\infty} + \|y\|_{\infty}^2 + \varepsilon).$$

Putting things together, we get

$$\begin{aligned}
|\Upsilon_{1,\mathbb{R}}^{\varepsilon}(y)| &\leq K\varepsilon(\|y\|_{\infty} + \varepsilon), \\
\|\nabla \Upsilon_{1,\mathbb{R}}^{\varepsilon}(y)\| &\leq K(\varepsilon\|y\|_{\infty} + \|y\|_{\infty}^2 + \varepsilon).
\end{aligned}$$

Let's now bound $\Upsilon_{2,s}^{\varepsilon}$, $s \in \{\mathbb{L}, \mathbb{R}\}$. Let $\Upsilon(z) \stackrel{\text{def}}{=} u_{\mathbb{R}}^{\circ}(z) - u_{\mathbb{L}}^{\circ}(z)$ for $z \in \mathbb{R}$. For $\varepsilon \in (0, 1)$ and $y \in \square$, define

$$\hat{J}(y) \stackrel{\text{def}}{=} -\varepsilon \int_{z=-\infty}^0 \mathfrak{s}(y_1) \Upsilon(\mathfrak{s}(y_1) \tilde{\vartheta}_2(y)z) \mathfrak{G}\left(-\frac{|\tilde{\vartheta}_1(y)|}{\varepsilon} + z\right) dz.$$

Then, using the fact that \mathfrak{G} is even, one can see that $\Upsilon_{2,\mathbb{R}}^{\varepsilon}(y) = 2(1-\alpha(y))\hat{J}(y)$ on $\mathcal{C}_{\mathbb{R}}$ and $\Upsilon_{2,\mathbb{L}}^{\varepsilon}(y) = 2\alpha(y)\hat{J}(y)$ on $\mathcal{C}_{\mathbb{L}}$. Also, define

$$\mathfrak{G}^{(-1)}(z) \stackrel{\text{def}}{=} \int_{t=-\infty}^z \mathfrak{G}(t) dt, \quad \mathfrak{G}^{(-2)}(z) \stackrel{\text{def}}{=} \int_{t=-\infty}^z \mathfrak{G}^{(-1)}(t) dt, \quad \mathfrak{G}^{(-3)}(z) \stackrel{\text{def}}{=} \int_{t=-\infty}^z \mathfrak{G}^{(-2)}(t) dt.$$

It is easily seen using the observations regarding integrals of Gaussians above, that the functions $z \mapsto e^{z^2/2}\mathfrak{G}^{(-1)}(z)$, $z \mapsto e^{z^2/2}\mathfrak{G}^{(-2)}(z)$ and $z \mapsto e^{z^2/2}\mathfrak{G}^{(-3)}(z)$ are bounded for $z \in (-\infty, 0)$. Note also that on integrating by parts,

$$\int_{t=-\infty}^z t\mathfrak{G}^{(-1)}(t)dt = z\mathfrak{G}^{(-2)}(z) - \mathfrak{G}^{(-3)}(z).$$

Hence, the absolute value of the integral can be bounded above by $Ke^{-z^2/4}$. Integrating by parts, we get

$$\hat{J}(y) = -\varepsilon\mathfrak{s}(y_1)\Upsilon(0)\mathfrak{G}^{(-1)}\left(-\frac{|\tilde{\vartheta}_1(y)|}{\varepsilon}\right) + \varepsilon\tilde{\vartheta}_2(y)\int_{z=-\infty}^0 \dot{\Upsilon}(\mathfrak{s}(y_1)\tilde{\vartheta}_2(y)z)\mathfrak{G}^{(-1)}\left(-\frac{|\tilde{\vartheta}_1(y)|}{\varepsilon} + z\right) dz.$$

Hence,

$$\begin{aligned} |\hat{J}(y)| &\leq K\varepsilon e^{-\frac{|\tilde{\vartheta}_1(y)|^2}{2\varepsilon^2}} + K\varepsilon\|y\|_\infty \int_{w=-\infty}^{-\frac{|\tilde{\vartheta}_1(y)|}{\varepsilon}} \mathfrak{G}^{(-1)}(w)dw \\ &\leq K\varepsilon e^{-\frac{\|y\|_\infty^2}{2\varepsilon^2}}. \end{aligned}$$

It now easily follows that

$$|\Upsilon_{2,s}^\varepsilon(y)| \leq K\varepsilon \exp\left[-\frac{\|y\|_\infty^2}{K\varepsilon^2}\right]$$

for suitable $K > 0$.

Differentiating the last expression for $\hat{J}(y)$, we get

$$\begin{aligned} \nabla\hat{J}(y) &= \Upsilon(0)\mathfrak{G}\left(-\frac{|\tilde{\vartheta}_1(y)|}{\varepsilon}\right)\nabla\tilde{\vartheta}_1(y) \\ &\quad + \varepsilon\nabla\tilde{\vartheta}_2(y)\int_{z=-\infty}^0 \dot{\Upsilon}(\mathfrak{s}(y_1)\tilde{\vartheta}_2(y)z)\mathfrak{G}^{(-1)}\left(-\frac{|\tilde{\vartheta}_1(y)|}{\varepsilon} + z\right) dz \\ &\quad + \varepsilon\mathfrak{s}(y_1)\tilde{\vartheta}_2(y)\nabla\tilde{\vartheta}_2(y)\int_{z=-\infty}^0 \ddot{\Upsilon}(\mathfrak{s}(y_1)\tilde{\vartheta}_2(y)z)z\mathfrak{G}^{(-1)}\left(-\frac{|\tilde{\vartheta}_1(y)|}{\varepsilon} + z\right) dz \\ &\quad - \mathfrak{s}(y_1)\tilde{\vartheta}_2(y)\nabla\tilde{\vartheta}_1(y)\int_{z=-\infty}^0 \dot{\Upsilon}(\mathfrak{s}(y_1)\tilde{\vartheta}_2(y)z)\mathfrak{G}\left(-\frac{|\tilde{\vartheta}_1(y)|}{\varepsilon} + z\right) dz. \end{aligned}$$

Making the change of variables $w = -|\tilde{\vartheta}_1(y)|/\varepsilon + z$, we get

$$\begin{aligned} \nabla\hat{J}(y) &= \Upsilon(0)\mathfrak{G}\left(-\frac{|\tilde{\vartheta}_1(y)|}{\varepsilon}\right)\nabla\tilde{\vartheta}_1(y) \\ &\quad + \varepsilon\nabla\tilde{\vartheta}_2(y)\int_{w=-\infty}^{-\frac{|\tilde{\vartheta}_1(y)|}{\varepsilon}} \dot{\Upsilon}\left(\frac{\tilde{\mathfrak{H}}(y)}{\varepsilon} + \mathfrak{s}(y_1)\tilde{\vartheta}_2(y)w\right)\mathfrak{G}^{(-1)}(w)dw \\ &\quad + \varepsilon\mathfrak{s}(y_1)\tilde{\vartheta}_2(y)\nabla\tilde{\vartheta}_2(y)\int_{w=-\infty}^{-\frac{|\tilde{\vartheta}_1(y)|}{\varepsilon}} \ddot{\Upsilon}\left(\frac{\tilde{\mathfrak{H}}(y)}{\varepsilon} + \mathfrak{s}(y_1)\tilde{\vartheta}_2(y)w\right)w\mathfrak{G}^{(-1)}(w)dw \\ &\quad + \tilde{\mathfrak{H}}(y)\nabla\tilde{\vartheta}_2(y)\int_{w=-\infty}^{-\frac{|\tilde{\vartheta}_1(y)|}{\varepsilon}} \ddot{\Upsilon}\left(\frac{\tilde{\mathfrak{H}}(y)}{\varepsilon} + \mathfrak{s}(y_1)\tilde{\vartheta}_2(y)w\right)\mathfrak{G}^{(-1)}(w)dw \\ &\quad - \mathfrak{s}(y_1)\tilde{\vartheta}_2(y)\nabla\tilde{\vartheta}_1(y)\int_{w=-\infty}^{-\frac{|\tilde{\vartheta}_1(y)|}{\varepsilon}} \dot{\Upsilon}\left(\frac{\tilde{\mathfrak{H}}(y)}{\varepsilon} + \mathfrak{s}(y_1)\tilde{\vartheta}_2(y)w\right)\mathfrak{G}(w)dw. \end{aligned}$$

Now,

$$\begin{aligned} \|\nabla \hat{J}(y)\| &\leq K e^{-\frac{|\bar{\vartheta}_1(y)|^2}{2\varepsilon^2}} + K\varepsilon e^{-\frac{|\bar{\vartheta}_1(y)|^2}{2\varepsilon^2}} + K\varepsilon \|y\|_\infty e^{-\frac{|\bar{\vartheta}_1(y)|^2}{4\varepsilon^2}} + K \|y\|_\infty^2 e^{-\frac{|\bar{\vartheta}_1(y)|^2}{2\varepsilon^2}} + K \|y\|_\infty e^{-\frac{|\bar{\vartheta}_1(y)|^2}{2\varepsilon^2}} \\ &\leq K e^{-\frac{\|y\|_\infty^2}{4\varepsilon^2}}. \end{aligned}$$

Hence,

$$\|\nabla \Upsilon_{2,s}^\varepsilon(y)\| \leq K \exp \left[-\frac{\|y\|_\infty^2}{K\varepsilon^2} \right]$$

for suitable $K > 0$.

Combining things, we get the stated result. \square

Lemma 3.12. *There is a constant $K > 0$ such that for $\delta \in (0, \bar{\delta})$, $\varepsilon \in (0, 1)$, $y \in (\mathcal{C}_U \cup \mathcal{C}_D) \cap (\tilde{\mathcal{B}}^\varepsilon(2\delta) \setminus \tilde{\mathcal{B}}^\varepsilon(\delta))$, we have*

$$\begin{aligned} |\tilde{\Psi}_B^\varepsilon(y) - \tilde{\Psi}_A^\varepsilon(y)| &\leq \frac{K\varepsilon^2}{\sqrt{\delta}}, \\ \|\nabla(\tilde{\Psi}_B^\varepsilon - \tilde{\Psi}_A^\varepsilon)(y)\| &\leq \frac{K\varepsilon \|y\|_\infty}{\delta} \quad \text{for } y_1 \neq 0. \end{aligned}$$

Proof. We will prove the bounds for the case $y \in \mathcal{C}_U$. The calculations for the case $y \in \mathcal{C}_D$ are similar. Note that for $y = (y_1, y_2) \in \mathcal{C}_U \cup \mathcal{C}_D$, $|\tilde{H}(y)| \geq y_1^2$. For convenience, define $\tilde{\Theta}^C(y_1, y_2) \stackrel{\text{def}}{=} \tilde{\Theta}(0, y_2)$ for all $(y_1, y_2) \in \square$. Then

$$\tilde{\Psi}_B^\varepsilon(y) - \tilde{\Psi}_A^\varepsilon(y) = \varepsilon \left\{ u_U \left(-\tilde{\Theta}^C(y), \frac{\tilde{H}(y)}{\varepsilon} \right) - u_U \left(-\tilde{\Theta}(y), \frac{\tilde{H}(y)}{\varepsilon} \right) \right\}.$$

It is easily seen from the explicit formulas for $\tilde{\Theta}$ and $\tilde{\Theta}^C$ that $-\tilde{\Theta}^C(y) \geq -\tilde{\Theta}(y) > 0$ for $y \in \mathcal{C}_U \cup \mathcal{C}_D$. For $y \in \mathcal{C}_U$ such that $y_1 \neq 0$, we have

$$\tilde{\Psi}_B^\varepsilon(y) - \tilde{\Psi}_A^\varepsilon(y) = \varepsilon \int_{t=-\tilde{\Theta}(y)}^{-\tilde{\Theta}^C(y)} \frac{\partial u_U}{\partial t} \left(t, \frac{\tilde{H}(y)}{\varepsilon} \right) dt.$$

Recalling Proposition 3.3 and (22) and noting that the right-hand side of (22) is decreasing in $|\theta|$, we get that

$$\begin{aligned} |\tilde{\Psi}_B^\varepsilon(y) - \tilde{\Psi}_A^\varepsilon(y)| &\leq \frac{K\varepsilon}{|\tilde{\Theta}(y)| + \left| \frac{\tilde{H}(y)}{\varepsilon} \right|^2} \exp \left[-\sqrt{\frac{\pi}{G_O}} \left| \frac{\tilde{H}(y)}{\varepsilon} \right| \right] |\tilde{\Theta}(y) - \tilde{\Theta}^C(y)| \\ &\leq \frac{K\varepsilon y_1^2}{\tilde{\mathcal{D}}^\varepsilon(y)} \\ &\leq \frac{K\varepsilon}{[\tilde{\mathcal{D}}^\varepsilon(y)]^{1/2}} \left| \frac{\tilde{H}(y)}{[\tilde{\mathcal{D}}^\varepsilon(y)]^{1/2}} \right| \\ &\leq \frac{K\varepsilon^2}{\sqrt{\delta}}. \end{aligned}$$

By continuity, the bound extends to the case $y_1 = 0$.

Differentiating, we have

$$\nabla(\tilde{\Psi}_B^\varepsilon - \tilde{\Psi}_A^\varepsilon)(y) = J_1^\varepsilon(y) + J_2^\varepsilon(y) + J_3^\varepsilon(y)$$

where

$$\begin{aligned} J_1^\varepsilon(y) &\stackrel{\text{def}}{=} -\varepsilon \frac{\partial u_{\mathbb{U}}}{\partial t} \left(-\tilde{\Theta}^C(y), \frac{\tilde{H}(y)}{\varepsilon} \right) \nabla \tilde{\Theta}^C(y), \\ J_2^\varepsilon(y) &\stackrel{\text{def}}{=} \varepsilon \frac{\partial u_{\mathbb{U}}}{\partial t} \left(-\tilde{\Theta}(y), \frac{\tilde{H}(y)}{\varepsilon} \right) \nabla \tilde{\Theta}(y), \\ J_3^\varepsilon(y) &\stackrel{\text{def}}{=} \left\{ \frac{\partial u_{\mathbb{U}}}{\partial h} \left(-\tilde{\Theta}^C(y), \frac{\tilde{H}(y)}{\varepsilon} \right) - \frac{\partial u_{\mathbb{U}}}{\partial h} \left(-\tilde{\Theta}(y), \frac{\tilde{H}(y)}{\varepsilon} \right) \right\} \nabla \tilde{H}(y). \end{aligned}$$

Using Proposition 3.3 and (22), we get

$$\begin{aligned} \|J_1^\varepsilon(y)\| &\leq \frac{K\varepsilon\|y\|_\infty}{|\tilde{\Theta}^C(y)| + \left| \frac{\tilde{H}(y)}{\varepsilon} \right|^2} \exp \left[-\sqrt{\frac{\pi}{G_O}} \left| \frac{\tilde{H}(y)}{\varepsilon} \right| \right] \\ &\leq \frac{K\varepsilon\|y\|_\infty}{|\tilde{\Theta}(y)| + \left| \frac{\tilde{H}(y)}{\varepsilon} \right|^2} \exp \left[-\sqrt{\frac{\pi}{G_O}} \left| \frac{\tilde{H}(y)}{\varepsilon} \right| \right] \\ &\leq \frac{K\varepsilon\|y\|_\infty}{\tilde{\mathfrak{D}}^\varepsilon(y)} \exp \left[-\sqrt{\frac{\pi}{G_O}} \left| \frac{\tilde{H}(y)}{\varepsilon} \right| \right] \\ &\leq \frac{K\varepsilon\|y\|_\infty}{\delta}. \end{aligned}$$

Similarly, one can show that

$$\|J_2^\varepsilon(y)\| \leq \frac{K\varepsilon\|y\|_\infty}{\delta}.$$

Now J_3^ε can be written as

$$J_3^\varepsilon(y) = \left(\int_{t=-\tilde{\Theta}(y)}^{-\tilde{\Theta}^C(y)} \frac{\partial^2 u_{\mathbb{U}}}{\partial t \partial h} \left(t, \frac{\tilde{H}(y)}{\varepsilon} \right) dt \right) \nabla \tilde{H}(y)$$

Using Proposition 3.3, (22) and reasoning as before, we get

$$\begin{aligned} \|J_3^\varepsilon(y)\| &\leq \frac{K}{\left(|\tilde{\Theta}(y)| + \left| \frac{\tilde{H}(y)}{\varepsilon} \right|^2 \right)^{3/2}} \exp \left[-\sqrt{\frac{\pi}{G_O}} \left| \frac{\tilde{H}(y)}{\varepsilon} \right| \right] |\tilde{\Theta}(y) - \tilde{\Theta}^C(y)| \|\nabla \tilde{H}(y)\| \\ &\leq \frac{Ky_1^2}{[\tilde{\mathfrak{D}}^\varepsilon(y)]^{3/2}} \|y\|_\infty \\ &\leq \frac{K\|y\|_\infty}{\tilde{\mathfrak{D}}^\varepsilon(y)} \left| \frac{\tilde{H}(y)}{[\tilde{\mathfrak{D}}^\varepsilon(y)]^{1/2}} \right| \\ &\leq \frac{K\varepsilon\|y\|_\infty}{\delta}. \end{aligned}$$

Combining things, we get the stated result. \square

Lemma 3.13. *The function $\tilde{\mathcal{L}}^\varepsilon \tilde{\Psi}_B^\varepsilon$ has a continuous extension to \square .*

Proof. Let's start by marshaling a few facts. First, for $y \in \square \setminus \tilde{X}$, $(\tilde{\mathcal{L}}^\varepsilon \tilde{F})(y) = 0$. Hence, $(\tilde{\mathcal{L}}^\varepsilon \tilde{\Psi}_B^\varepsilon)(y) = \tilde{\mathcal{L}}^\varepsilon [\tilde{\Psi}_B^\varepsilon + \tilde{F}](y)$ can be obtained by applying the operator $\tilde{\mathcal{L}}^\varepsilon$ to the integral representations for $\tilde{\Psi}_B^\varepsilon + \tilde{F}$ in

the statement of Lemma 3.7. Next, for $y = (y_1, y_2) \in \square$, we have

$$(48) \quad \begin{aligned} (\nabla^\perp \tilde{H}, \nabla \tilde{\vartheta}_1)(y) &= \tilde{\vartheta}_1(y) = \sqrt{2}y_1, & (\nabla^\perp \tilde{H}, \nabla \tilde{\vartheta}_2)(y) &= -\tilde{\vartheta}_2(y) = -\frac{y_2}{\sqrt{2}}, \\ \|\nabla \tilde{\vartheta}_1(y)\| &= \sqrt{2}, & \|\nabla \tilde{\vartheta}_2(y)\| &= \frac{1}{\sqrt{2}}. \end{aligned}$$

Finally, note that $\ddot{u}_R^\circ(0+) = \ddot{u}_R^\circ(0-)$ and that $\ddot{u}_L^\circ(0+) = \ddot{u}_L^\circ(0-)$; we denote these quantities by $\ddot{u}_R^\circ(0)$ and $\ddot{u}_L^\circ(0)$ respectively.

Direct computation yields, for $y_2 > 0$,

$$\begin{aligned} \lim_{y_1 \searrow 0} (\tilde{\mathcal{L}}^\varepsilon \tilde{\Psi}_B^\varepsilon)(y) &= \lim_{y_1 \nearrow 0} (\tilde{\mathcal{L}}^\varepsilon \tilde{\Psi}_B^\varepsilon)(y) \\ &= 2 \frac{\beta_O}{\beta_O + \beta_L} \frac{1}{\varepsilon} \int_{z=0}^{\infty} \mathfrak{G}''(z) u_R^\circ(\tilde{\vartheta}_2(y)z) dz + 2 \frac{\beta_L}{\beta_O + \beta_L} \frac{1}{\varepsilon} \int_{z=-\infty}^0 \mathfrak{G}''(z) u_L^\circ(\tilde{\vartheta}_2(y)z) dz \\ &\quad - 2 \frac{\beta_O}{\beta_O + \beta_L} \frac{\tilde{\vartheta}_2(y)}{\varepsilon} \int_{z=0}^{\infty} z \mathfrak{G}(z) \ddot{u}_R^\circ(\tilde{\vartheta}_2(y)z) dz - 2 \frac{\beta_L}{\beta_O + \beta_L} \frac{\tilde{\vartheta}_2(y)}{\varepsilon} \int_{z=-\infty}^0 z \mathfrak{G}(z) \ddot{u}_L^\circ(\tilde{\vartheta}_2(y)z) dz \\ &\quad + 2 \frac{\beta_O}{\beta_O + \beta_L} \frac{\varepsilon}{4} \int_{z=0}^{\infty} z^2 \mathfrak{G}(z) \ddot{u}_R^\circ(\tilde{\vartheta}_2(y)z) dz + 2 \frac{\beta_L}{\beta_O + \beta_L} \frac{\varepsilon}{4} \int_{z=-\infty}^0 z^2 \mathfrak{G}(z) \ddot{u}_L^\circ(\tilde{\vartheta}_2(y)z) dz. \end{aligned}$$

This establishes continuity along $\{(y_1, y_2) \in \square : y_1 = 0, y_2 > 0\}$, i.e. the positive y_2 -axis.

Let's now establish continuity along $\{(y_1, y_2) \in \square : y_2 = 0, y_1 > 0\}$, i.e. the positive y_1 -axis. Careful computation yields, for $y_1 > 0$,

$$(49) \quad \begin{aligned} \lim_{y_2 \searrow 0} (\tilde{\mathcal{L}}^\varepsilon \tilde{\Psi}_B^\varepsilon)(y) &= \frac{\tilde{\vartheta}_1(y)}{\varepsilon^2} u_R^\circ(0) \int_{z=0}^{\infty} \mathfrak{G}'\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z\right) dz + \frac{1}{\varepsilon} u_R^\circ(0) \int_{z=0}^{\infty} \mathfrak{G}''\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z\right) dz \\ &\quad + \frac{\varepsilon}{4} \ddot{u}_R^\circ(0) \int_{z=0}^{\infty} \mathfrak{G}\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z\right) z^2 dz + \frac{\beta_O - \beta_L}{\beta_O + \beta_L} \frac{\tilde{\vartheta}_1(y)}{\varepsilon^2} u_R^\circ(0) \int_{z=0}^{\infty} \mathfrak{G}'\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} + z\right) dz \\ &\quad + \frac{\beta_O - \beta_L}{\beta_O + \beta_L} \frac{1}{\varepsilon} u_R^\circ(0) \int_{z=0}^{\infty} \mathfrak{G}''\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} + z\right) dz + \frac{\beta_O - \beta_L}{\beta_O + \beta_L} \frac{\varepsilon}{4} \ddot{u}_R^\circ(0) \int_{z=0}^{\infty} \mathfrak{G}\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} + z\right) z^2 dz \\ &\quad + 2 \frac{\beta_L}{\beta_O + \beta_L} \frac{\tilde{\vartheta}_1(y)}{\varepsilon^2} u_L^\circ(0) \int_{z=-\infty}^0 \mathfrak{G}'\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z\right) dz \\ &\quad + 2 \frac{\beta_L}{\beta_O + \beta_L} \frac{1}{\varepsilon} u_L^\circ(0) \int_{z=-\infty}^0 \mathfrak{G}''\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z\right) dz \\ &\quad + 2 \frac{\beta_L}{\beta_O + \beta_L} \frac{\varepsilon}{4} \ddot{u}_L^\circ(0) \int_{z=-\infty}^0 \mathfrak{G}\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z\right) z^2 dz \end{aligned}$$

and
(50)

$$\begin{aligned}
\lim_{y_2 \nearrow 0} (\mathcal{L}^\varepsilon \tilde{\Psi}_B^\varepsilon)(y) &= \frac{\tilde{\vartheta}_1(y)}{\varepsilon^2} u_R^\circ(0) \int_{z=0}^\infty \mathfrak{G}' \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) dz + \frac{1}{\varepsilon} u_R^\circ(0) \int_{z=0}^\infty \mathfrak{G}'' \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) dz \\
&+ \frac{\varepsilon}{4} \ddot{u}_R^\circ(0) \int_{z=0}^\infty \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) z^2 dz + \frac{\beta_R - \beta_O}{\beta_O + \beta_R} \frac{\tilde{\vartheta}_1(y)}{\varepsilon^2} u_R^\circ(0) \int_{z=0}^\infty \mathfrak{G}' \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} + z \right) dz \\
&+ \frac{\beta_R - \beta_O}{\beta_O + \beta_R} \frac{1}{\varepsilon} u_R^\circ(0) \int_{z=0}^\infty \mathfrak{G}'' \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} + z \right) dz + \frac{\beta_R - \beta_O}{\beta_O + \beta_R} \frac{\varepsilon}{4} \ddot{u}_R^\circ(0) \int_{z=0}^\infty \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} + z \right) z^2 dz \\
&+ 2 \frac{\beta_O}{\beta_O + \beta_R} \frac{\tilde{\vartheta}_1(y)}{\varepsilon^2} u_L^\circ(0) \int_{z=-\infty}^0 \mathfrak{G}' \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) dz \\
&+ 2 \frac{\beta_O}{\beta_O + \beta_R} \frac{1}{\varepsilon} u_L^\circ(0) \int_{z=-\infty}^0 \mathfrak{G}'' \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) dz \\
&+ 2 \frac{\beta_O}{\beta_O + \beta_R} \frac{\varepsilon}{4} \ddot{u}_L^\circ(0) \int_{z=-\infty}^0 \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) z^2 dz.
\end{aligned}$$

For equality of the limits in (49) and (50), it suffices to have

$$\frac{\beta_O - \beta_L}{\beta_O + \beta_L} = \frac{\beta_R - \beta_O}{\beta_O + \beta_R} \quad \text{and} \quad \frac{\beta_L}{\beta_O + \beta_L} = \frac{\beta_O}{\beta_O + \beta_R}.$$

Both these equations reduce to

$$\beta_O^2 = \beta_L \beta_R,$$

which is precisely (4). This proves continuity along the positive y_1 -axis.

Continuity along the negative y_1 and y_2 axes is established by similar calculations. Finally, it is easily checked that $\lim_{(y_1, y_2) \rightarrow (0, 0)} (\mathcal{L}^\varepsilon \tilde{\Psi}_B^\varepsilon)(y)$ exists and is given by

$$\begin{aligned}
\lim_{(y_1, y_2) \rightarrow (0, 0)} (\mathcal{L}^\varepsilon \tilde{\Psi}_B^\varepsilon)(y) &= \left[2 \frac{\beta_O}{\beta_O + \beta_L} u_R^\circ(0) + 2 \frac{\beta_L}{\beta_O + \beta_L} u_L^\circ(0) \right] \frac{1}{\varepsilon} \int_{z=0}^\infty \mathfrak{G}''(z) dz \\
&+ \left[2 \frac{\beta_O}{\beta_O + \beta_L} \ddot{u}_R^\circ(0) + 2 \frac{\beta_L}{\beta_O + \beta_L} \ddot{u}_L^\circ(0) \right] \frac{\varepsilon}{4} \int_{z=0}^\infty z^2 \mathfrak{G}(z) dz;
\end{aligned}$$

this proves continuity at $(0, 0)$. \square

Lemma 3.14. *There is a constant $K > 0$ such that*

$$|\mathcal{L}^\varepsilon \tilde{\Psi}_B^\varepsilon(y)| \leq K \left(\varepsilon + \frac{\|y\|_\infty^2}{\varepsilon} \right)$$

for $y \in \square \setminus \tilde{\mathbf{X}}$, $\varepsilon \in (0, 1)$.

Proof. Suppose \mathfrak{l} is either \mathbb{R}^+ or \mathbb{R}^- and φ is a real-valued function defined on \mathfrak{l} such that $\dot{\varphi}$, $\ddot{\varphi}$ are bounded. For $\varepsilon \in (0, 1)$, $y \in \square \setminus \tilde{\mathbf{X}}$, define

$$\begin{aligned}
\Upsilon^\varepsilon(y) &\stackrel{\text{def}}{=} \varepsilon \int_{z \in \mathfrak{l}} \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) \varphi(\tilde{\vartheta}_2(y)z) dz, \\
\Gamma^\varepsilon(y) &\stackrel{\text{def}}{=} \varepsilon \int_{z \in \mathfrak{l}} \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} + z \right) \varphi(\tilde{\vartheta}_2(y)z) dz.
\end{aligned}$$

Recalling Lemma 3.7 and the comments at the beginning of the proof of Lemma 3.13, it suffices to understand how to bound $|\mathcal{L}^\varepsilon \Upsilon^\varepsilon(y)|$ and $|\mathcal{L}^\varepsilon \Gamma^\varepsilon(y)|$.

We have

$$\tilde{\mathcal{L}}^\varepsilon \Upsilon^\varepsilon(y) = \sum_{i=1}^4 \mathsf{K}_i^\varepsilon(y)$$

where

$$\begin{aligned} \mathsf{K}_1^\varepsilon(y) &\stackrel{\text{def}}{=} \frac{\tilde{\vartheta}_1(y)}{\varepsilon^2} \int_{z \in \mathbb{I}} \mathfrak{G}' \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) \varphi(\tilde{\vartheta}_2(y)z) dz, \\ \mathsf{K}_2^\varepsilon(y) &\stackrel{\text{def}}{=} \frac{1}{\varepsilon} \int_{z \in \mathbb{I}} \mathfrak{G}'' \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) \varphi(\tilde{\vartheta}_2(y)z) dz, \\ \mathsf{K}_3^\varepsilon(y) &\stackrel{\text{def}}{=} -\frac{\tilde{\vartheta}_2(y)}{\varepsilon} \int_{z \in \mathbb{I}} z \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) \dot{\varphi}(\tilde{\vartheta}_2(y)z) dz, \\ \mathsf{K}_4^\varepsilon(y) &\stackrel{\text{def}}{=} \frac{\varepsilon}{4} \int_{z \in \mathbb{I}} \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) z^2 \ddot{\varphi}(\tilde{\vartheta}_2(y)z) dz. \end{aligned}$$

Let $\mathfrak{s}(\mathbb{I}) = 1$ if $\mathbb{I} = \mathbb{R}^+$ and $\mathfrak{s}(\mathbb{I}) = -1$ if $\mathbb{I} = \mathbb{R}^-$. We then have the following integration by parts formula: for $i \in \{1, 2\}$,

$$\begin{aligned} \int_{z \in \mathbb{I}} \mathfrak{G}^{(i)} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) \varphi(\tilde{\vartheta}_2(y)z) dz &= \mathfrak{s}(\mathbb{I}) \varphi(0) \mathfrak{G}^{(i-1)} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} \right) \\ &\quad + \tilde{\vartheta}_2(y) \int_{z \in \mathbb{I}} \mathfrak{G}^{(i-1)} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) \dot{\varphi}(\tilde{\vartheta}_2(y)z) dz. \end{aligned}$$

We will repeatedly use the fact that $\mathfrak{G}'(z) = -z\mathfrak{G}(z)$ for all $z \in \mathbb{R}$. Integrating by parts, we have

$$\mathsf{K}_1^\varepsilon(y) = \mathfrak{s}(\mathbb{I}) \frac{\tilde{\vartheta}_1(y)}{\varepsilon^2} \varphi(0) \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} \right) + \frac{\tilde{\mathsf{H}}(y)}{\varepsilon^2} \int_{z \in \mathbb{I}} \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) \dot{\varphi}(\tilde{\vartheta}_2(y)z) dz$$

and

$$\begin{aligned} \mathsf{K}_2^\varepsilon(y) &= \mathfrak{s}(\mathbb{I}) \frac{1}{\varepsilon} \varphi(0) \mathfrak{G}' \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} \right) + \frac{\tilde{\vartheta}_2(y)}{\varepsilon} \int_{z \in \mathbb{I}} \mathfrak{G}' \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) \dot{\varphi}(\tilde{\vartheta}_2(y)z) dz \\ &= -\mathfrak{s}(\mathbb{I}) \frac{\tilde{\vartheta}_1(y)}{\varepsilon^2} \varphi(0) \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} \right) + \frac{\tilde{\vartheta}_2(y)}{\varepsilon} \int_{z \in \mathbb{I}} z \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) \dot{\varphi}(\tilde{\vartheta}_2(y)z) dz \\ &\quad - \frac{\tilde{\mathsf{H}}(y)}{\varepsilon^2} \int_{z \in \mathbb{I}} \mathfrak{G} \left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z \right) \dot{\varphi}(\tilde{\vartheta}_2(y)z) dz. \end{aligned}$$

Hence,

$$\sum_{i=1}^3 \mathsf{K}_i^\varepsilon(y) = 0$$

and $\tilde{\mathcal{L}}^\varepsilon \Upsilon^\varepsilon(y) = K_4^\varepsilon(y)$. Using the substitution $w = z - \tilde{\vartheta}_1(y)/\varepsilon$ and recalling that $\dot{\varphi}$ is bounded, we see that there are constants \tilde{K} , $K > 0$ such that

$$\begin{aligned} |\tilde{\mathcal{L}}^\varepsilon \Upsilon^\varepsilon(y)| &\leq \tilde{K}\varepsilon \int_{z \in \mathbb{R}} \mathfrak{G}\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} - z\right) z^2 dz \\ &\leq \tilde{K}\varepsilon \int_{w \in \mathbb{R}} \mathfrak{G}(w) \left(w + \frac{\tilde{\vartheta}_1(y)}{\varepsilon}\right)^2 dw \\ &\leq 4\tilde{K}\varepsilon \int_{w \in \mathbb{R}} \mathfrak{G}(w) \left[w^2 + \left|\frac{\tilde{\vartheta}_1(y)}{\varepsilon}\right|^2\right] dw \\ &\leq 4\tilde{K}\varepsilon + 4\tilde{K}\varepsilon \left|\frac{\tilde{\vartheta}_1(y)}{\varepsilon}\right|^2 \\ &\leq K\varepsilon + K \frac{\|y\|_\infty^2}{\varepsilon}. \end{aligned}$$

Similarly, one can show that there exists $K > 0$ such that

$$|\tilde{\mathcal{L}}^\varepsilon \Gamma^\varepsilon(y)| \leq K\varepsilon + K \frac{\|y\|_\infty^2}{\varepsilon}.$$

The calculations are virtually identical, except that the integration by parts formula takes the form: for $i \in \{1, 2\}$,

$$\begin{aligned} \int_{z \in \mathbb{I}} \mathfrak{G}^{(i)}\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} + z\right) \varphi(\tilde{\vartheta}_2(y)z) dz &= -\mathfrak{s}(\mathbb{I})\varphi(0)\mathfrak{G}^{(i-1)}\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon}\right) \\ &\quad - \tilde{\vartheta}_2(y) \int_{z \in \mathbb{I}} \mathfrak{G}^{(i-1)}\left(\frac{\tilde{\vartheta}_1(y)}{\varepsilon} + z\right) \dot{\varphi}(\tilde{\vartheta}_2(y)z) dz. \end{aligned}$$

□

Lemma 3.15. *There exists a constant $K > 0$ such that*

$$\begin{aligned} \|\nabla \eta^{\delta, \varepsilon}(x)\| &\leq \frac{K}{\varepsilon} \left\{ 1 + \left(\frac{\varepsilon}{\sqrt{\delta}}\right)^2 \right\} \chi_{B^\varepsilon(2\delta)}(x), \\ |\mathcal{L}^\varepsilon \eta^{\delta, \varepsilon}(x)| &\leq \frac{K}{\varepsilon^2} \left\{ 1 + \left(\frac{\varepsilon}{\sqrt{\delta}}\right)^4 \right\} \chi_{B^\varepsilon(2\delta)}(x) \end{aligned}$$

for $x \in \bar{\mathbf{S}}$, $\varepsilon \in (0, 1)$, $\delta \in (0, \bar{\delta})$.

Proof. The proof is similar to that of Lemma 4.9 in [Sow05].

□

Lemma 3.16. *For $\delta \in (0, \bar{\delta})$, $\varepsilon \in (0, 1)$, the function $F + \Psi^{\delta, \varepsilon}$ satisfies the skewness conditions at $\mathbf{H}^{-1}(0) \setminus \{\mathbf{o}\}$.*

Proof. Recalling Lemma 3.6, it suffices to establish the claim for $x \in (\mathbf{H}^{-1}(0) \setminus \{\mathbf{o}\}) \cap \tilde{\varphi}(\square)$; we naturally switch to (y_1, y_2) -coordinates. For $\varepsilon \in (0, 1)$, $\delta \in (0, \bar{\delta})$, define $\tilde{\eta}^{\delta, \varepsilon} : \square \rightarrow \mathbb{R}$ by

$$\tilde{\eta}^{\delta, \varepsilon}(y) \stackrel{\text{def}}{=} \mathfrak{c}_0 \left(\frac{\tilde{\mathcal{D}}^\varepsilon(y)}{\delta} \right) \quad \text{for } y \in \square.$$

Also define

$$\mathfrak{D}^*(y) \stackrel{\text{def}}{=} |\tilde{\Theta}(y)|^2 + \left| \frac{\tilde{H}(y)}{\varepsilon} \right|^4 \quad \text{for } y \in \square,$$

$$\mathfrak{c}^*(z) \stackrel{\text{def}}{=} \mathfrak{c}_0(\sqrt{z}) \quad \text{for } z \in [0, \infty).$$

For $y \in \square$, we have

$$\tilde{\Psi}^{\delta, \varepsilon}(y) = \tilde{\Psi}_A^\varepsilon(y)(1 - \tilde{\eta}^{\delta, \varepsilon}(y)) + \tilde{\Psi}_B^\varepsilon(y)\tilde{\eta}^{\delta, \varepsilon}(y).$$

Differentiating, we get

$$(\nabla \tilde{\Psi}^{\delta, \varepsilon})(y) = -\tilde{\Psi}_A^\varepsilon(y)(\nabla \tilde{\eta}^{\delta, \varepsilon})(y) + (\nabla \tilde{\Psi}_A^\varepsilon)(y)(1 - \tilde{\eta}^{\delta, \varepsilon}(y)) + \tilde{\Psi}_B^\varepsilon(y)(\nabla \tilde{\eta}^{\delta, \varepsilon})(y) + (\nabla \tilde{\Psi}_B^\varepsilon)(y)\tilde{\eta}^{\delta, \varepsilon}(y).$$

Since $(\nabla \tilde{H}, \nabla \tilde{\Theta})(y) \equiv 0$ for $y \in \square$, we have

$$(\nabla \tilde{\eta}^{\delta, \varepsilon}, \nabla \tilde{H})(y) = \frac{1}{\delta^2} \mathfrak{c}^* \left(\frac{\mathfrak{D}^*(y)}{\delta^2} \right) \frac{4}{\varepsilon} \left(\frac{\tilde{H}(y)}{\varepsilon} \right)^3 \|\nabla \tilde{H}(y)\|^2,$$

which implies that

$$(\nabla \tilde{\eta}^{\delta, \varepsilon}, \nabla \tilde{H})(y) = 0 \quad \text{for } y \in \tilde{\mathbf{X}}.$$

Consequently, for $y \in \square$,

$$(\nabla[\tilde{F} + \tilde{\Psi}^{\delta, \varepsilon}], \nabla \tilde{H})(y) = (\nabla[\tilde{F} + \tilde{\Psi}_A^\varepsilon], \nabla \tilde{H})(y)(1 - \tilde{\eta}^{\delta, \varepsilon}(y)) + (\nabla[\tilde{F} + \tilde{\Psi}_B^\varepsilon], \nabla \tilde{H})(y)\tilde{\eta}^{\delta, \varepsilon}(y).$$

The result now easily follows from Lemmas 3.6 and 3.7. \square

Lemma 3.17. *There is a constant $K > 0$ such that for $\delta \in (0, \bar{\delta} \wedge \frac{1}{8})$, $\varepsilon \in (0, 1)$ satisfying $\varepsilon/\sqrt{\delta} \leq (1/2)^{1/4}$, we have*

$$|\Psi^{\delta, \varepsilon}(x)| \leq K\varepsilon \quad \text{for } x \in \tilde{\mathbf{S}}$$

and

$$\begin{aligned} |(\mathcal{L}^\varepsilon \Psi^{\delta, \varepsilon})(x)| &\leq \frac{K}{\sqrt{\delta}} \mathcal{E} \left(\frac{H(x)}{\varepsilon} \right) + K \left(\frac{\sqrt{\delta}}{\varepsilon} + \frac{\varepsilon}{\delta} \right) \mathfrak{c}_0 \left(\frac{H(x)}{\varepsilon\sqrt{2\delta}} \right) \\ &\quad + \frac{K}{\varepsilon} \exp \left[-\frac{\sqrt{\delta}}{K\varepsilon} \right] \mathfrak{c}_0 \left(\frac{H(x)}{\varepsilon\sqrt{2\delta}} \right) \quad \text{for } x \in \tilde{\mathbf{S}} \setminus H^{-1}(0). \end{aligned}$$

Proof. Let's start with the bound on $\Psi^{\delta, \varepsilon}$. Using the exponential decay of the $\frac{\partial \Psi_\ell^K}{\partial h}$'s as $|h| \rightarrow \infty$, it follows that $\Psi_\ell^K(\theta, h)$ is bounded as $|h| \rightarrow \infty$. Now using Proposition 3.3 and Lemma 3.6, it follows that there exists $K > 0$ with

$$|\Psi_A^\varepsilon(x)| \leq K\varepsilon \quad \text{for } x \in \tilde{\mathbf{S}} \setminus \{\mathfrak{o}\}.$$

Recalling the comments following the definition of $\tilde{\Psi}_B^\varepsilon$, together with the fact that $\tilde{\Psi}_B^\varepsilon$ has a continuous extension to \square , it follows that for suitable $K > 0$,

$$|\Psi_B^\varepsilon(x)| \leq K\varepsilon \quad \text{for } x \in \tilde{\phi}(\square).$$

The bound on $\Psi^{\delta, \varepsilon}$ now follows.

To establish the bound on $\mathcal{L}^\varepsilon \Psi^{\delta, \varepsilon}$, we start with a couple of useful estimates. First, if $x \in \mathcal{B}^\varepsilon(2\delta)$, then $\left| \frac{H(x)}{\varepsilon\sqrt{2\delta}} \right| \leq 1$. Hence,

$$\chi_{\mathcal{B}^\varepsilon(2\delta)}(x) \leq \mathfrak{c}_0 \left(\frac{H(x)}{\varepsilon\sqrt{2\delta}} \right) \quad \text{for } x \in \tilde{\mathbf{S}}.$$

Next,

$$\begin{aligned}
\mathfrak{c}_0 \left(\frac{\mathbf{H}(x)}{\varepsilon\sqrt{2\delta}} \right) &\leq \chi_{[-1,1]} \left(\frac{\mathbf{H}(x)}{2\varepsilon\sqrt{2\delta}} \right) \\
&\leq \exp \left[\frac{1}{2} \sqrt{\frac{\pi}{\mathbf{G}_O}} \left\{ 1 - \left| \frac{\mathbf{H}(x)}{2\varepsilon\sqrt{2\delta}} \right| \right\} \right] \\
&\leq \exp \left[\frac{1}{2} \sqrt{\frac{\pi}{\mathbf{G}_O}} \right] \exp \left[-\frac{1}{2} \sqrt{\frac{\pi}{\mathbf{G}_O}} \left| \frac{\mathbf{H}(x)}{2\varepsilon\sqrt{2\delta}} \right| \right] \\
&\leq K\mathcal{E} \left(\frac{\mathbf{H}(x)}{2\varepsilon\sqrt{2\delta}} \right)
\end{aligned}$$

for suitable $K > 0$. If, in addition, $\delta \leq 1/8$, then $2\varepsilon\sqrt{2\delta} \leq \varepsilon$; since $\mathcal{E}(z)$ is even and decreasing on $[0, \infty)$, it now follows that

$$\mathcal{E} \left(\frac{\mathbf{H}(x)}{2\varepsilon\sqrt{2\delta}} \right) \leq \mathcal{E} \left(\frac{\mathbf{H}(x)}{\varepsilon} \right) \quad \text{for } x \in \bar{\mathbf{S}}.$$

Finally, we derive an upper bound for $\exp \left[-\frac{\|y\|_\infty^2}{K\varepsilon^2} \right]$ when $y \in \tilde{\mathcal{B}}^\varepsilon(2\delta) \setminus \tilde{\mathcal{B}}_0^\varepsilon(\delta)$. If $\tilde{\mathcal{D}}^\varepsilon(y) \geq \delta$, then we must have

$$|\tilde{\Theta}(y)|^2 \geq \frac{\delta^2}{2} \quad \text{or} \quad \left| \frac{\tilde{\mathbf{H}}(y)}{\varepsilon} \right|^4 \geq \frac{\delta^2}{2}.$$

Easy manipulations involving the relations

$$|\tilde{\Theta}(y)| \leq \|y\|^2, \quad |\tilde{\mathbf{H}}(y)| \leq \|y\|^2 \quad \text{and} \quad \frac{1}{4}\|y\| \leq \|y\|_\infty \leq 4\|y\|$$

now imply that either

$$\|y\|_\infty \geq \frac{\sqrt{\delta}}{4(2^{1/4})} \quad \text{or} \quad \|y\|_\infty \geq \frac{\sqrt{\varepsilon}\delta^{1/4}}{4(2)^{1/8}}.$$

If $\varepsilon/\sqrt{\delta} \leq (1/2)^{1/4}$, then the first inequality implies the second. Hence, for $y \in \tilde{\mathcal{B}}^\varepsilon(2\delta) \setminus \tilde{\mathcal{B}}_0^\varepsilon(\delta)$, we must have

$$\|y\|_\infty \geq \frac{\sqrt{\varepsilon}\delta^{1/4}}{4(2)^{1/8}}$$

which implies

$$\exp \left[-\frac{\|y\|_\infty^2}{K\varepsilon^2} \right] \leq \exp \left[-\frac{\sqrt{\delta}}{\tilde{K}\varepsilon} \right]$$

for suitable $\tilde{K} > 0$.

Recall the notation of Lemma 3.9. We will repeatedly use the assumption that $\varepsilon/\sqrt{\delta} \leq (1/2)^{1/4}$. Using Lemma 3.10, we get

$$|I_1^{\delta,\varepsilon}(x)| \leq \frac{K}{\sqrt{\delta}} \mathcal{E} \left(\frac{\mathbf{H}(x)}{\varepsilon} \right) \quad \text{for } x \in \bar{\mathbf{S}} \setminus \mathbf{H}^{-1}(0).$$

Lemmas 3.14 and 3.8 imply that

$$|(\mathcal{L}^\varepsilon \Psi_B^\varepsilon)(x)| \leq K \left(\varepsilon + \frac{\delta}{\varepsilon} \right) \quad \text{for } x \in \tilde{\phi}(\square) \setminus \mathbf{H}^{-1}(0).$$

Since $\eta^{\delta,\varepsilon}(x) \leq \chi_{\mathcal{B}^\varepsilon(2\delta)}(x)$ for $x \in \bar{\mathbf{S}}$, we have

$$|I_2^{\delta,\varepsilon}(x)| \leq K \left(\varepsilon + \frac{\delta}{\varepsilon} \right) \mathfrak{c}_0 \left(\frac{\mathbf{H}(x)}{\varepsilon\sqrt{2\delta}} \right) \quad \text{for } x \in \bar{\mathbf{S}} \setminus \mathbf{H}^{-1}(0).$$

Combining Lemmas 3.11 and 3.12, and using Lemma 3.8 to simplify, we get

$$\begin{aligned} |\Psi_B^\varepsilon(x) - \Psi_A^\varepsilon(x)| &\leq K \left(\frac{\varepsilon^2}{\sqrt{\delta}} + \varepsilon\sqrt{\delta} \right) + K\varepsilon \exp \left[-\frac{\sqrt{\delta}}{K\varepsilon} \right], \\ \|\nabla(\Psi_B^\varepsilon - \Psi_A^\varepsilon)(x)\| &\leq K \left(\frac{\varepsilon^2}{\delta} + \frac{\varepsilon}{\sqrt{\delta}} + \varepsilon + \delta \right) + K \exp \left[-\frac{\sqrt{\delta}}{K\varepsilon} \right] \end{aligned}$$

for $x \in \mathcal{B}^\varepsilon(2\delta) \setminus (\mathcal{B}_0^\varepsilon(\delta) \cup \mathbf{H}^{-1}(0))$. Using Lemma 3.15, we easily get

$$\begin{aligned} |I_3^{\delta,\varepsilon}(x)| &\leq K \left(\frac{1}{\sqrt{\delta}} + \frac{\sqrt{\delta}}{\varepsilon} \right) \mathfrak{c}_0 \left(\frac{\mathbf{H}(x)}{\varepsilon\sqrt{2\delta}} \right) + \frac{K}{\varepsilon} \exp \left[-\frac{\sqrt{\delta}}{K\varepsilon} \right] \mathfrak{c}_0 \left(\frac{\mathbf{H}(x)}{\varepsilon\sqrt{2\delta}} \right), \\ |I_4^{\delta,\varepsilon}(x)| &\leq K \left(\frac{1}{\sqrt{\delta}} + \frac{\varepsilon}{\delta} + \frac{\delta}{\varepsilon} \right) \mathfrak{c}_0 \left(\frac{\mathbf{H}(x)}{\varepsilon\sqrt{2\delta}} \right) + \frac{K}{\varepsilon} \exp \left[-\frac{\sqrt{\delta}}{K\varepsilon} \right] \mathfrak{c}_0 \left(\frac{\mathbf{H}(x)}{\varepsilon\sqrt{2\delta}} \right) \end{aligned}$$

for $x \in \bar{\mathbf{S}} \setminus \mathbf{H}^{-1}(0)$. Putting things together, we get the stated result. \square

It is often useful in averaging calculations to have a bound on $\|\nabla\Psi^{\delta,\varepsilon}\|$.

Lemma 3.18. *There is a constant $K > 0$ such that for $\delta \in (0, \bar{\delta})$, $\varepsilon \in (0, 1)$ satisfying $\varepsilon/\sqrt{\delta} \leq 1$, and $x \in \mathbf{S}$ with $|\mathbf{H}(x)| \geq \varepsilon\sqrt{2\delta}$, we have*

$$\|\nabla\Psi^{\delta,\varepsilon}(x)\| \leq \frac{K}{\sqrt{\delta}} \mathcal{E} \left(\frac{\mathbf{H}(x)}{\varepsilon} \right).$$

Proof. Let δ, ε, x be as required. Since $|\mathbf{H}(x)| \geq \varepsilon\sqrt{2\delta}$, it easily follows that $\Psi^{\delta,\varepsilon}(x) = \Psi_A^\varepsilon(x)$. For $x \in \mathbf{S}_\ell \setminus \mathbf{X}$, $\ell \in \Lambda$, we have

$$(\nabla\Psi_A^\varepsilon)(x) = \varepsilon \frac{\partial\Psi_\ell^K}{\partial\theta} \left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon} \right) \nabla\Theta(x) + \frac{\partial\Psi_\ell^K}{\partial h} \left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon} \right) \nabla\mathbf{H}(x).$$

Using Proposition 3.3 and equation (22), we have

$$\begin{aligned} \left\| \varepsilon \frac{\partial\Psi_\ell^K}{\partial\theta} \left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon} \right) \nabla\Theta(x) \right\| &\leq K\varepsilon\mathcal{S}_2 \left(\mathbf{d}_\ell(\Theta(x)), \frac{\mathbf{H}(x)}{\varepsilon} \right) \\ &\leq \frac{K\varepsilon}{\left(|\mathbf{d}_\ell(\Theta(x))| + \left| \frac{\mathbf{H}(x)}{\varepsilon} \right|^2 \right)} \exp \left[-\sqrt{\frac{\pi}{\mathbf{G}_O}} \left| \frac{\mathbf{H}(x)}{\varepsilon} \right| \right] \\ &\leq \frac{K\varepsilon}{\delta} \mathcal{E} \left(\frac{\mathbf{H}(x)}{\varepsilon} \right) \end{aligned}$$

where the last inequality follows from the fact that $|\mathbf{H}(x)/\varepsilon| \geq \sqrt{2\delta}$. Identical reasoning yields

$$\begin{aligned} \left\| \frac{\partial\Psi_\ell^K}{\partial h} \left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon} \right) \nabla\mathbf{H}(x) \right\| &\leq K\mathcal{S}_1 \left(\mathbf{d}_\ell(\Theta(x)), \frac{\mathbf{H}(x)}{\varepsilon} \right) \\ &\leq \frac{K}{\left(|\mathbf{d}_\ell(\Theta(x))| + \left| \frac{\mathbf{H}(x)}{\varepsilon} \right|^2 \right)^{1/2}} \exp \left[-\sqrt{\frac{\pi}{\mathbf{G}_O}} \left| \frac{\mathbf{H}(x)}{\varepsilon} \right| \right] \\ &\leq \frac{K}{\sqrt{\delta}} \mathcal{E} \left(\frac{\mathbf{H}(x)}{\varepsilon} \right). \end{aligned}$$

Using the fact that $\varepsilon/\sqrt{\delta} \leq 1$, we now get the stated bound. By Lemma 3.6, the bound extends to $x \in \mathbf{X}$ satisfying $|\mathbf{H}(x)| \geq \varepsilon\sqrt{2\delta}$. \square

3.6. Proof of Theorem 3.1.

Proof of Theorem 3.1. For $\varepsilon \in (0, 1)$, set

$$\delta_\varepsilon \stackrel{\text{def}}{=} \varepsilon^{1/2} \quad \text{and let} \quad \bar{\varepsilon}_{3.1} \stackrel{\text{def}}{=} \bar{\delta}^2 \wedge \left(\frac{1}{8}\right)^2.$$

Define

$$\Psi^\varepsilon \stackrel{\text{def}}{=} \Psi^{\delta_\varepsilon, \varepsilon} \quad \text{for } \varepsilon \in (0, \bar{\varepsilon}_{3.1}).$$

It follows from Lemma 3.16 that $F + \Psi^\varepsilon$ satisfies the skewness conditions at $\mathbf{H}^{-1}(0) \setminus \{\mathbf{o}\}$. The bounds on Ψ^ε and $\mathcal{L}^\varepsilon \Psi^\varepsilon$ follow easily from Lemma 3.17. □

4. CORRECTORS (II)

4.1. Introduction. Let's now construct the correctors $\{\Phi^\varepsilon : \varepsilon \in (0, \bar{\varepsilon})\}$. Recall that the Φ^ε 's serve to correct small discontinuities in $\mathcal{L}^\varepsilon(F + \Psi^\varepsilon)$ at $\mathbf{H}^{-1}(0)$. To identify the nature of these discontinuities, we start by defining

$$f_1(\theta, h) \stackrel{\text{def}}{=} \sum_{\ell \in \Lambda} (\mathbf{v}_\ell h + \Psi_\ell^K(\theta, h)) \chi_{\mathbb{1}_\ell \times \mathcal{R}_\ell}(\theta, h)$$

and

$$f_1^\varepsilon(x) \stackrel{\text{def}}{=} \varepsilon f_1\left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon}\right)$$

for $x \in \bar{\mathbf{S}} \setminus (\mathbf{H}^{-1}(0) \cup \mathbf{X})$. Then

$$\begin{aligned} (\mathcal{L}^\varepsilon f_1^\varepsilon)(x) &= \frac{1}{\varepsilon} \left\{ \frac{\partial f_1}{\partial \theta}(\nabla^\perp \mathbf{H}, \nabla \Theta)(x) + \frac{1}{2} \frac{\partial^2 f_1}{\partial h^2} \|\nabla \mathbf{H}\|^2(x) \right\} \\ &\quad + \left\{ \frac{\partial f_1}{\partial h}(\mathcal{L} \mathbf{H})(x) + \frac{\partial^2 f_1}{\partial h \partial \theta}(\nabla \mathbf{H}, \nabla \Theta)(x) \right\} + \varepsilon \left\{ \frac{\partial f_1}{\partial \theta}(\mathcal{L} \Theta)(x) + \frac{1}{2} \frac{\partial^2 f_1}{\partial \theta^2} \|\nabla \Theta\|^2(x) \right\} \end{aligned}$$

where the various derivatives of f_1 are evaluated at $\left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon}\right)$. Note that $(\nabla \mathbf{H}, \nabla \Theta) = 0$ at $\mathbf{H}^{-1}(0) \setminus \{\mathbf{o}\}$. Recalling Proposition 3.3, we see that the problematic term is

$$\frac{\partial f_1}{\partial h}\left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon}\right) (\mathcal{L} \mathbf{H})(x)$$

which has a discontinuity at $\mathbf{H}^{-1}(0) \setminus \{\mathbf{o}\}$ (unless the β_ℓ 's are equal). We would like to replace this term by a smooth approximation. Note that we only need to do this *outside* \mathcal{U} ; $(\mathcal{L} \mathbf{H})(x) = 0$ for $x \in \mathcal{U}$.

4.2. Construction. Let's first understand, in (θ, h) coordinates, the smooth approximation to $\frac{\partial f_1}{\partial h}$ that we intend to use. By the observation at the end of the preceding paragraph, we can restrict attention to compact subintervals of \mathbb{I}_L and \mathbb{I}_R . Define

$$\mathbf{a}_L \stackrel{\text{def}}{=} \frac{\varpi^2}{2}, \quad \mathbf{b}_L \stackrel{\text{def}}{=} \mathbf{G}_L - \frac{\varpi^2}{2}, \quad \mathbf{a}_R \stackrel{\text{def}}{=} \mathbf{G}_L + \frac{\varpi^2}{2}, \quad \mathbf{b}_R \stackrel{\text{def}}{=} \mathbf{G}_O - \frac{\varpi^2}{2}$$

and for $\ell \in \{L, R\}$, set $\mathbb{J}_\ell \stackrel{\text{def}}{=} [\mathbf{a}_\ell, \mathbf{b}_\ell]$. Define $\mathbb{J}_O \stackrel{\text{def}}{=} \mathbb{J}_L \cup \mathbb{J}_R$. Clearly $\mathbb{J}_\ell \subset \subset \mathbb{I}_\ell$, $\ell \in \Lambda$. Set

$$\mathbf{g}(\theta, h) \stackrel{\text{def}}{=} \begin{cases} \frac{\partial f_1}{\partial h}(\theta, h) & (\theta, h) \in \cup_{\ell \in \Lambda} \mathbb{J}_\ell \times \mathcal{R}_\ell \\ 0 & \text{otherwise.} \end{cases}$$

Clearly

$$\mathbf{g}(\theta, h) = \sum_{\ell \in \Lambda} \left(\mathbf{v}_\ell + \frac{\partial \Psi_\ell^K}{\partial h}(\theta, h) \right) \chi_{\mathbb{J}_\ell \times \mathcal{R}_\ell}(\theta, h).$$

For $\nu \in (0, 1)$, define

$$\mathbf{g}^\nu(\theta, h) \stackrel{\text{def}}{=} \int_{h' \in \mathbb{R}} \mathbf{g}(\theta, h') \eta^\nu(h - h') dh'.$$

Lemma 4.1. *Fix $\ell \in \{L, R\}$. Then there is a constant $K > 0$ such that*

$$\begin{aligned} (51) \quad & |\mathbf{g}^\nu(\theta, h) - \mathbf{g}(\theta, h)| \leq K\nu \exp\left[-\frac{|h|}{K}\right] \chi_{\{|h| \geq \nu\}} + K \chi_{\{0 < |h| < \nu\}}, \\ & \left| \frac{\partial \mathbf{g}^\nu}{\partial \theta}(\theta, h) - \frac{\partial \mathbf{g}}{\partial \theta}(\theta, h) \right| \leq K\nu \exp\left[-\frac{|h|}{K}\right] \chi_{\{|h| \geq \nu\}} + K \chi_{\{0 < |h| < \nu\}}, \\ & \left| \frac{\partial^2 \mathbf{g}^\nu}{\partial \theta^2}(\theta, h) - \frac{\partial^2 \mathbf{g}}{\partial \theta^2}(\theta, h) \right| \leq K\nu \exp\left[-\frac{|h|}{K}\right] \chi_{\{|h| \geq \nu\}} + K \chi_{\{0 < |h| < \nu\}} \end{aligned}$$

for $\theta \in \mathbb{J}_\ell$, $h \in \mathbb{R} \setminus \{0\}$.

Proof. We start by noting that since $d_\ell(\theta)$ is bounded away from zero, by Proposition 3.3, there exists $K > 0$ such that

$$\begin{aligned} \left| \frac{\partial \Psi_\ell^K}{\partial h}(\theta, h) \right| &\leq K \exp \left[-\frac{|h|}{K} \right] && \text{for } (\theta, h) \in \mathbf{J}_\ell \times \mathcal{R}_\ell, \\ \left| \frac{\partial \Psi_O^K}{\partial h}(\theta, h) \right| &\leq K \exp \left[-\frac{|h|}{K} \right] && \text{for } (\theta, h) \in \mathbf{J}_\ell \times \mathcal{R}_O. \end{aligned}$$

For $\theta \in \mathbf{J}_\ell$, $h \neq 0$, we have

$$\begin{aligned} \mathbf{g}'(\theta, h) - \mathbf{g}(\theta, h) &= \chi_{\{h \geq \nu\}} \left[\int_{h'=0}^{\infty} \left(\frac{\partial \Psi_O^K}{\partial h}(\theta, h') - \frac{\partial \Psi_O^K}{\partial h}(\theta, h) \right) \eta^\nu(h-h') dh' \right] \\ &+ \chi_{\{0 < h < \nu\}} \left[\int_{h'=h-\nu}^0 \left(\frac{\partial \Psi_\ell^K}{\partial h}(\theta, h') - \frac{\partial \Psi_O^K}{\partial h}(\theta, h) + \mathbf{v}_\ell - \mathbf{v}_O \right) \eta^\nu(h-h') dh' \right. \\ &\quad \left. + \int_{h'=0}^{h+\nu} \left(\frac{\partial \Psi_O^K}{\partial h}(\theta, h') - \frac{\partial \Psi_O^K}{\partial h}(\theta, h) \right) \eta^\nu(h-h') dh' \right] \\ &+ \chi_{\{-\nu < h < 0\}} \left[\int_{h'=h-\nu}^0 \left(\frac{\partial \Psi_\ell^K}{\partial h}(\theta, h') - \frac{\partial \Psi_\ell^K}{\partial h}(\theta, h) \right) \eta^\nu(h-h') dh' \right. \\ &\quad \left. + \int_{h'=0}^{h+\nu} \left(\frac{\partial \Psi_O^K}{\partial h}(\theta, h') - \frac{\partial \Psi_\ell^K}{\partial h}(\theta, h) + \mathbf{v}_O - \mathbf{v}_\ell \right) \eta^\nu(h-h') dh' \right] \\ &+ \chi_{\{h \leq -\nu\}} \left[\int_{h'=-\infty}^0 \left(\frac{\partial \Psi_\ell^K}{\partial h}(\theta, h') - \frac{\partial \Psi_\ell^K}{\partial h}(\theta, h) \right) \eta^\nu(h-h') dh' \right] \end{aligned}$$

Hence, there exists $K > 0$ such that

$$\begin{aligned} |\mathbf{g}'(\theta, h) - \mathbf{g}(\theta, h)| &\leq K\nu \exp \left[-\frac{|h-\nu|}{K} \right] \leq K\nu \exp \left[-\frac{h}{K} \right] \exp \left[\frac{\nu}{K} \right] && \text{for } h \geq \nu \\ |\mathbf{g}'(\theta, h) - \mathbf{g}(\theta, h)| &\leq K && \text{for } 0 < |h| < \nu \\ |\mathbf{g}'(\theta, h) - \mathbf{g}(\theta, h)| &\leq K\nu \exp \left[-\frac{|h+\nu|}{K} \right] \leq K\nu \exp \left[-\frac{|h|}{K} \right] \exp \left[\frac{\nu}{K} \right] && \text{for } h \leq -\nu \end{aligned}$$

where we have used the mean-value theorem to get the first and third inequalities. Putting things together, we now get the first estimate in (51). The calculations for the other two estimates in (51) are very similar (in fact, slightly easier due to the absence of the \mathbf{v}_ℓ 's). \square

Define

$$\check{\mathbf{S}} \stackrel{\text{def}}{=} \{x \in \bar{\mathbf{S}} : \Theta(x) \in \mathbf{J}_O\}$$

and let $T : \check{\mathbf{S}} \rightarrow \mathbf{J}_O \times [-h, h]$ be the bijection given by $T(x) \stackrel{\text{def}}{=} (\Theta(x), \mathbf{H}(x))$. Then, T is smooth with smooth inverse T^{-1} satisfying

$$T^{-1}(\Theta(x), \mathbf{H}(x)) = x$$

for $x \in \check{\mathbf{S}}$. If we now define

$$I(\theta, h) \stackrel{\text{def}}{=} \frac{(\mathcal{L}\mathbf{H})(T^{-1}(\theta, h))}{\|\nabla \mathbf{H}\|^2(T^{-1}(\theta, h))} \quad \text{for } (\theta, h) \in \mathbf{J}_O \times [-h, h],$$

then I is smooth and there is a constant $K > 0$ such that

$$|I(\theta, h)| \leq K, \quad \left| \frac{\partial I}{\partial \theta}(\theta, h) \right| \leq K, \quad \left| \frac{\partial^2 I}{\partial \theta^2}(\theta, h) \right| \leq K$$

for $(\theta, h) \in \mathbf{J}_O \times [-h, h]$. Finally, for $\varepsilon \in (0, 1)$, define

$$I^\varepsilon(\theta, h) \stackrel{\text{def}}{=} I(\theta, \varepsilon h)$$

for $(\theta, h) \in \mathcal{J}_O \times [-h/\varepsilon, h/\varepsilon]$. Note that $|I^\varepsilon(\theta, h)|$, $|\frac{\partial I^\varepsilon}{\partial \theta}(\theta, h)|$, $|\frac{\partial^2 I^\varepsilon}{\partial \theta^2}(\theta, h)|$ are bounded as well (with the bound independent of ε), and

$$I^\varepsilon \left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon} \right) = \frac{(\mathcal{L}\mathbf{H})(x)}{\|\nabla \mathbf{H}\|^2(x)}$$

for $x \in \bar{\mathbf{S}}$.

We can now provide the desired correctors. For ν, ε in $(0, 1)$, define

$$\mathbf{A}^{\nu, \varepsilon}(\theta, h) \stackrel{\text{def}}{=} 2 \int_{z=0}^h \int_{s=0}^z [\mathbf{g}^\nu(\theta, s) - \mathbf{g}(\theta, s)] I^\varepsilon(\theta, s) ds dz \quad \text{for } (\theta, h) \in \mathcal{J}_O \times [-h/\varepsilon, h/\varepsilon].$$

Let $h_0 > 0$ be small enough that $\{x \in \bar{\mathbf{S}} : \Theta(x) \in \{\mathbf{a}_L, \mathbf{b}_L, \mathbf{a}_R, \mathbf{b}_R\}, |\mathbf{H}(x)| \leq 2h_0\} \subset \mathcal{U}$. Then there exists $\delta_0 > 0$ such that $\mathbf{A}^{\nu, \varepsilon}(\theta, h) \equiv 0$ for $\theta \in ([\mathbf{a}_L, \mathbf{a}_L + \delta_0] \cup [\mathbf{b}_L - \delta_0, \mathbf{b}_L] \cup [\mathbf{a}_R, \mathbf{a}_R + \delta_0] \cup [\mathbf{b}_R - \delta_0, \mathbf{b}_R])$, $|h| \leq 2h_0$. We now define

$$(52) \quad \Phi^{\nu, \varepsilon}(x) \stackrel{\text{def}}{=} \varepsilon^2 \mathbf{A}^{\nu, \varepsilon} \left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon} \right) \mathbf{c}_0 \left(\frac{\mathbf{H}(x)}{h_0} \right) \chi_{\bar{\mathbf{S}}}(x) \quad \text{for } x \in \bar{\mathbf{S}}.$$

Note that $\text{supp}(\Phi^{\nu, \varepsilon}) \subset (\bar{\mathbf{S}})^\circ$ where $^\circ$ denotes the interior of a set. This ensures that $\Phi^{\nu, \varepsilon}$ is smooth at points $x \in \bar{\mathbf{S}}$ with $\Theta(x) \in \{\mathbf{a}_L, \mathbf{b}_L, \mathbf{a}_R, \mathbf{b}_R\}$. For $x \in \bar{\mathbf{S}} \setminus \mathbf{H}^{-1}(0)$, we have

$$(53) \quad \begin{aligned} (\mathcal{L}^\varepsilon \Phi^{\nu, \varepsilon})(x) &= \varepsilon^2 \mathcal{L}^\varepsilon \left[\mathbf{A}^{\nu, \varepsilon} \left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon} \right) \right] \mathbf{c}_0 \left(\frac{\mathbf{H}(x)}{h_0} \right) \chi_{\bar{\mathbf{S}}}(x) \\ &\quad + \varepsilon^2 \left(\nabla \left[\mathbf{A}^{\nu, \varepsilon} \left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon} \right) \right], \nabla \left[\mathbf{c}_0 \left(\frac{\mathbf{H}(x)}{h_0} \right) \right] \right) \chi_{\bar{\mathbf{S}}}(x) \\ &\quad + \varepsilon^2 \mathbf{A}^{\nu, \varepsilon} \left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon} \right) \mathcal{L}^\varepsilon \left[\mathbf{c}_0 \left(\frac{\mathbf{H}(x)}{h_0} \right) \right] \chi_{\bar{\mathbf{S}}}(x). \end{aligned}$$

Note that

$$(54) \quad \begin{aligned} \varepsilon^2 \mathcal{L}^\varepsilon \left[\mathbf{A}^{\nu, \varepsilon} \left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon} \right) \right] &= \left\{ \frac{\partial \mathbf{A}^{\nu, \varepsilon}}{\partial \theta} (\nabla^\perp \mathbf{H}, \nabla \Theta)(x) + \frac{1}{2} \frac{\partial^2 \mathbf{A}^{\nu, \varepsilon}}{\partial h^2} \|\nabla \mathbf{H}\|^2(x) \right\} \\ &\quad + \varepsilon \left\{ \frac{\partial \mathbf{A}^{\nu, \varepsilon}}{\partial h} (\mathcal{L}\mathbf{H})(x) + \frac{\partial^2 \mathbf{A}^{\nu, \varepsilon}}{\partial h \partial \theta} (\nabla \mathbf{H}, \nabla \Theta)(x) \right\} \\ &\quad + \varepsilon^2 \left\{ \frac{\partial \mathbf{A}^{\nu, \varepsilon}}{\partial \theta} (\mathcal{L}\Theta)(x) + \frac{1}{2} \frac{\partial^2 \mathbf{A}^{\nu, \varepsilon}}{\partial \theta^2} \|\nabla \Theta\|^2(x) \right\}, \\ \varepsilon^2 \left(\nabla \left[\mathbf{A}^{\nu, \varepsilon} \left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon} \right) \right], \nabla \left[\mathbf{c}_0 \left(\frac{\mathbf{H}(x)}{h_0} \right) \right] \right) &= \frac{\varepsilon^2}{h_0} \frac{\partial \mathbf{A}^{\nu, \varepsilon}}{\partial \theta} (\nabla \mathbf{H}, \nabla \Theta)(x) \dot{\mathbf{c}}_0 \left(\frac{\mathbf{H}(x)}{h_0} \right) \\ &\quad + \frac{\varepsilon}{h_0} \frac{\partial \mathbf{A}^{\nu, \varepsilon}}{\partial h} \|\nabla \mathbf{H}\|^2(x) \dot{\mathbf{c}}_0 \left(\frac{\mathbf{H}(x)}{h_0} \right), \\ \varepsilon^2 \mathcal{L}^\varepsilon \left[\mathbf{c}_0 \left(\frac{\mathbf{H}(x)}{h_0} \right) \right] &= \frac{\varepsilon^2}{h_0} \dot{\mathbf{c}}_0 \left(\frac{\mathbf{H}(x)}{h_0} \right) (\mathcal{L}\mathbf{H})(x) + \frac{\varepsilon^2}{2h_0^2} \ddot{\mathbf{c}}_0 \left(\frac{\mathbf{H}(x)}{h_0} \right) \|\nabla \mathbf{H}\|^2(x), \end{aligned}$$

where the partial derivatives of $\mathbf{A}^{\nu, \varepsilon}$ with respect to θ and h are evaluated at $(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon})$.

We will take

$$\Phi^\varepsilon(x) \stackrel{\text{def}}{=} \Phi^{\nu_\varepsilon, \varepsilon}(x)$$

for suitable ν_ε .

Proposition 4.2. *There exists a constant $K > 0$ such that*

$$(55) \quad \begin{aligned} |A^{\nu,\varepsilon}(\theta, h)| &\leq K\nu|h|, & \left| \frac{\partial A^{\nu,\varepsilon}}{\partial \theta}(\theta, h) \right| &\leq K\nu|h|, & \left| \frac{\partial^2 A^{\nu,\varepsilon}}{\partial \theta^2}(\theta, h) \right| &\leq K\nu|h|, \\ \left| \frac{\partial A^{\nu,\varepsilon}}{\partial h}(\theta, h) \right| &\leq K|h|\chi_{\{0 < |h| < \nu\}} + K\nu(1 + |h|)\chi_{\{|h| \geq \nu\}}, \\ \left| \frac{\partial^2 A^{\nu,\varepsilon}}{\partial h \partial \theta}(\theta, h) \right| &\leq K|h|\chi_{\{0 < |h| < \nu\}} + K\nu(1 + |h|)\chi_{\{|h| \geq \nu\}}, \\ \left| \frac{\partial^2 A^{\nu,\varepsilon}}{\partial h^2}(\theta, h) \right| &\leq K\chi_{\{0 < |h| < \nu\}} + K\nu e^{-|h|/K}\chi_{\{|h| \geq \nu\}} \end{aligned}$$

for $(\theta, h) \in J_O \times [-h/\varepsilon, h/\varepsilon]$, ν, ε in $(0, 1)$.

Proof. We will repeatedly use Lemma 4.1 (for each estimate) without explicit mention. We start with the bound on $A^{\nu,\varepsilon}$. Fix $\theta \in J_O$. For $0 < h < \nu$, we have

$$|A^{\nu,\varepsilon}(\theta, h)| \leq \int_{z=0}^h \int_{s=0}^z K ds dz \leq Kh^2 \leq K\nu h.$$

For $h \geq \nu$, we have

$$A^{\nu,\varepsilon}(\theta, h) = \sum_{i=1}^3 A_i^{\nu,\varepsilon}(\theta, h)$$

where

$$\begin{aligned} A_1^{\nu,\varepsilon}(\theta, h) &\stackrel{\text{def}}{=} 2 \int_{z=0}^{\nu} \int_{s=0}^z [\mathbf{g}^{\nu}(\theta, s) - \mathbf{g}(\theta, s)] I^{\varepsilon}(\theta, s) ds dz, \\ A_2^{\nu,\varepsilon}(\theta, h) &\stackrel{\text{def}}{=} 2 \int_{z=\nu}^h \int_{s=0}^{\nu} [\mathbf{g}^{\nu}(\theta, s) - \mathbf{g}(\theta, s)] I^{\varepsilon}(\theta, s) ds dz, \\ A_3^{\nu,\varepsilon}(\theta, h) &\stackrel{\text{def}}{=} 2 \int_{z=\nu}^h \int_{s=\nu}^z [\mathbf{g}^{\nu}(\theta, s) - \mathbf{g}(\theta, s)] I^{\varepsilon}(\theta, s) ds dz. \end{aligned}$$

Now

$$\begin{aligned} |A_1^{\nu,\varepsilon}(\theta, h)| &\leq \int_{z=0}^{\nu} \int_{s=0}^z K ds dz \leq K\nu^2 \leq K\nu h, \\ |A_2^{\nu,\varepsilon}(\theta, h)| &\leq \int_{z=\nu}^h \int_{s=0}^{\nu} K ds dz \leq K\nu(h - \nu) \leq K\nu h, \\ |A_3^{\nu,\varepsilon}(\theta, h)| &\leq \int_{z=\nu}^h \int_{s=\nu}^z K\nu e^{-s/K} ds dz \leq K^2\nu \int_{z=\nu}^h (1 - e^{-z/K}) dz \leq K^2\nu h. \end{aligned}$$

This easily gives the stated bound on $A^{\nu,\varepsilon}$.

We have

$$\frac{\partial A^{\nu,\varepsilon}}{\partial h}(\theta, h) = 2 \int_{s=0}^h [\mathbf{g}^{\nu}(\theta, s) - \mathbf{g}(\theta, s)] I^{\varepsilon}(\theta, s) ds$$

For $0 < h < \nu$, we easily get

$$\left| \frac{\partial A^{\nu,\varepsilon}}{\partial h}(\theta, h) \right| \leq Kh.$$

For $h \geq \nu$,

$$\begin{aligned} \left| \frac{\partial A^{\nu,\varepsilon}}{\partial h}(\theta, h) \right| &\leq \left| 2 \int_{s=0}^{\nu} [\mathbf{g}^{\nu}(\theta, s) - \mathbf{g}(\theta, s)] I^{\varepsilon}(\theta, s) ds \right| + \left| 2 \int_{s=\nu}^h [\mathbf{g}^{\nu}(\theta, s) - \mathbf{g}(\theta, s)] I^{\varepsilon}(\theta, s) ds \right| \\ &\leq \int_{s=0}^{\nu} K ds + \int_{s=\nu}^h K\nu e^{-s/K} ds \\ &\leq K\nu + K\nu h. \end{aligned}$$

We now get the appropriate bound.

Differentiating, we get

$$\frac{\partial^2 \mathbf{A}^{\nu, \varepsilon}}{\partial \theta \partial h}(\theta, h) = 2 \int_{s=0}^h \left[\frac{\partial \mathbf{g}^\nu}{\partial \theta}(\theta, s) - \frac{\partial \mathbf{g}}{\partial \theta}(\theta, s) \right] I^\varepsilon(\theta, s) ds + 2 \int_{s=0}^h [\mathbf{g}^\nu(\theta, h) - \mathbf{g}(\theta, h)] \frac{\partial I^\varepsilon}{\partial \theta}(\theta, s) ds.$$

The calculations are now identical to those for $\frac{\partial \mathbf{A}^{\nu, \varepsilon}}{\partial h}$.

Now

$$\begin{aligned} \frac{\partial \mathbf{A}^{\nu, \varepsilon}}{\partial \theta}(\theta, h) &= 2 \int_{z=0}^h \int_{s=0}^z \left[\frac{\partial \mathbf{g}^\nu}{\partial \theta}(\theta, s) - \frac{\partial \mathbf{g}}{\partial \theta}(\theta, s) \right] I^\varepsilon(\theta, s) ds dz \\ &\quad + 2 \int_{z=0}^h \int_{s=0}^z [\mathbf{g}^\nu(\theta, s) - \mathbf{g}(\theta, s)] \frac{\partial I^\varepsilon}{\partial \theta}(\theta, s) ds dz \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 \mathbf{A}^{\nu, \varepsilon}}{\partial \theta^2}(\theta, h) &= 2 \int_{z=0}^h \int_{s=0}^z \left[\frac{\partial^2 \mathbf{g}^\nu}{\partial \theta^2}(\theta, s) - \frac{\partial^2 \mathbf{g}}{\partial \theta^2}(\theta, s) \right] I^\varepsilon(\theta, s) ds dz \\ &\quad + 4 \int_{z=0}^h \int_{s=0}^z \left[\frac{\partial \mathbf{g}^\nu}{\partial \theta}(\theta, s) - \frac{\partial \mathbf{g}}{\partial \theta}(\theta, s) \right] \frac{\partial I^\varepsilon}{\partial \theta}(\theta, s) ds dz + 2 \int_{z=0}^h \int_{s=0}^z [\mathbf{g}^\nu(\theta, s) - \mathbf{g}(\theta, s)] \frac{\partial^2 I^\varepsilon}{\partial \theta^2}(\theta, s) ds dz. \end{aligned}$$

The calculations are now identical to those for $\mathbf{A}^{\nu, \varepsilon}$.

Finally,

$$\frac{\partial^2 \mathbf{A}^{\nu, \varepsilon}}{\partial h^2}(\theta, h) = 2[\mathbf{g}^\nu(\theta, h) - \mathbf{g}(\theta, h)] I^\varepsilon(\theta, h)$$

which easily yields the claimed bound. \square

Proposition 4.3. *There exists a constant $K > 0$ such that*

(56)

$$\begin{aligned} |\Phi^{\nu, \varepsilon}(x)| &\leq K \varepsilon \nu |\mathbf{H}(x)| && \text{for } x \in \bar{\mathbf{S}}, \\ \|\nabla \Phi^{\nu, \varepsilon}(x)\| &\leq K |\mathbf{H}(x)| \chi_{\{0 < |\mathbf{H}(x)| < \varepsilon \nu\}} + K \nu (\varepsilon + |\mathbf{H}(x)|) \chi_{\{|\mathbf{H}(x)| \geq \varepsilon \nu\}} && \text{for } x \in \bar{\mathbf{S}} \setminus \mathbf{H}^{-1}(0), \\ |(\mathcal{L}^\varepsilon \Phi^{\nu, \varepsilon})(x)| &\leq K \nu \left| \frac{\mathbf{H}(x)}{\varepsilon} \right| + K \chi_{\{0 < |\mathbf{H}(x)| < \varepsilon \nu\}} + K \nu e^{-|\mathbf{H}(x)|/K\varepsilon} \chi_{\{|\mathbf{H}(x)| \geq \varepsilon \nu\}} + K \varepsilon \nu && \text{for } x \in \bar{\mathbf{S}} \setminus \mathbf{H}^{-1}(0). \end{aligned}$$

Proof. We will use Proposition 4.2 here. The bound on $|\Phi^{\nu, \varepsilon}(x)|$ in the first line of (56) is trivial.

For $x \in \mathbf{S} \setminus \mathbf{H}^{-1}(0)$, we have

$$\begin{aligned} \nabla \Phi^{\nu, \varepsilon}(x) &= \varepsilon^2 \frac{\partial \mathbf{A}^{\nu, \varepsilon}}{\partial \theta} \nabla \Theta(x) \mathbf{c}_0 \left(\frac{\mathbf{H}(x)}{h_0} \right) \chi_{\bar{\mathbf{S}}}(x) + \varepsilon \frac{\partial \mathbf{A}^{\nu, \varepsilon}}{\partial h} \nabla \mathbf{H}(x) \mathbf{c}_0 \left(\frac{\mathbf{H}(x)}{h_0} \right) \chi_{\bar{\mathbf{S}}}(x) \\ &\quad + \frac{\varepsilon^2}{h_0} \mathbf{A}^{\nu, \varepsilon} \dot{\mathbf{c}}_0 \left(\frac{\mathbf{H}(x)}{h_0} \right) \nabla \mathbf{H}(x) \chi_{\bar{\mathbf{S}}}(x) \end{aligned}$$

where $\mathbf{A}^{\nu, \varepsilon}$ and its partial derivatives are evaluated at $\left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon} \right)$. There now exists $K > 0$ such that

$$\begin{aligned} \|\nabla \Phi^{\nu, \varepsilon}(x)\| &\leq K \varepsilon^2 \nu \left| \frac{\mathbf{H}(x)}{\varepsilon} \right| + K \varepsilon \left| \frac{\mathbf{H}(x)}{\varepsilon} \right| \chi_{\{0 < |\mathbf{H}(x)|/\varepsilon < \nu\}} + K \varepsilon \nu \left(1 + \left| \frac{\mathbf{H}(x)}{\varepsilon} \right| \right) \chi_{\{|\mathbf{H}(x)|/\varepsilon \geq \nu\}} \\ &\quad + K \varepsilon^2 \nu \left| \frac{\mathbf{H}(x)}{\varepsilon} \right| \end{aligned}$$

which easily yields the second line in (56).

For the bound on $|\mathcal{L}^\varepsilon \Phi^{\nu, \varepsilon}(x)|$, we start by recalling (53) and (54). There exists $K_1 > 0$ such that for $x \in \bar{\mathbf{S}} \setminus \mathbf{H}^{-1}(0)$,

$$\begin{aligned} \left| \varepsilon^2 \mathcal{L}^\varepsilon \left[\mathbf{A}^{\nu, \varepsilon} \left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon} \right) \right] \right| &\leq K_1 \nu \left| \frac{\mathbf{H}(x)}{\varepsilon} \right| + K_1 \chi_{\{0 < |\mathbf{H}(x)|/\varepsilon < \nu\}} + K_1 \nu e^{-|\mathbf{H}(x)|/K\varepsilon} \chi_{\{|\mathbf{H}(x)|/\varepsilon \geq \nu\}} \\ &\quad + K_1 \varepsilon \nu \left(1 + \left| \frac{\mathbf{H}(x)}{\varepsilon} \right| \right) + K_1 \varepsilon^2 \nu \left| \frac{\mathbf{H}(x)}{\varepsilon} \right|, \end{aligned}$$

and hence a constant $K > 0$ such that

$$\left| \varepsilon^2 \mathcal{L}^\varepsilon \left[\mathbf{A}^{\nu, \varepsilon} \left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon} \right) \right] \mathbf{c}_0 \left(\frac{\mathbf{H}(x)}{h_0} \right) \right| \leq K \nu \left| \frac{\mathbf{H}(x)}{\varepsilon} \right| + K \chi_{\{0 < |\mathbf{H}(x)| < \varepsilon \nu\}} + K \nu e^{-|\mathbf{H}(x)|/K\varepsilon} \chi_{\{|\mathbf{H}(x)| \geq \varepsilon \nu\}} + K \varepsilon \nu.$$

Again, there exist $K_1, K > 0$ such that

$$\begin{aligned} \left| \varepsilon^2 \left(\nabla \left[\mathbf{A}^{\nu, \varepsilon} \left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon} \right) \right], \nabla \left[\mathbf{c}_0 \left(\frac{\mathbf{H}(x)}{h_0} \right) \right] \right) \right| &\leq K_1 \varepsilon^2 \nu \left| \frac{\mathbf{H}(x)}{\varepsilon} \right| + K_1 \varepsilon \nu \left(1 + \left| \frac{\mathbf{H}(x)}{\varepsilon} \right| \right) \\ &\leq K \varepsilon \nu + K \nu |\mathbf{H}(x)|. \end{aligned}$$

Finally,

$$\left| \varepsilon^2 \mathbf{A}^{\nu, \varepsilon} \left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon} \right) \mathcal{L}^\varepsilon \left[\mathbf{c}_0 \left(\frac{\mathbf{H}(x)}{h_0} \right) \right] \right| \leq K \varepsilon^2 \nu \left| \frac{\mathbf{H}(x)}{\varepsilon} \right|$$

for suitable $K > 0$. Putting things together, we get the third line in (56). \square

4.3. Proof of Theorem 2.6.

We now provide

Proof of Theorem 2.6. Set

$$\nu_\varepsilon \stackrel{\text{def}}{=} \varepsilon^2 \quad \text{for } \varepsilon \in (0, 1)$$

and

$$\bar{\varepsilon}_{2.6} \stackrel{\text{def}}{=} \bar{\varepsilon}_{3.1}.$$

For $\varepsilon \in (0, \bar{\varepsilon}_{2.6})$, define

$$\Phi^\varepsilon \stackrel{\text{def}}{=} \Phi^{\nu_\varepsilon, \varepsilon} \quad \text{and} \quad \Pi^\varepsilon \stackrel{\text{def}}{=} \Psi^\varepsilon + \Phi^\varepsilon.$$

Clearly, $F + \Pi^\varepsilon \in C(\bar{\mathbf{S}}) \cap C^2(\bar{\mathbf{S}} \setminus \mathbf{H}^{-1}(0))$, and hence $F + \Pi^\varepsilon$ has a continuous extension which belongs to $C_0(\mathbb{R}^2) \cap C_0^2(\mathbb{R}^2 \setminus \mathbf{H}^{-1}(0))$. Using Theorem 3.1 and the bound on $\|\nabla \Phi^\varepsilon\|$ in the second line of equation (56), it follows that $F + \Pi^\varepsilon$ satisfies the skewness conditions at $\mathbf{H}^{-1}(0) \setminus \{\mathbf{o}\}$. To conclude that $F + \Pi^\varepsilon \in \mathcal{D}(\mathcal{A}^\varepsilon)$, it only remains to show that $\mathcal{L}^\varepsilon(F + \Pi^\varepsilon)$ has a continuous extension to $\bar{\mathbf{S}}$ (and hence an extension to all of \mathbb{R}^2 which belongs to C_0). Recalling equations (53) and (54), and using Proposition 4.2, we see that the only term in $\mathcal{L}^\varepsilon \Phi^\varepsilon$ which is possibly non-zero at $\mathbf{H}^{-1}(0)$ is

$$\frac{1}{2} \frac{\partial^2 \mathbf{A}^{\nu_\varepsilon, \varepsilon}}{\partial h^2} \left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon} \right) \|\nabla \mathbf{H}\|^2(x) = \left[\mathbf{g}^{\nu_\varepsilon} \left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon} \right) - \mathbf{g} \left(\Theta(x), \frac{\mathbf{H}(x)}{\varepsilon} \right) \right] (\mathcal{L}\mathbf{H})(x)$$

which is precisely what is needed to correct for the discontinuity in $\mathcal{L}^\varepsilon(F + \Psi^\varepsilon)$ at $\mathbf{H}^{-1}(0)$. Hence, $\mathcal{L}^\varepsilon(F + \Pi^\varepsilon)$ has a continuous extension to $\bar{\mathbf{S}}$.

The bound on Π^ε follows directly from Theorem 3.1 and the first line of equation (56). To obtain the bound on $\mathcal{L}^\varepsilon \Pi^\varepsilon$, note that

$$\chi_{\{0 < |\mathbf{H}(x)| < \varepsilon^3\}} \leq \mathbf{c}_0 \left(\frac{\mathbf{H}(x)}{\varepsilon^3} \right) \leq \mathbf{c}_0 \left(\frac{\mathbf{H}(x)}{\sqrt{2}\varepsilon^{5/4}} \right).$$

We now easily get the stated bound by combining Theorem 3.1 and the third line of equation (56). \square

5. FUNCTIONAL-ANALYTIC CALCULATIONS

5.1. **Fourier Series Solutions.** We would like to solve the PDE's

$$(57) \quad \frac{\partial \psi_\ell}{\partial \theta}(\theta, h) + \frac{1}{2} \frac{\partial^2 \psi_\ell}{\partial h^2}(\theta, h) = 0 \quad \text{for} \quad (\theta, h) \in \mathbb{I}_\ell \times \mathcal{R}_\ell$$

subject to the boundary conditions

$$(58) \quad \lim_{\substack{h \rightarrow 0 \\ h \in \mathcal{R}_\ell}} \psi_\ell(\cdot, h) = \xi, \quad \psi_\ell(\mathbf{A}_\ell, h) = \psi_\ell(\mathbf{B}_\ell, h) \quad \text{for all} \quad h \in \mathcal{R}_\ell$$

such that the skewness conditions (15) are satisfied. Note, however, that the Dirichlet data ξ is not given in advance. In light of the periodicity requirement in (58), we expect to find a solution that is given by a Fourier expansion in θ .

For $\xi \in L^2(\mathbb{I}_\ell)$, $\ell \in \Lambda$, $k \in \mathbb{Z}$, define the Fourier coefficient

$$\mathbf{c}_{\ell, k}[\xi] \stackrel{\text{def}}{=} \frac{1}{\mathbf{G}_\ell} \int_{\theta \in \mathbb{I}_\ell} \xi(\theta) \exp \left[-\frac{2\pi k i \theta}{\mathbf{G}_\ell} \right] d\theta.$$

If ξ is real-valued, then we have $\mathbf{c}_{\ell, -k}[\xi] = \mathbf{c}_{\ell, k}^*[\xi]$ for $k \in \mathbb{Z}$, where x^* denotes the complex conjugate of x for any $x \in \mathbb{C}$. By Bessel's inequality, it follows that

$$(59) \quad \sum_{k \in \mathbb{Z}} |\mathbf{c}_{\ell, k}[\xi]|^2 \leq \frac{1}{\mathbf{G}_\ell} \int_{\theta \in \mathbb{I}_\ell} |\xi(\theta)|^2 d\theta < \infty \quad \text{for} \quad \xi \in L^2(\mathbb{I}_\ell), \ell \in \Lambda.$$

Also, ξ is the sum (in $L^2(\mathbb{I}_\ell)$) of its Fourier series.

Let

$$\mathbf{s}_O \stackrel{\text{def}}{=} 1 \quad \text{and} \quad \mathbf{s}_L = \mathbf{s}_R \stackrel{\text{def}}{=} -1.$$

Define also some eigenvalues $\lambda_{\ell, k}^2$ by

$$\lambda_{\ell, k} \stackrel{\text{def}}{=} \begin{cases} \sqrt{\frac{2\pi}{\mathbf{G}_\ell}} \sqrt{|k|} (1 - i) & k > 0 \\ 0 & k = 0 \\ \sqrt{\frac{2\pi}{\mathbf{G}_\ell}} \sqrt{|k|} (1 + i) & k < 0 \end{cases}$$

Then

$$\lambda_{\ell, k}^2 = -\frac{4\pi k i}{\mathbf{G}_\ell}, \quad |\lambda_{\ell, k}| = \sqrt{\frac{4\pi}{\mathbf{G}_\ell}} \sqrt{|k|}, \quad \text{and} \quad \Re(\lambda_{\ell, k}) = \sqrt{\frac{2\pi}{\mathbf{G}_\ell}} \sqrt{|k|}$$

for $\ell \in \Lambda$, $k \in \mathbb{Z}$, where $\Re(x)$ denotes the real part of a complex number x . Note that for $k \in \mathbb{Z}$, we have $\lambda_{\ell, -k} = \lambda_{\ell, k}^*$. To start, let's solve the PDE's (57), (58) for smooth Dirichlet data ξ . Define, for each $\ell \in \Lambda$,

$$C_p^\infty(\overline{\mathbb{I}_\ell}) \stackrel{\text{def}}{=} \left\{ \xi \in C^\infty(\overline{\mathbb{I}_\ell}) : \frac{d^n \xi}{dx^n}(\mathbf{A}_\ell) = \frac{d^n \xi}{dx^n}(\mathbf{B}_\ell) \text{ for all } n \geq 0 \right\}.$$

The functions in $C_p^\infty(\overline{\mathbb{I}_\ell})$ can thus be thought of as the restrictions to $\overline{\mathbb{I}_\ell}$ of \mathbf{G}_ℓ -periodic elements of $C^\infty(\mathbb{R})$. It easily follows (see [Fol92]) that for $\xi \in C_p^\infty(\overline{\mathbb{I}_\ell})$, we have

$$(60) \quad \sum_{k \in \mathbb{Z}} |\mathbf{c}_{\ell, k}[\xi]|^2 (1 + |k|^n) < \infty \quad \text{for any } n \in \mathbb{N}.$$

For $\xi \in C_p^\infty(\overline{\mathbb{I}_\ell})$, $\ell \in \Lambda$, define

$$(61) \quad \begin{aligned} \mathfrak{B}_\ell[\xi](\theta, h) &\stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z} \setminus \{0\}} \mathbf{c}_{\ell, k}[\xi] \exp \left[\frac{2\pi k i \theta}{\mathbf{G}_\ell} - \mathbf{s}_\ell \lambda_{\ell, k} h \right] + \mathbf{c}_{\ell, 0}[\xi], \\ \mathfrak{b}_\ell[\xi](\theta) &\stackrel{\text{def}}{=} -\mathbf{s}_\ell \sum_{k \in \mathbb{Z} \setminus \{0\}} \lambda_{\ell, k} \mathbf{c}_{\ell, k}[\xi] \exp \left[\frac{2\pi k i \theta}{\mathbf{G}_\ell} \right]. \end{aligned}$$

Lemma 5.1. Fix $\ell \in \Lambda$ and $\xi \in C_p^\infty(\overline{\mathbf{l}_\ell})$. Then $\mathfrak{B}_\ell[\xi]$ exists as a pointwise sum and $\mathfrak{B}_\ell[\xi] \in C^\infty(\overline{\mathbf{l}_\ell} \times \overline{\mathcal{R}_\ell})$. Also, $\mathfrak{b}_\ell[\xi]$ exists as a sum in $L^2(\mathbf{l}_\ell)$; thus \mathfrak{b}_ℓ is a linear operator from $C_p^\infty(\overline{\mathbf{l}_\ell})$ to $L^2(\mathbf{l}_\ell)$. Moreover,

$$(62) \quad \begin{aligned} \frac{\partial \mathfrak{B}_\ell[\xi]}{\partial \theta}(\theta, h) + \frac{1}{2} \frac{\partial^2 \mathfrak{B}_\ell[\xi]}{\partial h^2}(\theta, h) &= 0, & (\theta, h) \in \mathbf{l}_\ell \times \mathcal{R}_\ell \\ \lim_{\substack{h \rightarrow 0 \\ h \in \mathcal{R}_\ell}} \|\mathfrak{B}_\ell[\xi](\cdot, h) - \xi\|_{L^2(\mathbf{l}_\ell)} &= 0, \\ \lim_{\substack{h \rightarrow 0 \\ h \in \mathcal{R}_\ell}} \left\| \frac{\partial \mathfrak{B}_\ell[\xi]}{\partial h}(\cdot, h) - \mathfrak{b}_\ell[\xi] \right\|_{L^2(\mathbf{l}_\ell)} &= 0, \\ \mathfrak{B}_\ell[\xi](\mathbf{A}_\ell, h) &= \mathfrak{B}_\ell[\xi](\mathbf{B}_\ell, h), & h \in \mathcal{R}_\ell. \end{aligned}$$

There is a constant $K > 0$ such that

$$(63) \quad |\mathfrak{B}_\ell[\xi](\theta, h) - \mathfrak{c}_{\ell,0}[\xi]| \leq K e^{-|h|/K} \quad \text{for } (\theta, h) \in \mathbf{l}_\ell \times \mathcal{R}_\ell$$

and for each multiindex α such that $|\alpha| \geq 1$, there is a constant $K > 0$ such that

$$(64) \quad |D^\alpha \mathfrak{B}_\ell[\xi](\theta, h)| \leq K e^{-|h|/K} \quad \text{for } (\theta, h) \in \mathbf{l}_\ell \times \mathcal{R}_\ell.$$

Proof. Straightforward PDE calculations using (60) and the rate of growth of the $\lambda_{\ell,k}$'s. \square

5.2. Lax-Milgram Analysis. For $\xi \in L^2(\mathbf{l}_O)$, define $\xi_\ell \stackrel{\text{def}}{=} \xi|_{\mathbf{l}_\ell}$ for each $\ell \in \Lambda$. We would like the Dirichlet data for the PDEs (57), (58) to agree at $h = 0$. To this end, we define a class of ‘‘admissible’’ functions \mathfrak{U} by

$$\mathfrak{U} \stackrel{\text{def}}{=} \{ \xi \in L^2(\mathbf{l}_O) : \xi_\ell \in C_p^\infty(\overline{\mathbf{l}_\ell}) \text{ for each } \ell \in \Lambda \}.$$

The skewness conditions (15) now become

$$(65) \quad \begin{aligned} \beta_L \{ \mathbf{v}_L + \mathfrak{b}_L[\xi_L](\theta) \} &= \beta_O \{ \mathbf{v}_O + \mathfrak{b}_O[\xi_O](\theta) \} & \text{for } \theta \in \mathbf{l}_L, \\ \beta_R \{ \mathbf{v}_R + \mathfrak{b}_R[\xi_R](\theta) \} &= \beta_O \{ \mathbf{v}_O + \mathfrak{b}_O[\xi_O](\theta) \} & \text{for } \theta \in \mathbf{l}_R. \end{aligned}$$

We would like to cast the conditions (65) in the language of solvability for a certain Dirichlet-to-Neumann operator. For $\xi \in \mathfrak{U}$, define $\mathbf{K}[\xi] : \mathbf{l}_O \rightarrow \mathbb{R}$ by

$$\mathbf{K}[\xi](\theta) \stackrel{\text{def}}{=} \beta_L \mathfrak{b}_L[\xi_L](\theta) \chi_{\mathbf{l}_L} + \beta_R \mathfrak{b}_R[\xi_R](\theta) \chi_{\mathbf{l}_R} - \beta_O \mathfrak{b}_O[\xi_O](\theta)$$

and define $\mathbf{p} : \mathbf{l}_O \rightarrow \mathbb{R}$ by

$$\mathbf{p}(\theta) \stackrel{\text{def}}{=} \beta_O \mathbf{v}_O - \beta_L \mathbf{v}_L \chi_{\mathbf{l}_L} - \beta_R \mathbf{v}_R \chi_{\mathbf{l}_R}.$$

Lemma 5.1 tells us that $\mathbf{K}[\xi]$ is in fact an element of $L^2(\mathbf{l}_O)$. We now want to use the Lax-Milgram Theorem to find a weak solution of the equation

$$\mathbf{K}[\xi] = \mathbf{p}$$

in a suitable function space. To identify the appropriate function space, we start with a *coercivity* result and a *boundedness* result. Note that constants are both in the kernel of the \mathfrak{b}_ℓ 's as well as orthogonal to the range of the \mathfrak{b}_ℓ 's.

The proofs of Lemmas 5.2 and 5.3 are virtually identical to the proofs of Lemmas 9.2 and 9.3 in [Sow03].

Lemma 5.2 (Coercivity). For $\xi \in \mathfrak{U}$, we have,

$$\langle \mathbf{K}[\xi], \xi \rangle_{L^2(\mathbf{l}_O)} = \sum_{\ell \in \Lambda} \beta_\ell \int_{\theta \in \mathbf{l}_\ell} \int_{h \in \mathcal{R}_\ell} \left(\frac{\partial \mathfrak{B}_\ell[\xi_\ell]}{\partial h}(\theta, h) \right)^2 dh d\theta = \frac{1}{\sqrt{2}} \sum_{\substack{\ell \in \Lambda \\ k \in \mathbb{Z} \setminus \{0\}}} \beta_\ell \mathbf{G}_\ell |\lambda_{\ell,k}| |\mathfrak{c}_{\ell,k}[\xi_\ell]|^2.$$

Lemma 5.3 (Boundedness). For $\xi, \zeta \in \mathfrak{U}$, we have

$$\langle \mathbf{K}[\xi], \zeta \rangle_{L^2(\mathbf{l}_O)} \leq \left\{ \sum_{\substack{\ell \in \Lambda \\ k \in \mathbb{Z} \setminus \{0\}}} \beta_\ell \mathbf{G}_\ell |\lambda_{\ell,k}| |\mathfrak{c}_{\ell,k}[\xi_\ell]|^2 \right\}^{\frac{1}{2}} \left\{ \sum_{\substack{\ell \in \Lambda \\ k \in \mathbb{Z} \setminus \{0\}}} \beta_\ell \mathbf{G}_\ell |\lambda_{\ell,k}| |\mathfrak{c}_{\ell,k}[\zeta_\ell]|^2 \right\}^{\frac{1}{2}}$$

Define an inner product $\langle \cdot, \cdot \rangle_{\mathbf{H}}$ on \mathfrak{U} by

$$(66) \quad \langle \xi, \zeta \rangle_{\mathbf{H}} \stackrel{\text{def}}{=} \sum_{\substack{\ell \in \Lambda \\ k \in \mathbb{Z} \setminus \{0\}}} \beta_{\ell} \mathbf{G}_{\ell} |\lambda_{\ell, k}| \mathbf{c}_{\ell, k} [\xi_{\ell}] \mathbf{c}_{\ell, k}^* [\zeta_{\ell}] + \langle \xi, \zeta \rangle_{L^2(\mathfrak{l}_O)} \quad \text{for } \xi, \zeta \in \mathfrak{U}$$

and let $\| \cdot \|_{\mathbf{H}}$ be the corresponding norm, i.e.

$$(67) \quad \|\xi\|_{\mathbf{H}} \stackrel{\text{def}}{=} \sqrt{\langle \xi, \xi \rangle_{\mathbf{H}}} = \left\{ \sum_{\substack{\ell \in \Lambda \\ k \in \mathbb{Z} \setminus \{0\}}} \beta_{\ell} \mathbf{G}_{\ell} |\lambda_{\ell, k}| \mathbf{c}_{\ell, k} [\xi_{\ell}]^2 + \|\xi\|_{L^2(\mathfrak{l}_O)}^2 \right\}^{\frac{1}{2}} \quad \text{for } \xi \in \mathfrak{U}.$$

Using the estimate (60) together with the Cauchy-Schwarz inequality, we see that $\langle \xi, \zeta \rangle_{\mathbf{H}}$ is in fact finite for $\xi, \zeta \in \mathfrak{U}$. Also, the fact that $\mathbf{c}_{\ell, k} [\xi_{\ell}] = \mathbf{c}_{\ell, -k}^* [\xi_{\ell}]$, $\lambda_{\ell, k} = \lambda_{\ell, -k}^*$ for $\xi \in \mathfrak{U}$, $k \in \mathbb{Z}$ implies that $\langle \xi, \zeta \rangle_{\mathbf{H}}$ is real-valued for all $\xi, \zeta \in \mathfrak{U}$. Let \mathbf{H} be the closure of \mathfrak{U} with respect to $\| \cdot \|_{\mathbf{H}}$.

The following Lemma is taken from [Sow03].

Lemma 5.4. *The pair $(\mathbf{H}, \langle \cdot, \cdot \rangle_{\mathbf{H}})$ is a real Hilbert space which is compactly embedded in $L^2(\mathfrak{l}_O)$, written $\mathbf{H} \subset\subset L^2(\mathfrak{l}_O)$. The latter condition means the following:*

- (1) *There exists a constant C such that $\|\xi\|_{L^2(\mathfrak{l}_O)} \leq C \|\xi\|_{\mathbf{H}}$ for all $\xi \in \mathbf{H}$,*
- (2) *Every bounded sequence in \mathbf{H} is precompact in $L^2(\mathfrak{l}_O)$.*

Define now the bilinear mapping $\mathbf{B}_{\mathbf{K}} : \mathfrak{U} \times \mathfrak{U} \rightarrow \mathbb{R}$ by

$$(68) \quad \mathbf{B}_{\mathbf{K}}[\xi, \zeta] \stackrel{\text{def}}{=} \langle \mathbf{K}[\xi], \zeta \rangle_{L^2(\mathfrak{l}_O)} \quad \text{for } \xi, \zeta \in \mathfrak{U}.$$

By Lemma 5.3, it follows that

$$|\mathbf{B}_{\mathbf{K}}[\xi, \zeta]| = |\langle \mathbf{K}[\xi], \zeta \rangle_{L^2(\mathfrak{l}_O)}| \leq \|\xi\|_{\mathbf{H}} \|\zeta\|_{\mathbf{H}} \quad \text{for } \xi, \zeta \in \mathfrak{U}.$$

Lemma 5.5. *$\mathbf{B}_{\mathbf{K}}$ can be extended to yield a bilinear mapping on $\mathbf{H} \times \mathbf{H}$ satisfying*

$$|\mathbf{B}_{\mathbf{K}}[\xi, \zeta]| \leq \|\xi\|_{\mathbf{H}} \|\zeta\|_{\mathbf{H}} \quad \text{for } \xi, \zeta \in \mathbf{H}.$$

Moreover, we have

$$\mathbf{B}_{\mathbf{K}}[\xi, \xi] = \frac{1}{\sqrt{2}} \sum_{\substack{\ell \in \Lambda \\ k \in \mathbb{Z} \setminus \{0\}}} \beta_{\ell} \mathbf{G}_{\ell} |\lambda_{\ell, k}| \mathbf{c}_{\ell, k} [\xi_{\ell}]^2 \quad \text{for } \xi \in \mathbf{H}.$$

Proof. Note that the stated bound on $|\mathbf{B}_{\mathbf{K}}[\xi, \zeta]|$ and the expression for $\mathbf{B}_{\mathbf{K}}[\xi, \xi]$ hold for $\xi, \zeta \in \mathfrak{U}$. The proof easily follows by approximating ξ, ζ in \mathbf{H} by sequences $\{\xi_n\}_{n=1}^{\infty}, \{\zeta_n\}_{n=1}^{\infty}$ in \mathfrak{U} . \square

Proposition 5.6 (Solvability). *If $\mathbf{p} \in L^2(\mathfrak{l}_O)$ satisfies $\int_{\theta \in \mathfrak{l}_O} \mathbf{p}(\theta) d\theta = 0$, then there exists $\xi \in \mathbf{H}$ such that*

$$(69) \quad \mathbf{B}_{\mathbf{K}}[\xi, \zeta] = \langle \mathbf{p}, \zeta \rangle_{L^2(\mathfrak{l}_O)}$$

for all $\zeta \in \mathbf{H}$ and

$$(70) \quad \mathbf{B}_{\mathbf{K}}[\xi, \xi] \leq \left(\frac{\mathbf{G}_O}{2\pi\beta_O^2} \right)^{\frac{1}{2}} \|\mathbf{p}\|_{L^2(\mathfrak{l}_O)}^2.$$

Proof. For each $f \in L^2(\mathfrak{l}_O)$, define $L_f : \mathbf{H} \rightarrow \mathbb{R}$ by $L_f[\xi] \stackrel{\text{def}}{=} \langle \xi, f \rangle_{L^2(\mathfrak{l}_O)}$ for $\xi \in \mathbf{H}$. Then, $L_f \in \mathbf{H}^*$ and $\|L_f\| \leq \|f\|_{L^2(\mathfrak{l}_O)}$. Define the bilinear mapping $\bar{\mathbf{B}} : \mathbf{H} \times \mathbf{H} \rightarrow \mathbb{R}$ by

$$\bar{\mathbf{B}}[\xi, \zeta] \stackrel{\text{def}}{=} \mathbf{B}_{\mathbf{K}}[\xi, \zeta] + \frac{1}{\sqrt{2}} \langle \xi, \zeta \rangle_{L^2(\mathfrak{l}_O)} \quad \text{for } (\xi, \zeta) \in \mathbf{H} \times \mathbf{H}.$$

Now for $\xi, \zeta \in \mathbf{H}$, by Lemma 5.5,

$$|\bar{\mathbf{B}}[\xi, \zeta]| \leq \left(1 + \frac{1}{\sqrt{2}} \right) \|\xi\|_{\mathbf{H}} \|\zeta\|_{\mathbf{H}}$$

and

$$\bar{\mathbf{B}}[\xi, \xi] = \mathbf{B}_{\mathbf{K}}[\xi, \xi] + \frac{1}{\sqrt{2}} \|\xi\|_{L^2(\mathfrak{l}_O)}^2 = \frac{1}{\sqrt{2}} \|\xi\|_{\mathbf{H}}^2.$$

Hence, by the Lax-Milgram Theorem (see [Eva98]), for each $f \in L^2(\mathfrak{l}_O)$, there exists a unique element $\mathbf{A}[f] \in \mathbf{H}$ such that

$$\bar{\mathbf{B}}[\mathbf{A}[f], \zeta] = L_f[\zeta] \quad \text{for all } \zeta \in \mathbf{H}.$$

Using the coercivity result of Lemma 5.2, the boundedness of L_f , and the compact embedding result of Lemma 5.4, \mathbf{A} is seen to be a compact linear operator on $L^2(\mathfrak{l}_O)$. We would thus like to find $\xi \in L^2(\mathfrak{l}_O)$ such that

$$(71) \quad \xi = \mathbf{A} \left[\mathbf{p} + \frac{1}{\sqrt{2}} \xi \right] = \mathbf{A}[\mathbf{p}] + \frac{1}{\sqrt{2}} \mathbf{A}[\xi].$$

Note that any $\xi \in L^2(\mathfrak{l}_O)$ that solves (71) is automatically in \mathbf{H} . The equation (71) has a solution $\xi \in L^2(\mathfrak{l}_O)$ precisely when $\mathbf{A}[\mathbf{p}] \in R\left(I - \frac{1}{\sqrt{2}}\mathbf{A}\right)$. If we let \mathbf{A}^* be the $L^2(\mathfrak{l}_O)$ -adjoint of \mathbf{A} , then by the Fredholm alternative (see [Eva98]), we have $R\left(I - \frac{1}{\sqrt{2}}\mathbf{A}\right) = N\left(I - \frac{1}{\sqrt{2}}\mathbf{A}^*\right)^\perp$. Using the fact that $\langle \mathbf{A}f, g \rangle_{L^2(\mathfrak{l}_O)} = \langle f, \mathbf{A}^*g \rangle_{L^2(\mathfrak{l}_O)}$ for all f, g in $L^2(\mathfrak{l}_O)$, we require \mathbf{p} to be orthogonal in $L^2(\mathfrak{l}_O)$ to the subspace

$$M \stackrel{\text{def}}{=} \left\{ \mathbf{A}^*\zeta : \zeta \in L^2(\mathfrak{l}_O), \zeta = \frac{1}{\sqrt{2}}\mathbf{A}^*\zeta \right\} = \left\{ \zeta \in L^2(\mathfrak{l}_O) : \zeta = \frac{1}{\sqrt{2}}\mathbf{A}^*\zeta \right\}.$$

To complete the proof, we need to show that M consists only of constants. By [Sow03] and [Sow05], it follows that \mathbf{A}^* takes $L^2(\mathfrak{l}_O)$ into \mathbf{H} . Hence we have $M \subset \mathbf{H}$. Let $\zeta \in M$. We start with the observation that for any $u \in L^2(\mathfrak{l}_O)$, we have

$$\begin{aligned} \mathbf{B}_{\mathbf{K}}[\mathbf{A}[u], \zeta] &= \bar{\mathbf{B}}[\mathbf{A}[u], \zeta] - \frac{1}{\sqrt{2}} \langle \mathbf{A}[u], \zeta \rangle_{L^2(\mathfrak{l}_O)} \\ &= \langle u, \zeta \rangle_{L^2(\mathfrak{l}_O)} - \frac{1}{\sqrt{2}} \langle u, \mathbf{A}^*[\zeta] \rangle_{L^2(\mathfrak{l}_O)} = 0. \end{aligned}$$

Thus, if $\zeta \in R(\mathbf{A})$, then $\mathbf{B}_{\mathbf{K}}[\zeta, \zeta] = 0$. If not, there exists a sequence $\{\zeta_n\}_{n=1}^\infty \subset \mathfrak{U}$ such that $\|\zeta - \zeta_n\|_{\mathbf{H}} \rightarrow 0$. We claim that $\zeta_n \in R(\mathbf{A})$. Indeed, it is easily seen that $\zeta_n = \mathbf{A} \left[\mathbf{K}[\zeta_n] + \frac{1}{\sqrt{2}}\zeta_n \right]$. Hence, by Lemma 5.5,

$$\mathbf{B}_{\mathbf{K}}[\zeta, \zeta] = \lim_{n \rightarrow \infty} \mathbf{B}_{\mathbf{K}}[\zeta_n, \zeta] = 0$$

which implies that $c_{\ell, k}[\zeta_\ell] = 0$ for $k \in \mathbb{Z} \setminus \{0\}$, $\ell \in \Lambda$. Hence, ζ is constant.

Let's now prove (70). By (69), we have

$$\mathbf{B}_{\mathbf{K}}[\xi, \xi] = \langle \mathbf{p}, \xi \rangle_{L^2(\mathfrak{l}_O)}.$$

Let $\tilde{\xi} = \xi - c_{O,0}[\xi]$. Then $c_{O,0}[\tilde{\xi}] = 0$ and $c_{O,k}[\tilde{\xi}] = c_{O,k}[\xi]$ for $k \neq 0$. Recalling that \mathbf{p} is orthogonal in $L^2(\mathfrak{l}_O)$ to constants, we get

$$\mathbf{B}_{\mathbf{K}}[\xi, \xi] = \left\langle \mathbf{p}, \tilde{\xi} \right\rangle_{L^2(\mathfrak{l}_O)} \leq \|\mathbf{p}\|_{L^2(\mathfrak{l}_O)} \|\tilde{\xi}\|_{L^2(\mathfrak{l}_O)}.$$

By Parseval's Identity,

$$\|\tilde{\xi}\|_{L^2(\mathfrak{l}_O)}^2 = \mathbf{G}_O \sum_{k \in \mathbb{Z} \setminus \{0\}} |c_{O,k}[\tilde{\xi}]|^2.$$

Since $|\lambda_{O,k}| = (4\pi/\mathbf{G}_O)^{1/2} \sqrt{|k|}$, we have

$$1 = \sqrt{\frac{\mathbf{G}_O}{4\pi}} \frac{|\lambda_{O,k}|}{\sqrt{|k|}} \leq \sqrt{\frac{\mathbf{G}_O}{4\pi}} |\lambda_{O,k}| \quad \text{for } |k| \geq 1.$$

Recalling the explicit expression for $\mathbf{B}_K[\xi, \xi]$ given in Lemma 5.5, we get

$$\begin{aligned} \mathbf{B}_K[\xi, \xi] &= \frac{1}{\sqrt{2}} \sum_{\substack{\ell \in \Lambda \\ k \in \mathbb{Z} \setminus \{0\}}} \beta_\ell \mathbf{G}_\ell |\lambda_{\ell, k}| |\mathbf{c}_{\ell, k}[\xi_\ell]|^2 \leq \left(\frac{\mathbf{G}_O}{4\pi\beta_O^2} \right)^{\frac{1}{4}} \|\mathbf{P}\|_{L^2(I_O)} \left(\sum_{k \in \mathbb{Z} \setminus \{0\}} \beta_O \mathbf{G}_O |\lambda_{O, k}| |\mathbf{c}_{O, k}[\xi]|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\frac{\mathbf{G}_O}{4\pi\beta_O^2} \right)^{\frac{1}{4}} \|\mathbf{P}\|_{L^2(I_O)} \left(\sum_{\substack{\ell \in \Lambda \\ k \in \mathbb{Z} \setminus \{0\}}} \beta_\ell \mathbf{G}_\ell |\lambda_{\ell, k}| |\mathbf{c}_{\ell, k}[\xi_\ell]|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Rearranging and squaring, we get (70). \square

We now provide

Proof of Proposition 3.2. By the glueing conditions, we have

$$\int_{\theta \in I_O} \mathbf{p}(\theta) d\theta = \beta_O \mathbf{G}_O \nu_O - \beta_L \mathbf{G}_L \nu_L - \beta_R \mathbf{G}_R \nu_R = 0.$$

Hence, by Proposition 5.6, there exists $\xi \in \mathbf{H}$ such that (69) holds. For $\ell \in \Lambda$, define

$$(72) \quad \Psi_\ell^K(\theta, h) \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z} \setminus \{0\}} \mathbf{c}_{\ell, k}[\xi_\ell] \exp \left[\frac{2\pi k i \theta}{\mathbf{G}_\ell} - s_\ell \lambda_{\ell, k} h \right] + \mathbf{c}_{\ell, 0}[\xi_\ell] \quad \text{for } (\theta, h) \in \mathbb{R} \times \mathcal{R}_\ell.$$

The proof now very closely follows the proof of Proposition 8.6 in [Sow03]. \square

5.3. Some PDE Calculations Related to Skew Brownian Motion. In order to better understand the regularity of the Ψ_ℓ^K 's near $h = 0$, let's start with some calculations involving the transition density of *skew* Brownian motion. Recall that for $\gamma \in (0, 1)$, a skew Brownian motion with skewness γ behaves like a regular Brownian motion away from zero. When it hits zero, it makes an excursion to the right with probability γ and an excursion to the left with probability $1 - \gamma$.

Let $\mathbf{g}(t, x)$ be the heat kernel

$$\mathbf{g}(t, x) = \frac{1}{\sqrt{2\pi t}} \exp \left[-\frac{x^2}{2t} \right] \quad \text{for } t > 0, x \in \mathbb{R}.$$

Note that for any multiindex $\alpha \in (\mathbb{Z}^+)^2$ and $\delta > 0$, there exists a constant $K = K(\alpha, \delta) > 0$ such that

$$(73) \quad |D^\alpha \mathbf{g}(t, x)| \leq K \quad \text{for } t \geq \delta, x \in \mathbb{R}.$$

The transition density $p(t, x, y)$ of skew Brownian motion is given by the equations (see [RY99] p. 87)

$$p(t, 0, y) = 2\gamma \mathbf{g}(t, y) \chi_{\{y>0\}} + 2(1 - \gamma) \mathbf{g}(t, y) \chi_{\{y<0\}}$$

$$\begin{aligned} p(t, x, y) &= \chi_{\{x>0\}} \left[\left(\mathbf{g}(t, x - y) + (2\gamma - 1) \mathbf{g}(t, x + y) \right) \chi_{\{y>0\}} + 2(1 - \gamma) \mathbf{g}(t, x - y) \chi_{\{y<0\}} \right] + \\ &\quad \chi_{\{x<0\}} \left[\left(\mathbf{g}(t, x - y) + (1 - 2\gamma) \mathbf{g}(t, x + y) \right) \chi_{\{y<0\}} + 2\gamma \mathbf{g}(t, x - y) \chi_{\{y>0\}} \right] \end{aligned}$$

for $x = 0$ and $x \neq 0$ respectively. Let's start by collecting together a few properties of $p(t, x, y)$ to be used in the sequel.

Lemma 5.7. *Fix $y \in \mathbb{R}$. Then $p(t, x, y)$ solves*

$$(74) \quad \begin{aligned} \frac{\partial p}{\partial t}(t, x, y) &= \frac{1}{2} \frac{\partial^2 p}{\partial x^2}(t, x, y), & (t, x) \in (0, \infty) \times \mathbb{R} \setminus \{0\} \\ p(t, 0+, y) &= p(t, 0-, y), & t > 0 \\ \gamma \lim_{x \searrow 0} \frac{\partial p}{\partial x}(t, x, y) &= (1 - \gamma) \lim_{x \nearrow 0} \frac{\partial p}{\partial x}(t, x, y), & t > 0 \end{aligned}$$

Also, the map $(t, x) \mapsto p(t, x, y)$ is continuous on $(0, \infty) \times \mathbb{R}$ and C^∞ on $(0, \infty) \times \mathbb{R} \setminus \{0\}$ and for $\alpha = (\alpha_1, \alpha_2) \in (\mathbb{Z}^+)^2$, we have

$$(75) \quad \begin{aligned} \lim_{x \searrow 0} D_x^\alpha p(t, x, y) &= \lim_{x \nearrow 0} D_x^\alpha p(t, x, y) && \text{if } \alpha_2 \text{ is even} \\ \gamma \lim_{x \searrow 0} D_x^\alpha p(t, x, y) &= (1 - \gamma) \lim_{x \nearrow 0} D_x^\alpha p(t, x, y) && \text{if } \alpha_2 \text{ is odd} \end{aligned}$$

Lastly, for $\alpha \in (\mathbb{Z}^+)^2$, $\delta > 0$, there exists a constant $K = K(\alpha, \delta) > 0$ (independent of y) such that

$$(76) \quad |D_x^\alpha p(t, x, y)| \leq K, \quad t \geq \delta, x \in \mathbb{R} \setminus \{0\}.$$

Proof. Straightforward calculations. For (75), we use the fact that for $\alpha = (\alpha_1, \alpha_2) \in (\mathbb{Z}^+)^2$, $D^\alpha \mathbf{g}(t, \cdot)$ is even if α_2 is even, and odd if α_2 is odd. \square

Lemma 5.8. Let $f \in L^1(\mathbb{R})$ and define a function u by

$$u(t, x) \stackrel{\text{def}}{=} \int_{y \in \mathbb{R}} f(y) p(t, x, y) dy \quad \text{for } t > 0, x \in \mathbb{R}.$$

Then, $u(t, x)$ is continuous on $(0, \infty) \times \mathbb{R}$, u is C^∞ on $(0, \infty) \times \mathbb{R} \setminus \{0\}$ and for every multiindex $\alpha \in (\mathbb{Z}^+)^2$, we have

$$(77) \quad D^\alpha u(t, x) = \int_{y \in \mathbb{R}} f(y) D_x^\alpha p(t, x, y) dy \quad \text{for } t > 0, x \in \mathbb{R} \setminus \{0\}.$$

For $t > 0$ and $\alpha \in (\mathbb{Z}^+)^2$, the limits

$$\lim_{x \searrow 0} D^\alpha u(t, x) \quad \text{and} \quad \lim_{x \nearrow 0} D^\alpha u(t, x)$$

exist and are given by

$$(78) \quad D^\alpha u(t, 0+) = \int_{y \in \mathbb{R}} f(y) D_x^\alpha p(t, 0+, y) dy \quad \text{and} \quad D^\alpha u(t, 0-) = \int_{y \in \mathbb{R}} f(y) D_x^\alpha p(t, 0-, y) dy$$

respectively. Moreover,

$$(79) \quad \begin{aligned} D^\alpha u(t, 0+) &= D^\alpha u(t, 0-) && \text{if } \alpha_2 \text{ is even} \\ \gamma D^\alpha u(t, 0+) &= (1 - \gamma) D^\alpha u(t, 0-) && \text{if } \alpha_2 \text{ is odd.} \end{aligned}$$

Finally, u solves

$$(80) \quad \begin{aligned} \frac{\partial u}{\partial t}(t, x) &= \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(t, x), && t > 0, x \in \mathbb{R} \setminus \{0\} \\ u(t, 0+) &= u(t, 0-), && t > 0 \\ \gamma \lim_{x \searrow 0} \frac{\partial u}{\partial x}(t, x) &= (1 - \gamma) \lim_{x \nearrow 0} \frac{\partial u}{\partial x}(t, x), && t > 0 \end{aligned}$$

Proof. Straightforward calculations using Lemma 5.7. \square

Lemma 5.9. Fix $T > 0$ and suppose $u \in C^\infty((0, T) \times \mathbb{R} \setminus \{0\}) \cap L_{loc}^1([0, T] \times \mathbb{R})$ satisfies

$$(81) \quad \begin{aligned} u_t(t, x) &= \frac{1}{2} u_{xx}(t, x) && t \in (0, T), x \in \mathbb{R} \setminus \{0\}, \\ u(t, 0+) &= u(t, 0-) && \text{a.e. } t \in (0, T), \\ \gamma \lim_{x \searrow 0} \int_{t=0}^T u_x(t, x) \varphi(t) dt &= (1 - \gamma) \lim_{x \nearrow 0} \int_{t=0}^T u_x(t, x) \varphi(t) dt && \text{for all } \varphi \in C_c^\infty((0, T)), \\ \int_{x=-1}^1 \int_{t=0}^T (u_x(t, x))^2 dt dx &< \infty. \end{aligned}$$

Then $u \in C((0, T] \times \mathbb{R})$, for any multiindex $\alpha \in (\mathbb{Z}^+)^2$, $t \in (0, T]$, the limits

$$\lim_{x \searrow 0} D^\alpha u(t, x) \quad \text{and} \quad \lim_{x \nearrow 0} D^\alpha u(t, x)$$

exist and

$$(82) \quad \gamma u_x(t, 0+) = (1 - \gamma)u_x(t, 0-).$$

Proof. Suppose $\varphi \in C_c^\infty((0, T))$ and $\psi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ such that $\lim_{x \searrow 0} \psi^{(n)}(x)$ and $\lim_{x \nearrow 0} \psi^{(n)}(x)$ exist for every $n \in \mathbb{N}$. Then, for $h > 0$, we get

$$\begin{aligned} \int_{t=0}^T \int_{|x| \geq h} \dot{\varphi}(t) u(t, x) \psi(x) dx dt &= \int_{|x| \geq h} \int_{t=0}^T \dot{\varphi}(t) u(t, x) \psi(x) dt dx \\ &= - \int_{|x| \geq h} \int_{t=0}^T \varphi(t) u_t(t, x) \psi(x) dt dx \\ &= -\frac{1}{2} \int_{|x| \geq h} \int_{t=0}^T \varphi(t) u_{xx}(t, x) \psi(x) dt dx \end{aligned}$$

where we have integrated by parts (in t) and used the fact that u solves the heat equation away from $x = 0$, and that φ has compact support in $(0, T)$. Now integrating by parts (twice) with respect to x , we get

$$\int_{t=0}^T \int_{|x| \geq h} \dot{\varphi}(t) u(t, x) \psi(x) dx dt = -\frac{1}{2} \int_{t=0}^T \int_{|x| \geq h} \varphi(t) u(t, x) \ddot{\psi}(x) dx dt + B_1(h) + B_2(h)$$

where

$$\begin{aligned} B_1(h) &\stackrel{\text{def}}{=} \frac{1}{2} \int_{t=0}^T \varphi(t) u_x(t, h) \psi(h) dt - \frac{1}{2} \int_{t=0}^T \varphi(t) u_x(t, -h) \psi(-h) dt \\ B_2(h) &\stackrel{\text{def}}{=} -\frac{1}{2} \int_{t=0}^T \varphi(t) u(t, h) \dot{\psi}(h) dt + \frac{1}{2} \int_{t=0}^T \varphi(t) u(t, -h) \dot{\psi}(-h) dt \end{aligned}$$

Taking the limit as $h \searrow 0$, and replacing x and t by y and s respectively, we get

$$(83) \quad \int_{s=0}^T \int_{y \in \mathbb{R}} \dot{\varphi}(s) u(s, y) \psi(y) dy ds = -\frac{1}{2} \int_{s=0}^T \int_{y \in \mathbb{R}} \varphi(s) u(s, y) \ddot{\psi}(y) dy ds + \lim_{h \searrow 0} B_1(h) + \lim_{h \searrow 0} B_2(h)$$

Now let $\varphi(s) \stackrel{\text{def}}{=} \eta^\delta(t - s)$ and $\psi(y) \stackrel{\text{def}}{=} p(\delta, x, y)$, where $\delta \in (0, T/2)$, $t \in (\delta, T - \delta)$, $x \in \mathbb{R} \setminus \{0\}$ and η is the standard mollifier. Then

$$\dot{\varphi}(s) = -\dot{\eta}^\delta(t - s) \quad \text{and} \quad \dot{\psi}(y) = p_y(\delta, x, y)$$

Using the explicit expression for $p(\delta, x, y)$, we get

$$(1 - \gamma)\psi(0+) = \gamma\psi(0-), \quad \dot{\psi}(0+) = \dot{\psi}(0-)$$

It is now easily seen using the second and third equations in (81) that $\lim_{h \searrow 0} B_1(h) = 0$, $\lim_{h \searrow 0} B_2(h) = 0$.

Noting that $p_{yy}(\delta, x, y) = p_{xx}(\delta, x, y)$ whenever $x \neq 0$, we get

$$\int_{s=0}^T \int_{y \in \mathbb{R}} \dot{\eta}^\delta(t - s) u(s, y) p(\delta, x, y) dy ds = \frac{1}{2} \int_{s=0}^T \int_{y \in \mathbb{R}} \eta^\delta(t - s) u(s, y) p_{xx}(\delta, x, y) dy ds$$

What we have proved, in essence, is that if we define

$$u^\delta(t, x) \stackrel{\text{def}}{=} \int_{s=0}^T \int_{y \in \mathbb{R}} \eta^\delta(t - s) u(s, y) p(\delta, x, y) dy ds \quad \text{for } \delta \in (0, T/2), (t, x) \in (\delta, T - \delta) \times \mathbb{R},$$

then $u^\delta(t, x)$ solves

$$(84) \quad \begin{aligned} u_t^\delta(t, x) &= \frac{1}{2} u_{xx}^\delta(t, x) & (t, x) \in (\delta, T - \delta) \times \mathbb{R} \setminus \{0\} \\ u^\delta(t, 0+) &= u^\delta(t, 0-) & t \in (\delta, T - \delta) \\ \gamma u_x^\delta(t, 0+) &= (1 - \gamma) u_x^\delta(t, 0-) & t \in (\delta, T - \delta). \end{aligned}$$

We now use Lemma 5.8 to get, for $\delta < s < t < T - \delta$,

$$u^\delta(t, x) = \int_{y \in \mathbb{R}} p(t - s, x, y) u^\delta(s, y) dy$$

and let $\delta \searrow 0$. □

Lemma 5.10. *Let u_+ and u_- be elements of $C^\infty(\mathbb{R})$ such that for every $n \in \mathbb{N}$, $u_+^{(n)}$ and $u_-^{(n)}$ are bounded. Let $u = u(t, x)$ solve the PDE*

$$(85) \quad \begin{aligned} u_t(t, x) &= \frac{1}{2} u_{xx}(t, x) & t > 0, x \in \mathbb{R} \setminus \{0\} \\ u(t, 0+) &= u(t, 0-) & t > 0 \\ \gamma u_x(t, 0+) &= (1 - \gamma) u_x(t, 0-) & t > 0 \\ \lim_{t \searrow 0} u(t, x) &= u_+(x) \chi_{(0, \infty)}(x) + u_-(x) \chi_{(-\infty, 0)}(x) & x \in \mathbb{R}. \end{aligned}$$

Then, for every $n \in \mathbb{N}$, there is a constant $K_n > 0$ such that

$$(86) \quad \left| \frac{\partial^n u}{\partial x^n}(t, x) \right| \leq K_n \left\{ \frac{1}{t^{n/2}} \exp \left[-\frac{x^2}{4t} \right] + 1 \right\}$$

for all $t > 0, x \in \mathbb{R} \setminus \{0\}$.

Proof. Let's first consider the case $t > 0, x > 0$. Then,

$$u(t, x) = \int_{y=0}^{\infty} u_+(y) p(t, x, y) dy + \int_{y=-\infty}^0 u_-(y) p(t, x, y) dy$$

Using the expression for $p(t, x, y)$, we get

$$u(t, x) = I_1(t, x) + (2\gamma - 1)I_2(t, x) + 2(1 - \gamma)I_3(t, x)$$

where

$$\begin{aligned} I_1(t, x) &\stackrel{\text{def}}{=} \int_{y=0}^{\infty} u_+(y) \frac{1}{\sqrt{t}} \mathfrak{G} \left(\frac{x-y}{\sqrt{t}} \right) dy, \\ I_2(t, x) &\stackrel{\text{def}}{=} \int_{y=0}^{\infty} u_+(y) \frac{1}{\sqrt{t}} \mathfrak{G} \left(\frac{x+y}{\sqrt{t}} \right) dy, \\ I_3(t, x) &\stackrel{\text{def}}{=} \int_{y=-\infty}^0 u_-(y) \frac{1}{\sqrt{t}} \mathfrak{G} \left(\frac{x-y}{\sqrt{t}} \right) dy. \end{aligned}$$

By making the change of variables $z = x - y$ in I_1 and I_3 and $z = x + y$ in I_2 , the above expressions become

$$\begin{aligned} I_1(t, x) &\stackrel{\text{def}}{=} \int_{z=-\infty}^x u_+(x-z) \frac{1}{\sqrt{t}} \mathfrak{G} \left(\frac{z}{\sqrt{t}} \right) dz, \\ I_2(t, x) &\stackrel{\text{def}}{=} \int_{z=x}^{\infty} u_+(-x+z) \frac{1}{\sqrt{t}} \mathfrak{G} \left(\frac{z}{\sqrt{t}} \right) dz, \\ I_3(t, x) &\stackrel{\text{def}}{=} \int_{z=x}^{\infty} u_-(x-z) \frac{1}{\sqrt{t}} \mathfrak{G} \left(\frac{z}{\sqrt{t}} \right) dz. \end{aligned}$$

We now claim that for $n \in \mathbb{N}$,

$$\frac{\partial^n I_1}{\partial x^n}(t, x) = \int_{z=-\infty}^x u_+^{(n)}(x-z) \frac{1}{\sqrt{t}} \mathfrak{G} \left(\frac{z}{\sqrt{t}} \right) dz + \sum_{j=0}^{n-1} \frac{1}{t^{(j+1)/2}} \mathfrak{G}^{(j)} \left(\frac{x}{\sqrt{t}} \right) u_+^{(n-j-1)}(0)$$

Indeed, for $n = 1$, we have

$$\frac{\partial I_1}{\partial x}(t, x) = \int_{z=-\infty}^x u_+'(x-z) \frac{1}{\sqrt{t}} \mathfrak{G} \left(\frac{z}{\sqrt{t}} \right) dz + u_+(0) \frac{1}{\sqrt{t}} \mathfrak{G} \left(\frac{x}{\sqrt{t}} \right)$$

and the claim now follows by induction. By exactly the same line of reasoning, we have for $n \in \mathbb{N}$,

$$\frac{\partial^n I_2}{\partial x^n}(t, x) = (-1)^n \left[\int_{z=x}^{\infty} u_+^{(n)}(-x+z) \frac{1}{\sqrt{t}} \mathfrak{G}\left(\frac{z}{\sqrt{t}}\right) dz + \sum_{j=0}^{n-1} (-1)^j \frac{1}{t^{(j+1)/2}} \mathfrak{G}^{(j)}\left(\frac{x}{\sqrt{t}}\right) u_+^{(n-j-1)}(0) \right]$$

and

$$\frac{\partial^n I_3}{\partial x^n}(t, x) = \int_{z=x}^{\infty} u_-^{(n)}(x-z) \frac{1}{\sqrt{t}} \mathfrak{G}\left(\frac{z}{\sqrt{t}}\right) dz - \sum_{j=0}^{n-1} \frac{1}{t^{(j+1)/2}} \mathfrak{G}^{(j)}\left(\frac{x}{\sqrt{t}}\right) u_-^{(n-j-1)}(0)$$

Now

$$\frac{\partial^n u}{\partial x^n}(t, x) = \frac{\partial^n I_1}{\partial x^n}(t, x) + (2\gamma - 1) \frac{\partial^n I_2}{\partial x^n}(t, x) + 2(1 - \gamma) \frac{\partial^n I_3}{\partial x^n}(t, x)$$

Since $u_+^{(n)}$ and $u_-^{(n)}$ are bounded, the Gaussian integral in the expression for each $\frac{\partial^n I_j}{\partial x^n}$, $j \in \{1, 2, 3\}$ can be bounded from above by the appropriate upper bound on the integrand. Note that for each $n \in \mathbb{N}$, there is a constant K_n (not necessarily the same one as in (86)) such that

$$|\mathfrak{G}^{(n)}(z)| \leq K_n e^{-z^2/4}$$

for $z \in \mathbb{R}$. Since the dominant term in the sum involves $t^{-n/2}$, (86) now follows.

The proof for $t > 0$, $x < 0$ is similar. □

6. APPENDIX: CONSTRUCTION OF THE PRE-LIMIT PROCESS

We now discuss the construction of the pre-limit process: for $\varepsilon \in (0, 1)$, $x_0 \in \mathbf{S} \setminus \{\mathbf{o}\}$, we would like to construct a probability measure $\mathbb{P}_{x_0}^\varepsilon \in \mathcal{P}(C([0, \infty); \mathbb{R}^2))$ which solves the original stopped martingale problem. This can be accomplished by *localization*; one considers separately the questions of existence away from the origin and existence near the origin. By working with the induced laws on the path space $C([0, \infty); \mathbb{R}^2)$, it should be possible to construct $\mathbb{P}_{x_0}^\varepsilon$ using the localization techniques in [SV79] (see Section 6.6).

There are several different, yet equivalent, probabilistic characterizations of *skewness* and it is useful to take some time to examine them (see [Lej06] for more details). For simplicity, we illustrate the case of skew Brownian motion. Let Y_t be a skew Brownian motion on \mathbb{R} with parameter $\gamma \in (0, 1)$. Then Y_t can be characterized in the following terms:

- (1) SDE with local time.

It is shown in [HS81] that Y_t is a semi-martingale which is a strong solution to an SDE with local time

$$Y_t = Y_0 + B_t + (2\gamma - 1)L_t^0(Y)$$

where B_t is a Brownian motion and $L_t^0(Y)$ is the local time of the process Y at 0.

- (2) Diffusion with generalized coefficients.

Y_t can be thought of as a diffusion generated by

$$L \stackrel{\text{def}}{=} \frac{1}{2} \frac{d^2}{dx^2} + (2\gamma - 1)\delta \frac{d}{dx}$$

where δ is the Dirac δ -function concentrated at $x = 0$. The skewness translates into generalized drift concentrated at $x = 0$ with the coefficient providing the biases.

- (3) Solution of a martingale problem.

The article [Lej06] provides other characterizations and constructions as well (e.g. Dirichlet forms, scale and speed measures, etc.), but we only mention the ones we'll be using. Obviously, the three approaches outlined above extend (under suitable conditions) to multidimensional problems and operators/SDE's with variable coefficients (see [Por90], [SV86]). The martingale approach is best suited to proving limit theorems.

We first address the issue of existence of the pre-limit process near the origin. Naturally, we change to (y_1, y_2) -coordinates. The generator now takes the form

$$\tilde{\mathcal{L}}^\varepsilon = \frac{1}{\varepsilon^2} y_1 \frac{\partial}{\partial y_1} - \frac{1}{\varepsilon^2} y_2 \frac{\partial}{\partial y_2} + \frac{1}{2} \Delta.$$

By the skewness conditions (5), one can see that the probability of making a positive excursion in the y_1 direction on hitting the y_2 -axis is $\frac{\beta_O}{\beta_O + \beta_L}$ if $y_2 > 0$, and $\frac{\beta_R}{\beta_O + \beta_R}$ if $y_2 < 0$. Similarly, the probability of making a positive excursion in the y_2 direction on hitting the y_1 -axis is $\frac{\beta_O}{\beta_O + \beta_R}$ if $y_1 > 0$, and $\frac{\beta_L}{\beta_O + \beta_L}$ if $y_1 < 0$. Using (4), it is easily seen that

$$\frac{\beta_O}{\beta_O + \beta_L} = \frac{\beta_R}{\beta_O + \beta_R} \quad \text{and} \quad \frac{\beta_O}{\beta_O + \beta_R} = \frac{\beta_L}{\beta_O + \beta_L}.$$

In other words, the pre-limit process (near the origin) decouples into two *independent* skew Ornstein-Uhlenbeck processes $y_t^{1,\varepsilon}$ and $y_t^{2,\varepsilon}$, governed according to

$$\begin{aligned} \mathcal{L}_1^\varepsilon &\stackrel{\text{def}}{=} \frac{1}{\varepsilon^2} y_1 \frac{\partial}{\partial y_1} + \frac{1}{2} \frac{\partial^2}{\partial y_1^2}, & \gamma_1 &= \frac{\beta_O}{\beta_O + \beta_L}, \\ \mathcal{L}_2^\varepsilon &\stackrel{\text{def}}{=} -\frac{1}{\varepsilon^2} y_2 \frac{\partial}{\partial y_2} + \frac{1}{2} \frac{\partial^2}{\partial y_2^2}, & \gamma_2 &= \frac{\beta_L}{\beta_O + \beta_L}, \end{aligned}$$

where γ_i represents the probability of making a positive excursion (in y_i) on hitting zero, $i \in \{1, 2\}$. An alternative characterization of each $y_t^{i,\varepsilon}$ is as the solution of an SDE with local time. The existence of the

processes $y_t^{1,\varepsilon}$, $y_t^{2,\varepsilon}$ should now follow from [LeG85] (see also [Lej06]). Of course, one should then take $x_t^\varepsilon = (x_t^{1,\varepsilon}, x_t^{2,\varepsilon}) \stackrel{\text{def}}{=} \tilde{\phi}(y_t^{1,\varepsilon}, y_t^{2,\varepsilon})$.

Away from the origin, the existence of the pre-limit process should follow from the theory of *generalized diffusion processes* (see [Por90]).

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