

Analysis of Single Domain Particles

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Preface

This paper came about because of my curiosity with magnets. I think every child begins to wonder about the magic within two pieces of metal that stick to the refrigerator and that also stick together or repel each other with just half a rotation. Feeling like I know so little about magnets, which have huge impacts on life as we know it as well as believing much, much more is to come from them, I sought out Dr. Carlos Garcia-Cervera to guide me in studying them. I had previously found that he knows a great deal about computer hard drives and therefore magnetism. The research was very interesting and challenging and what I'm writing about is unlike anything I've ever attempted before. Thank you very much for teaching, guiding, and pushing me through out the course Carlos.

The purpose of this paper is to take my work and present it in a way that is readable for anyone who would like to know some "basic" magnetic properties. There will be a lot of Calculus involved, but hopefully the ideas and results will be clear even if the reader has little or no experience with the math involved. Some things will be technical, but I will try to explain and simplify them because I've found that math equations don't always speak for themselves.

Single Domain Particles

What is a single domain particle? It is simply an object with one magnetic direction throughout that entire object. So no matter what point you pick on, or in, the object, the magnetic direction, \mathbf{m} , is always pointing the same way and at the same strength.

Single Domain Particles have the same magnetic strength and direction, \mathbf{m} , everywhere.

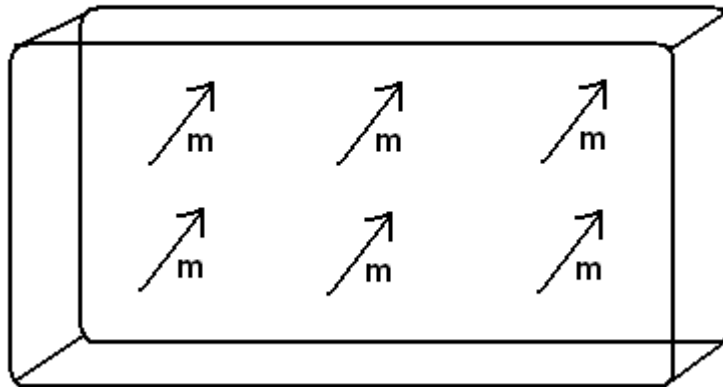


Figure 1

But how do we know which way the magnetization is pointing? This depends on a few factors like the objects shape and both internal and external magnetic energies. The shape we're interested in is the Ellipsoid because it can be similar to the shape of one bit of storage space on a computer's hard drive. From this we can try to understand a storage space's magnetic properties. Although it is not uncommon for a desktop computer to have a 120-gigabyte hard drive, which is 9.6×10^{11} bits or 960 billion bits¹, one should find a "one bit" hard drive's properties interesting, even though it is obviously less complicated.

Ellipsoid

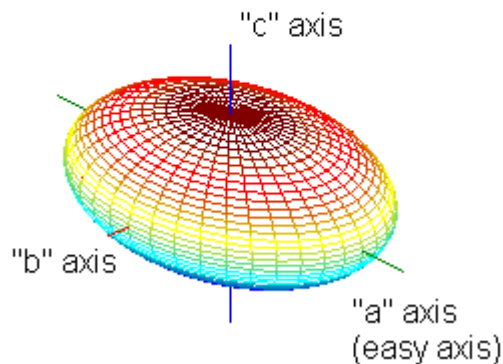


Figure 2

¹ Assuming a 120-gigabyte hard drive consists of 120 billion bytes, which isn't correct, but they are usually sold that way.

Our ellipsoid is given by the following formula: $\Omega = \frac{a^2}{a_0^2} + \frac{b^2}{b_0^2} + \frac{c^2}{c_0^2} \leq 1$. Where a_0 , b_0 , and c_0 are the lengths of the ellipsoid's radius in the "a", "b", or "c" axis direction, respectively. The "easy axis" is the longest axis and is the direction that magnetization will naturally point.

Energy

Now, let's look at the energies of our particle². There are four types:

- 1) Exchange Energy, denoted by E_x
- 2) Uniaxial Energy, denoted by E_{Ku}
- 3) External Field Energy, denoted by E_H
- 4) Stay Field Energy, denoted by E_d .

The total energy, E_{tot} , is the sum of these energies: $E_{tot} = E_x + E_{Ku} + E_H + E_d$ (all of their units are in Joules).

Exchange energy is given by: $E_x = A \int_{\Omega} \nabla^2 \mathbf{m}^2 dV$. It is the energy penalty for changing the direction of magnetization because of atomic interactions. Here $\mathbf{m} = \frac{J(r)}{J_s} = \frac{[m_1, m_2, m_3]}{J_s}$ is the strength and direction of magnetization of the ellipsoid and A is a material constant (different materials have bigger penalties than others). But why would changing the magnetic direction require energy? Well in nature, objects tend to be in a state that requires the least amount of energy according to its surrounding. It's like how soap bubbles are always spheres while floating around in the air. They only pop because something, like touching it with your finger, makes its shape change so that the new shape requires more energy than it available to hold the inner air, and so it pops. Similarly, there is a certain direction that the magnetization naturally wants to point in and it requires the input of external energy to change that direction. Recall that \mathbf{m} is the magnetization vector of the ellipsoid Ω (figure 1). Also, note that Ω is a single domain particle and \mathbf{m} is the same for all points in Ω . Therefore \mathbf{m} is constant.

Uniaxial Energy is given by: $E_{Ku} = \frac{1}{2} \int_{\Omega} Ku(m_2^2 + m_3^2) dV$. It is the magnetic energy of the object and depends upon the internal organization of the objects atoms or lattice structure but not on the overall shape of Ω . Here, Ku is a material constant and

² The following energy formulas were obtained from: *Magnetic Domains; The Analysis of Magnetic Microstructures* by Alex Hubert and Rudolf Schäfer.

m_2 and m_3 are the strengths of magnetization in the direction of the ellipsoid's "b" and "c" axis'.

External Field Energy is given by: $E_H = -J_s \int_{\Omega} \mathbf{H}_{ex} \cdot \mathbf{m} dV$. It is the interaction energy between the direction of the magnetization, \mathbf{m} , and the external magnetized field vector $\mathbf{H}_{ex} = [h_1, h_2, h_3]$. Here, J_s is a constant. Note that if the external field is uniform, then this energy only depends on the average magnetization and not on the shape of the magnetized object.

Stray Field Energy is given by: $E_d = -\frac{1}{2} \int_{\Omega} \mathbf{H}_d \cdot \mathbf{J} dV$. It is the energy of the magnetic field that Ω creates. The shape and size of Ω is what describes the Stray Field Energy. Here, $\mathbf{H}_d = -\mathbf{N} \cdot \frac{\mathbf{J}}{\mu_0} = [d_1, d_2, d_3]$ and $\mathbf{J} = \mathbf{m} J_s$, where J_s and μ_0 are constants. \mathbf{N} is called a symmetrical demagnetizing tensor. In this case \mathbf{N} is a 3x3 diagonal matrix, which will be looked at more carefully a little later. For the sake of information, when one adds up the External Field Energy with the Stray Field Energy, one gets the magnetic field energy.

Now, to make things easier to work with, let's convert every thing to polar coordinates and expand. Letting $\mathbf{m} = [\cos(\theta) \sin(\phi), \sin(\theta) \sin(\phi), \cos(\phi)]$, the previous formulas become:

$$E_x = A \int_{\Omega} (\nabla [\cos(\theta) \sin(\phi), \sin(\theta) \sin(\phi), \cos(\phi)])^2 dV = 0 \quad (\text{since } \mathbf{m} \text{ is constant})$$

$$E_{Ku} = \frac{1}{2} \int_{\Omega} Ku (\sin^2(\theta) \sin^2(\phi) + \cos^2(\phi)) dV$$

$$E_H = -J_s \int_{\Omega} h_1 \cos(\theta) \sin(\phi) + h_2 \sin(\theta) \sin(\phi) + h_3 \cos(\phi) dV$$

$$E_d = \frac{J_s^2}{2\mu_0} \int_{\Omega} d_1 \cos^2(\theta) \sin^2(\phi) + d_2 \sin^2(\theta) \sin^2(\phi) + d_3 \cos^2(\phi) dV$$

We can get the total energy by summing these energies. But first, notice that all the integrals are over the same domain and all integrands are constant (because \mathbf{m} , \mathbf{H}_d , and \mathbf{H}_{ex} are constant for our Ω). This means that we can pull out all the integrands from the integral, sum them, and end up multiplying all of them by the volume of Ω . Let the volume of $\Omega = W$ (i.e. $\int_{\Omega} dV = W$). Then the total energy involved with our object is:

$$E_{tot} = W \left[\frac{J_s^2 d_1}{2\mu_0} \cos^2(\theta) \sin^2(\phi) + \frac{Ku\mu_0 + J_s^2 d_2}{2\mu_0} \sin^2(\theta) \sin^2(\phi) + \frac{Ku\mu_0 + J_s^2 d_3}{2\mu_0} \cos^2(\phi) \right. \\ \left. - J_s h_1 \cos(\theta) \sin(\phi) - J_s h_2 \sin(\theta) \sin(\phi) - J_s h_3 \cos(\phi) \right]$$

Recall that the direction of magnetization points in the direction that requires the least amount of energy. Therefore we should minimize E_{tot} to find which directions are possible for the magnetization to point.

Energy Minimization

First, to simplify things somewhat, let's minimize E_{tot} with no External Field Energy. That is, let $\mathbf{H}_{ex} = [h_1, h_2, h_3] = [0, 0, 0]$. To check where E_{tot} is minimized, one must apply the 1st and 2nd partial derivative tests. So we have:

$$E_{tot} = \frac{W}{2} \left[\frac{J_s^2 d_1}{\mu_0} \cos^2(\theta) \sin^2(\phi) + \left(Ku + \frac{J_s^2 d_2}{\mu_0} \right) \sin^2(\theta) \sin^2(\phi) + \left(Ku + \frac{J_s^2 d_3}{\mu_0} \right) \cos^2(\phi) \right]$$

And we need to look at: $\frac{\partial E_{tot}}{\partial \theta} = 0$, $\frac{\partial E_{tot}}{\partial \phi} = 0$, and $Det \begin{pmatrix} \frac{\partial^2 E_{tot}}{\partial \theta^2} & \frac{\partial^2 E_{tot}}{\partial \theta \partial \phi} \\ \frac{\partial^2 E_{tot}}{\partial \theta \partial \phi} & \frac{\partial^2 E_{tot}}{\partial \phi^2} \end{pmatrix}$.

The reader could verify:

$$\frac{\partial E_{tot}}{\partial \theta} = \left(Ku - \frac{J_s^2 d_1}{\mu_0} + \frac{J_s^2 d_2}{\mu_0} \right) (\cos(\theta) \sin(\theta) \sin^2(\phi))$$

$$\frac{\partial^2 E_{tot}}{\partial \theta^2} = \left(Ku - \frac{J_s^2 d_1}{\mu_0} + \frac{J_s^2 d_2}{\mu_0} \right) (\cos^2(\theta) \sin^2(\phi) - \sin^2(\theta) \sin^2(\phi))$$

$$\frac{\partial E_{tot}}{\partial \phi} = \left[\frac{J_s^2 d_1}{\mu_0} \cos^2(\theta) + \left(Ku + \frac{J_s^2 d_2}{\mu_0} \right) \sin^2(\theta) - \left(Ku + \frac{J_s^2 d_3}{\mu_0} \right) \right] \cos(\phi) \sin(\phi)$$

$$\frac{\partial^2 E_{tot}}{\partial \phi^2} = \left[\frac{J_s^2 d_1}{\mu_0} \cos^2(\theta) + \left(Ku + \frac{J_s^2 d_2}{\mu_0} \right) \sin^2(\theta) - \left(Ku + \frac{J_s^2 d_3}{\mu_0} \right) \right] (\cos^2(\phi) - \sin^2(\phi))$$

$$\frac{\partial^2 E_{tot}}{\partial \theta \partial \phi} = 2(Ku - \frac{J_s^2 d_1}{\mu_0} + \frac{J_s^2 d_2}{\mu_0})(\cos(\theta) \sin(\theta) \cos(\phi) \sin(\phi))$$

Thus we find that:

$$\frac{\partial E_{tot}}{\partial \theta} = 0 \quad \text{for} \quad \begin{cases} \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \\ \phi \in [0, \pi] \end{cases} \quad \text{and} \quad \begin{cases} \phi = 0, \pi \\ \theta \in [0, 2\pi) \end{cases}$$

$$\frac{\partial E_{tot}}{\partial \phi} = 0 \quad \text{for} \quad \begin{cases} \theta = 0, \frac{\pi}{2}, \pi \\ \phi \in [0, \pi] \end{cases}$$

By the 1st partial derivative test, when $\frac{\partial E_{tot}}{\partial \theta} = 0$ and $\frac{\partial E_{tot}}{\partial \phi} = 0$, with standard angles for θ and ϕ , E_{tot} will have a minimum or a maximum. To check if it is a minimum, we must apply the 2nd partial derivative test. The 2nd partial derivative test

states that E_{tot} is minimized when $H(E_{tot}) = \begin{bmatrix} \frac{\partial^2 E_{tot}}{\partial \theta^2} & \frac{\partial^2 E_{tot}}{\partial \theta \partial \phi} \\ \frac{\partial^2 E_{tot}}{\partial \theta \partial \phi} & \frac{\partial^2 E_{tot}}{\partial \phi^2} \end{bmatrix}$ is positive definite. First

we need the determinant of $H(E_{tot})$ to be positive. Next since $\frac{\partial^2 E_{tot}}{\partial \theta \partial \phi} = 0$ for all the values that we must check, we will only need to find when $\frac{\partial^2 E_{tot}}{\partial \theta^2} > 0$ and $\frac{\partial^2 E_{tot}}{\partial \phi^2} > 0$ for

$$\begin{cases} \theta = 0 \\ \phi \in [0, 2\pi) \end{cases}, \quad \begin{cases} \theta = \pi \\ \phi \in [0, 2\pi) \end{cases}, \quad \begin{cases} \theta = 0 \\ \phi = \frac{\pi}{2} \end{cases}, \quad \begin{cases} \theta = \frac{\pi}{2} \\ \phi = \frac{\pi}{2} \end{cases}, \quad \begin{cases} \theta = \pi \\ \phi = \frac{\pi}{2} \end{cases}, \quad \text{and} \quad \begin{cases} \theta = \frac{3\pi}{2} \\ \phi = \frac{\pi}{2} \end{cases} \quad \text{to}$$

determine if we have a minimum. If $\frac{\partial^2 E_{tot}}{\partial \theta^2} = 0$ or $\frac{\partial^2 E_{tot}}{\partial \phi^2} = 0$, then the determinant of $H(E_{tot})$ will be zero and 2nd partial derivative test will be inconclusive.

First for $\begin{cases} \theta = 0 \text{ or } \pi \\ \phi \in [0, 2\pi) \end{cases}$, we have $\frac{\partial^2 E_{tot}}{\partial \theta^2} = 0$, so we have no conclusion.

For $\begin{cases} \theta = 0 \\ \phi = \frac{\pi}{2} \end{cases}$, we have $\frac{\partial^2 E_{tot}}{\partial \theta^2} > 0$ and $\frac{\partial^2 E_{tot}}{\partial \phi^2} > 0$ when $Ku + \frac{J_s^2 d_2}{\mu_0} > \frac{J_s^2 d_1}{\mu_0}$ and

$Ku + \frac{J_s^2 d_3}{\mu_0} > \frac{J_s^2 d_1}{\mu_0}$ respectively. Thus a minimum for E_{tot} has been found.

For $\begin{cases} \theta = \pi \\ \phi = \frac{\pi}{2} \end{cases}$, we have $\frac{\partial^2 E_{tot}}{\partial \theta^2} > 0$ and $\frac{\partial^2 E_{tot}}{\partial \phi^2} > 0$ when $Ku + \frac{J_s^2 d_2}{\mu_0} > \frac{J_s^2 d_1}{\mu_0}$ and $\frac{J_s^2 d_1}{\mu_0} > Ku + \frac{J_s^2 d_3}{\mu_0}$ respectively. Another minimum for E_{tot} has been found.

For $\begin{cases} \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \\ \phi = \frac{\pi}{2} \end{cases}$, we have $\frac{\partial^2 E_{tot}}{\partial \theta^2} > 0$ and $\frac{\partial^2 E_{tot}}{\partial \phi^2} > 0$ when $\frac{J_s^2 d_1}{\mu_0} > Ku + \frac{J_s^2 d_2}{\mu_0}$ and $\frac{J_s^2 d_3}{\mu_0} > \frac{J_s^2 d_2}{\mu_0}$ respectively. A 3rd minimum for E_{tot} has been found.

All the previous inequalities associated with $\frac{\partial^2 E_{tot}}{\partial \theta^2} > 0$ and $\frac{\partial^2 E_{tot}}{\partial \phi^2} > 0$ are dependant on the values of constants, and mainly on $\mathbf{H}_d = [d_1, d_2, d_3]$ of the Stray Field Energy. Recall that \mathbf{H}_d depends on the shape and size of Ω . According to this, E_{tot} is then minimized in either the "a" or "b" axis direction. Since we have already defined the a-axis as the "easy axis", d_1 will be small enough (as we will see later) for the inequalities to hold and have the direction of magnetization point in one of the directions of the "easy axis".

Now, let's minimize E_{tot} when an external field is applied in the direction of the "easy axis". That is, let $\mathbf{H}_{ex} = [h_1, h_2, h_3] = [h_1, 0, 0]$. Then:

$$E_{tot} = \frac{W}{2} \left[\frac{J_s^2 d_1}{\mu_0} \cos^2(\theta) \sin^2(\phi) + \left(Ku + \frac{J_s^2 d_2}{\mu_0} \right) \sin^2(\theta) \sin^2(\phi) \right. \\ \left. + \left(Ku + \frac{J_s^2 d_3}{\mu_0} \right) \cos^2(\phi) - J_s h_1 \cos(\theta) \sin(\phi) \right]$$

So let's find when $\frac{\partial E_{tot}}{\partial \theta} = 0$, $\frac{\partial E_{tot}}{\partial \phi} = 0$, and when $H(E_{tot})$ is positive definite.

We have:

$$\frac{\partial E_{tot}}{\partial \theta} = \left[\left(Ku - \frac{J_s^2 d_1}{\mu_0} + \frac{J_s^2 d_2}{\mu_0} \right) \cos(\theta) \sin(\phi) + J_s h_1 \right] \sin(\theta) \sin(\phi)$$

$$\frac{\partial^2 E_{tot}}{\partial \theta^2} = \left(Ku - \frac{J_s^2 d_1}{\mu_0} + \frac{J_s^2 d_2}{\mu_0} \right) (\cos^2(\theta) - \sin^2(\theta)) \sin^2(\phi) + J_s h_1 \cos(\theta) \sin(\phi)$$

$$\frac{\partial E_{tot}}{\partial \phi} = \left[\frac{J_s^2 d_1}{\mu_0} \cos^2(\theta) + \left(Ku + \frac{J_s^2 d_2}{\mu_0} \right) \sin^2(\theta) - \left(Ku + \frac{J_s^2 d_3}{\mu_0} \right) \right] \sin(\phi) - J_s h_1 \cos(\theta) \cos(\phi)$$

$$\frac{\partial^2 E_{tot}}{\partial \phi^2} = \left[\frac{J_s^2 d_1}{\mu_0} \cos^2(\theta) + \left(Ku + \frac{J_s^2 d_2}{\mu_0} \right) \sin^2(\theta) - \left(Ku + \frac{J_s^2 d_3}{\mu_0} \right) \right] (\cos^2(\phi) - \sin^2(\phi)) + J_s h_1 \cos(\theta) \sin(\phi)$$

$$\frac{\partial^2 E_{tot}}{\partial \theta \partial \phi} = 2 \left(Ku - \frac{J_s^2 d_1}{\mu_0} + \frac{J_s^2 d_2}{\mu_0} \right) \cos(\theta) \sin(\theta) \cos(\phi) \sin(\phi) + J_s h_1 \sin(\theta) \cos(\phi)$$

It directly follows that:

$$\frac{\partial E_{tot}}{\partial \theta} = 0 \quad \text{for} \quad \begin{cases} \theta = 0, \pi, \\ \phi \in [0, \pi] \end{cases} \quad \text{and} \quad \begin{cases} \phi = 0, \pi \\ \theta \in [0, 2\pi) \end{cases}$$

$$\frac{\partial E_{tot}}{\partial \phi} = 0 \quad \text{for} \quad \begin{cases} \theta = \frac{\pi}{2}, \frac{3\pi}{2} \\ \phi = 0, \pi \end{cases} \quad \text{and} \quad \begin{cases} \phi = \frac{\pi}{2} \\ \theta \in [0, 2\pi) \end{cases}$$

Recall that to check if E_{tot} is at a minimum, one must apply the 2nd partial derivative test.

So we must find when $Det \begin{pmatrix} \frac{\partial^2 E_{tot}}{\partial \theta^2} & \frac{\partial^2 E_{tot}}{\partial \theta \partial \phi} \\ \frac{\partial^2 E_{tot}}{\partial \theta \partial \phi} & \frac{\partial^2 E_{tot}}{\partial \phi^2} \end{pmatrix} > 0$, $\frac{\partial^2 E_{tot}}{\partial \theta^2} > 0$, and $\frac{\partial^2 E_{tot}}{\partial \phi^2} > 0$.

For $\begin{cases} \theta = 0 \text{ or } \pi \\ \phi = 0 \text{ or } \pi \end{cases}$, we have $Det \begin{pmatrix} 0 & 0 \\ 0 & \frac{J_s^2 d_1}{\mu_0} - Ku - \frac{J_s^2 d_3}{\mu_0} \end{pmatrix} = 0$ (it's undeterminable).

For $\begin{cases} \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \\ \phi = 0 \text{ or } \pi \end{cases}$, we have $Det \begin{pmatrix} 0 & \pm J_s h_1 \\ \pm J_s h_1 & \frac{J_s^2 d_2}{\mu_0} - \frac{J_s^2 d_3}{\mu_0} \end{pmatrix} = -(J_s h_1)^2 < 0$ (no minimum).

For $\begin{cases} \theta = 0 \\ \phi = \frac{\pi}{2} \end{cases}$, we have $Det \begin{pmatrix} Ku - \frac{J_s^2 d_1}{\mu_0} + \frac{J_s^2 d_2}{\mu_0} + J_s h_1 & 0 \\ 0 & Ku - \frac{J_s^2 d_3}{\mu_0} + \frac{J_s^2 d_1}{\mu_0} + J_s h_1 \end{pmatrix}$ which

means E_{tot} is minimized when $Ku + \frac{J_s^2 d_2}{\mu_0} + J_s h_1 > \frac{J_s^2 d_1}{\mu_0}$ and $Ku + \frac{J_s^2 d_1}{\mu_0} + J_s h_1 > \frac{J_s^2 d_3}{\mu_0}$.

$$\text{For } \begin{cases} \theta = \pi \\ \phi = \frac{\pi}{2} \end{cases} \text{ one gets } \text{Det} \begin{pmatrix} Ku - \frac{J_s^2 d_1}{\mu_0} + \frac{J_s^2 d_2}{\mu_0} - J_s h_1 & 0 \\ 0 & Ku - \frac{J_s^2 d_1}{\mu_0} + \frac{J_s^2 d_3}{\mu_0} - J_s h_1 \end{pmatrix} \text{ which}$$

means E_{tot} is minimized when $Ku + \frac{J_s^2 d_2}{\mu_0} > \frac{J_s^2 d_1}{\mu_0} + J_s h_1$ and $Ku + \frac{J_s^2 d_3}{\mu_0} > \frac{J_s^2 d_1}{\mu_0} + J_s h_1$.

$$\text{Finally, for } \begin{cases} \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \\ \phi = \frac{\pi}{2} \end{cases}, \text{ we have } \text{Det} \begin{pmatrix} \frac{J_s^2 d_1}{\mu_0} - Ku - \frac{J_s^2 d_2}{\mu_0} & 0 \\ 0 & \frac{J_s^2 d_3}{\mu_0} - \frac{J_s^2 d_2}{\mu_0} \end{pmatrix} \text{ which}$$

means E_{tot} is minimized when $\frac{J_s^2 d_1}{\mu_0} > Ku + \frac{J_s^2 d_2}{\mu_0}$ and $\frac{J_s^2 d_3}{\mu_0} > \frac{J_s^2 d_2}{\mu_0}$.

Combining these results of no external energy and an external field along the "easy axis", we find that the direction of magnetization, \mathbf{m} , can only be in the direction of

the "easy axis". The "easy axis" corresponds to the angles $\begin{cases} \theta = 0 \\ \phi = \frac{\pi}{2} \end{cases}$ and $\begin{cases} \theta = \pi \\ \phi = \frac{\pi}{2} \end{cases}$. But

what happens when we change the only thing that isn't a constant? That is, what happens when the applied field is increased, decreased, or reversed?

The Hysteresis Loop

From the previous minimization inequalities, we found that \mathbf{m} will be pointing in the direction $\begin{cases} \theta = 0 \\ \phi = \frac{\pi}{2} \end{cases}$, which is the positive "easy axis" direction, whenever the external

field is strong enough so $h_1 > \max[(d_1 - d_3) \frac{J_s}{\mu_0} - \frac{Ku}{J_s}, (d_1 - d_2) \frac{J_s}{\mu_0} - \frac{Ku}{J_s}]$ is satisfied.

Conversely, the direction of \mathbf{m} will be $\begin{cases} \theta = \pi \\ \phi = \frac{\pi}{2} \end{cases}$, the negative "easy axis" direction,

whenever $h_1 < \min[(d_3 - d_1) \frac{J_s}{\mu_0} + \frac{Ku}{J_s}, (d_2 - d_1) \frac{J_s}{\mu_0} + \frac{Ku}{J_s}]$ is satisfied.

This implies that whichever way \mathbf{m} is pointing, the direction will not change if h_1 is pointing in the same direction as \mathbf{m} or if h_1 equals 0. But if one applies a strong enough h_1 in the opposite direction of \mathbf{m} , h_1 will force \mathbf{m} to switch directions. This is because Ω would no longer be in a state of minimum energy. The end result is that if the external energy, h_1 , is made large enough, it will dominate the inequalities and force the direction of magnetization, \mathbf{m} , to be in the same direction as h_1 . A final situation worth describing is when a weak force of h_1 is applied in the opposite direction of \mathbf{m} . Based on the inequalities of minimization, a weak h_1 , will not affect the energy minimization and so \mathbf{m} will not change direction.

For the following figure, we adopt the convention of stating that if \mathbf{m} is pointing in the positive direction, then its direction is $\mathbf{1}$, and $-\mathbf{1}$ if it's pointing in the negative direction. Figure 3 is a graph of the physical property of magnets that is called the Hysteresis Loop.

The Hysteresis Loop

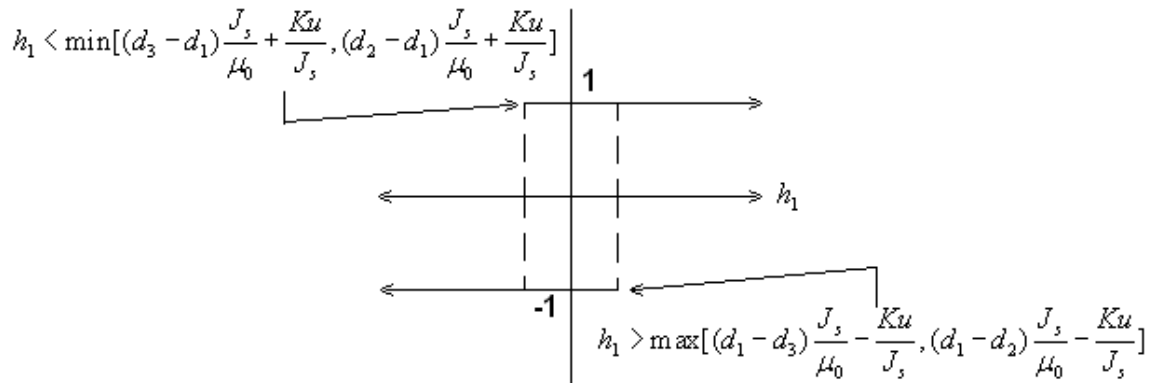


Figure 3

To explain the Hysteresis Loop, let's say \mathbf{m} 's direction is $\mathbf{1}$ and h_1 is largely positive. If we were to gradually take h_1 to zero, then \mathbf{m} 's direction will still be $\mathbf{1}$. Now as we start to make h_1 negative, \mathbf{m} 's direction will remain $\mathbf{1}$ until one reaches the value where $h_1 < \min[(d_3 - d_1) \frac{J_s}{\mu_0} + \frac{Ku}{J_s}, (d_2 - d_1) \frac{J_s}{\mu_0} + \frac{Ku}{J_s}]$. Then the direction of \mathbf{m} switches to $-\mathbf{1}$ (along the dashed line on the left) because the force of h_1 would force E_{tot} to no longer be in a state of minimum energy. As we keep increasing h_1 in the negative direction, the direction of \mathbf{m} will still be $-\mathbf{1}$. Then as we bring h_1 back to zero and start to get to where h_1 is positive again, \mathbf{m} still points in the $-\mathbf{1}$ direction. But as soon as we get to a value for h_1 where the inequality $h_1 > \max[(d_1 - d_3) \frac{J_s}{\mu_0} - \frac{Ku}{J_s}, (d_1 - d_2) \frac{J_s}{\mu_0} - \frac{Ku}{J_s}]$ holds, the direction of \mathbf{m} switches back to $\mathbf{1}$ (along the dashed line on the right). So, \mathbf{m}

doesn't follow the same path when it changes direction. This is how the "loop" of the Hysteresis Loop is created and why magnets have a "memory". That is to say: once a magnet is magnetized in a certain direction, it will point in that direction until an external force makes it point in another direction.

This fact forms the basis of how a computer hard drive works since it is what gives a computer its "memory". The writer of a hard drive creates an external magnetic field, h_1 , and tells the direction, \mathbf{m} , of the bit which way to point. One direction is read as a "1" and the other is read as a "0". The direction that a hard drive's bits are pointing is how a computer interprets these 1's or 0's as it reads a file.

Shape and Size

It is now time to explore the properties of \mathbf{N} (from Stray Field Energy). Specifically will examine why $|d_1|$ must be small when the a-axis is the "easy axis".

Recall that the Stray Field Energy is $E_d = -\frac{1}{2} \int_{\Omega} \mathbf{H}_d \cdot \mathbf{J} dV$ where

$$\mathbf{H}_d = -\mathbf{N} \cdot \frac{\mathbf{J}}{\mu_0} = [d_1, d_2, d_3]. \text{ Because of the dot product, } \mathbf{N} = \begin{pmatrix} N_a & 0 & 0 \\ 0 & N_b & 0 \\ 0 & 0 & N_c \end{pmatrix} \text{ is what}$$

assigns how much magnitude of \mathbf{J} goes into each entry of $[d_1, d_2, d_3]$. Each diagonal entry of \mathbf{N} , is dependent on the shape of the Ellipsoid. \mathbf{N} 's diagonal entries are given by the following formulas:

$$N_a = \frac{a_0 b_0 c_0}{2} \int_0^{\infty} \left[(a_0^2 + \eta) \sqrt{(a_0^2 + \eta)(b_0^2 + \eta)(c_0^2 + \eta)} \right]^{-1} d\eta$$

$$N_b = \frac{a_0 b_0 c_0}{2} \int_0^{\infty} \left[(b_0^2 + \eta) \sqrt{(a_0^2 + \eta)(b_0^2 + \eta)(c_0^2 + \eta)} \right]^{-1} d\eta$$

$$N_c = \frac{a_0 b_0 c_0}{2} \int_0^{\infty} \left[(c_0^2 + \eta) \sqrt{(a_0^2 + \eta)(b_0^2 + \eta)(c_0^2 + \eta)} \right]^{-1} d\eta$$

Let's briefly talk about size. If we set $a_0 = b_0 = c_0$ (making the ellipsoid a sphere), then we would find that $N_a = N_b = N_c = \frac{1}{3}$. Since we did not assign an actual length to the length of the sphere's radius, it suggests that shape, not size, is what affects the values of \mathbf{N} . To better study the shape of an object, we can study the ratio of lengths between a_0 , b_0 , and c_0 . To do this efficiently, we must apply a change of variables. Let

$$\eta = a_0^2 x \Rightarrow d\eta = a_0^2 dx \text{ and let } \lambda = \frac{b_0}{a_0} \text{ and } \mu = \frac{c_0}{a_0}. \text{ Thus:}$$

$$\begin{aligned}
N_a &= \frac{a_0 b_0 c_0}{2} \int_0^\infty \left[(a_0^2 + \eta) \sqrt{(a_0^2 + \eta)(b_0^2 + \eta)(c_0^2 + \eta)} \right]^{-1} d\eta \\
&= \frac{a_0^3 b_0 c_0}{2} \int_0^\infty \left[(a_0^2 + a_0^2 x) \sqrt{(a_0^2 + a_0^2 x)(b_0^2 + a_0^2 x)(c_0^2 + a_0^2 x)} \right]^{-1} dx \\
&= \frac{a_0^3 b_0 c_0}{2} \int_0^\infty \left[a_0^2 (1+x) \sqrt{a_0^6 (1+x) \left(\frac{b_0^2}{a_0^2} + x\right) \left(\frac{c_0^2}{a_0^2} + x\right)} \right]^{-1} dx \\
&= \frac{a_0^3 b_0 c_0}{2} \int_0^\infty \left[a_0^5 (1+x) \sqrt{(1+x)(\lambda^2 + x)(\mu^2 + x)} \right]^{-1} dx \\
&= \frac{a_0^3 b_0 c_0}{2a_0^5} \int_0^\infty \left[(1+x) \sqrt{(1+x)(\lambda^2 + x)(\mu^2 + x)} \right]^{-1} dx \\
&= \frac{b_0 c_0}{2a_0^2} \int_0^\infty \left[(1+x) \sqrt{(1+x)(\lambda^2 + x)(\mu^2 + x)} \right]^{-1} dx \\
&= \frac{1}{2} \frac{b_0}{a_0} \frac{c_0}{a_0} \int_0^\infty \left[(1+x) \sqrt{(1+x)(\lambda^2 + x)(\mu^2 + x)} \right]^{-1} dx \\
&= \frac{\lambda \mu}{2} \int_0^\infty \left[(1+x) \sqrt{(1+x)(\lambda^2 + x)(\mu^2 + x)} \right]^{-1} dx
\end{aligned}$$

A similar process can be performed for N_b and N_c with the following results:

$$\begin{aligned}
N_a &= \frac{\lambda \mu}{2} \int_0^\infty (1+x)^{-\frac{3}{2}} (\lambda^2 + x)^{-\frac{1}{2}} (\mu^2 + x)^{-\frac{1}{2}} dx \\
N_b &= \frac{\lambda \mu}{2} \int_0^\infty (1+x)^{-\frac{1}{2}} (\lambda^2 + x)^{-\frac{3}{2}} (\mu^2 + x)^{-\frac{1}{2}} dx \\
N_c &= \frac{\lambda \mu}{2} \int_0^\infty (1+x)^{-\frac{1}{2}} (\lambda^2 + x)^{-\frac{1}{2}} (\mu^2 + x)^{-\frac{3}{2}} dx
\end{aligned}$$

Now we are ready to see the effects of having different size ratios. There are two main properties of N_a , N_b , N_c , which are as follows:

- 1) $N_a + N_b + N_c = 1$
- 2) $0 < N_a, N_b, N_c < 1$

Property 1) implies that each entry of \mathbf{N} is dependent on the other entries. The following figures demonstrate this by fixing μ and letting λ vary.

(Note: In the following graphs, b/a is actually $\lambda = \frac{b_0}{a_0}$ and c/a is actually $\mu = \frac{c_0}{a_0}$.)

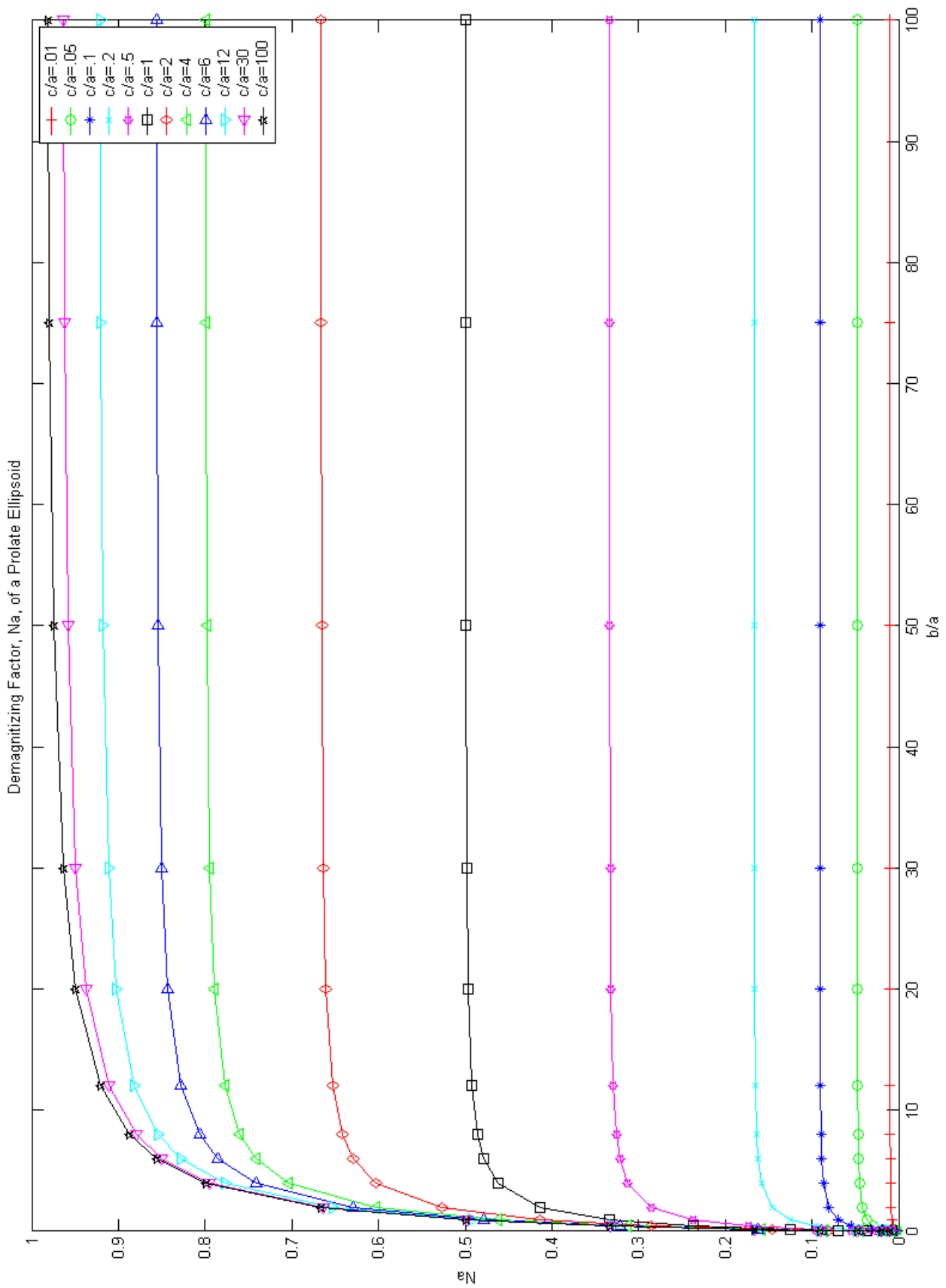


Figure 4

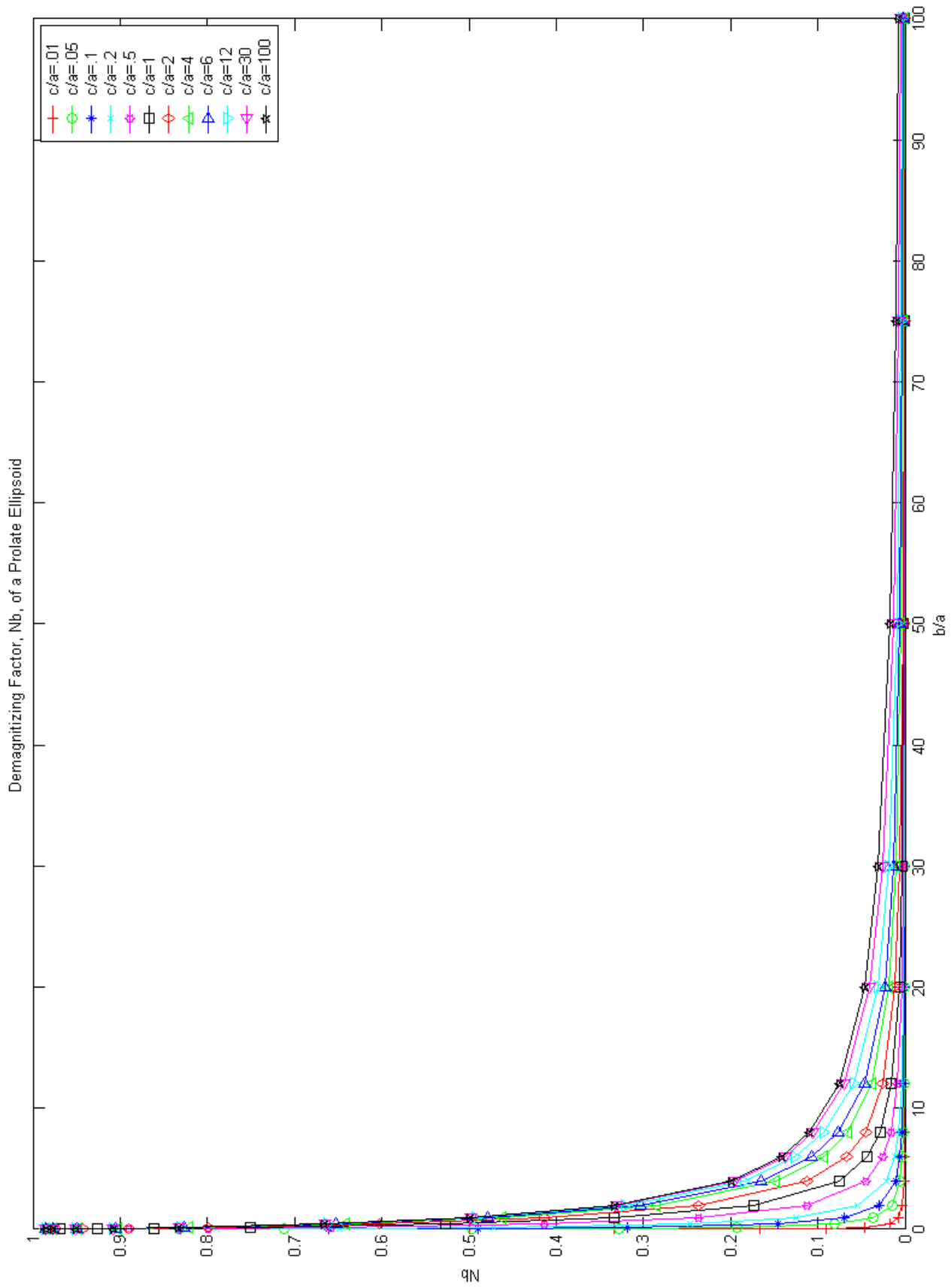


Figure 5

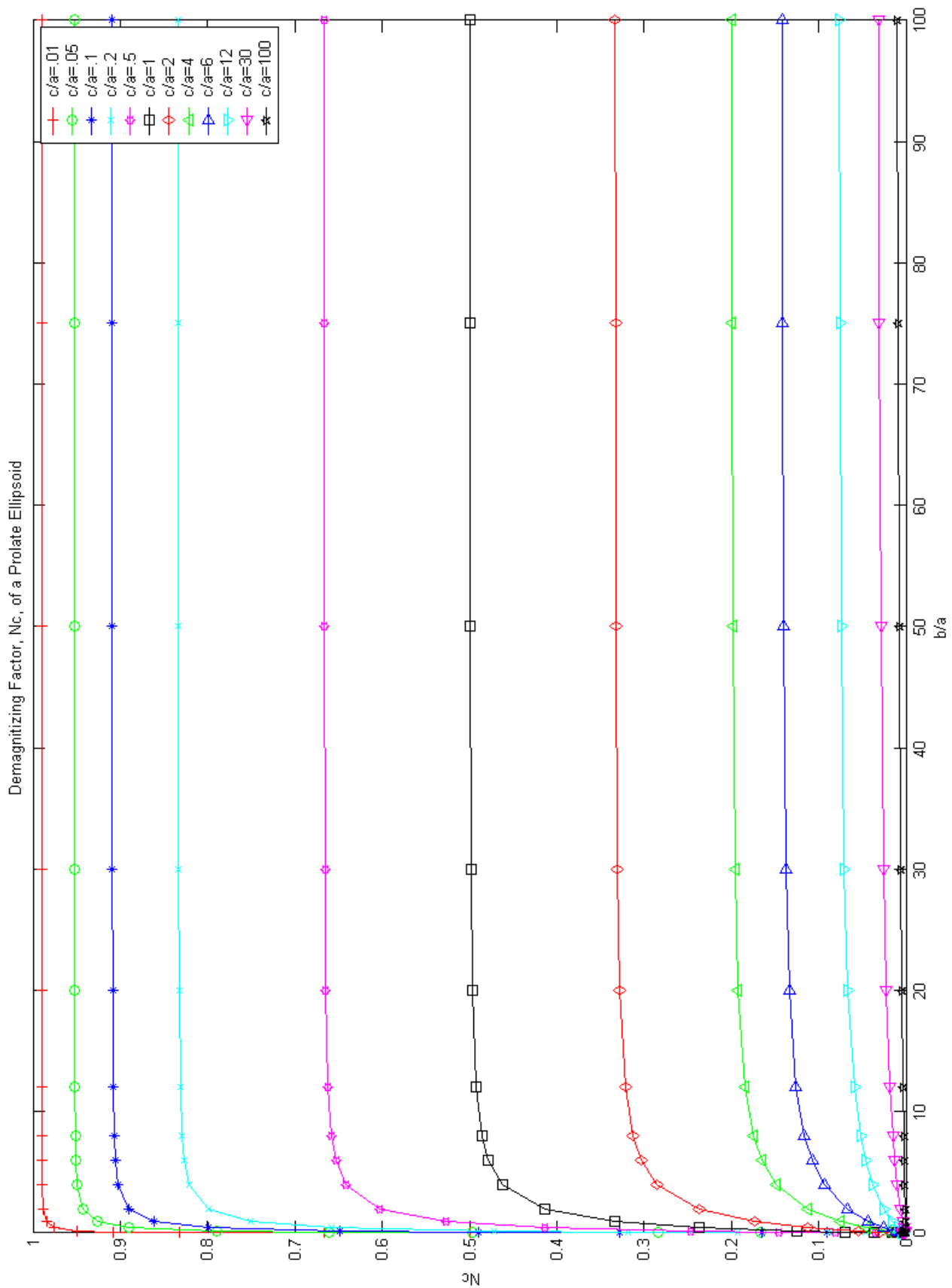


Figure 6

Since it may have been difficult to see properties 1) and 2) from the previous diagrams, here is a diagram of N_a , N_b , and N_c on the same page. Also, to make it easier to see, the following diagram considers only one value of μ and has had the logarithmic function applied to the values of λ , giving the graphs a gentler curve.

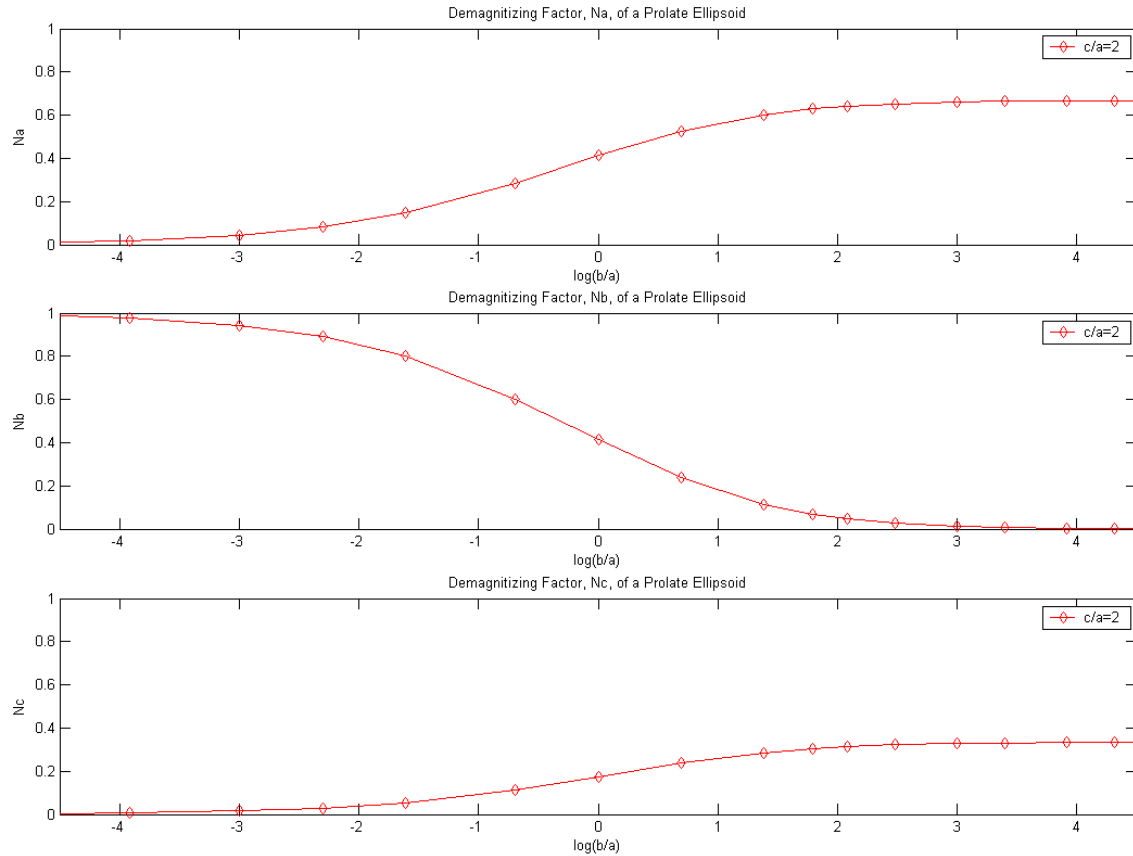


Figure 7

To see property 1), look at $\log(b/a) = 3$ on each curve. Notice that N_a is a little bit more .6, N_b is almost 0, and that N_c is a little less than .4. Summing these values give the value 1. To see property 2), simply note that each curve is between 0 and 1.

Now let's see why $|d_1|$, of the Stray Field Energy, is small when the a-axis is the "easy axis". Recall that $-\mathbf{N} \cdot \frac{\mathbf{J}}{\mu_0} = [d_1, d_2, d_3]$, where $\frac{\mathbf{J}}{\mu_0}$ is a constant,

$$\mathbf{N} = \begin{pmatrix} N_a & 0 & 0 \\ 0 & N_b & 0 \\ 0 & 0 & N_c \end{pmatrix}, \text{ and that the "easy axis" is the longest axis. Thus, since the a-axis}$$

is the "easy axis", b/a and c/a are both less than 1 and because of the dot product $|d_1|$ is entirely dependant on N_a . So recalling Figure 4, we can see that N_a , and therefore $|d_1|$,

is small when $b/a < 1$ and $c/a < 1$. The physical interpretation of this is that the smaller the value of N_a implies a small amount of energy, d_1 , is associated with "easy axis" when it is the same direction as the magnetization.

Summary of Basics

We have now seen that single domain particles have a uniform magnetic direction, they can be very small, and that there are several different forms of energy that determine the properties of its magnetization. We also found that the direction of magnetism will naturally point along the "easy axis", the longest line through the particle, when there is no external force. After that, we looked at the properties of the magnetic direction when an external force is applied along the "easy axis" and discovered of the Hysteresis Loop property. This loop is the property that allows a magnet to retain its direction despite an opposing external force. Finally, we saw how size has no effect on magnetic direction and that it is actually dependant on the particle's shape, or it's relative axis lengths described by the formulas N_a , N_b , and N_c . So the direction of magnetization mainly depends upon the magnet's composition, shape, and surrounding external forces.

Analysis

As just stated in the Summary of Basics, a single domain particle can be small. We are actually very interested in what happens when the "easy axis" is much longer than the others. That is what happens when $\lambda \rightarrow 0$ and $\mu \rightarrow 0$. We care about what happens when $\lambda \rightarrow 0$ or $\mu \rightarrow 0$ because this corresponds to procedures like decreasing the size of bits on a computer hard drive to allow for more memory in the same amount of space. However, this leads to a problem with the formulas:

$$N_a = \frac{\lambda\mu}{2} \int_0^\infty (1+x)^{-\frac{3}{2}} (\lambda^2+x)^{-\frac{1}{2}} (\mu^2+x)^{-\frac{1}{2}} dx$$

$$N_b = \frac{\lambda\mu}{2} \int_0^\infty (1+x)^{-\frac{1}{2}} (\lambda^2+x)^{-\frac{3}{2}} (\mu^2+x)^{-\frac{1}{2}} dx$$

$$N_c = \frac{\lambda\mu}{2} \int_0^\infty (1+x)^{-\frac{1}{2}} (\lambda^2+x)^{-\frac{1}{2}} (\mu^2+x)^{-\frac{3}{2}} dx$$

Notice that lower value of the integral is zero and that when we take $\lim_{\lambda \rightarrow 0}$ or $\lim_{\mu \rightarrow 0}$ of N_a ,

N_b , and N_c , the integrand is $\frac{1}{0}$. So we must make sure that the integral exist for each formula as $\lambda \rightarrow 0$ or $\mu \rightarrow 0$. Recall that from Figures 4-6, we had $N_a \rightarrow 0$, $N_b \rightarrow 1$, and $N_c \rightarrow 0$ when $\lambda \rightarrow 0$. Therefore, we will be looking for properties of the integrals that correspond with the preceding sentence and each formula's leading coefficients.

Consider N_a with out the coefficient $\frac{\lambda\mu}{2}$, fix μ , and assume $N_a(\lambda, \mu) = C\lambda^\alpha$, for some $C \in \mathbb{R}$, then we can solve for α using logarithmic properties:

$$N_a(\lambda, \mu) = C\lambda^\alpha \Rightarrow \frac{N_a(\lambda, \mu)}{N_a(\lambda/2, \mu)} = \frac{C\lambda^\alpha}{C(\lambda/2)^\alpha} = 2^\alpha \Rightarrow \log\left(\frac{N_a(\lambda, \mu)}{N_a(\lambda/2, \mu)}\right) = \alpha \log(2)$$

$$\therefore \frac{\log(N_a(\lambda, \mu)) - \log(N_a(\lambda/2, \mu))}{\log(2)} = \alpha$$

This can be done in the same manner for N_b and N_c . Furthermore, we only need to study the integrals using λ because λ and μ are symmetric between the formulas. (Notice that λ and μ are symmetric in N_a while λ and μ are symmetric between N_b and N_c).

Numerically we can solve for α , which will act as a guide as to what to expect for the asymptotic behavior of the integrals in N_a , N_b , and N_c as $\lambda \rightarrow 0$ or $\mu \rightarrow 0$. The next figure shows the behavior of α as $\lambda \rightarrow 0$ with $\mu = 1$. (Note: In the following graph, b/a is actually $\lambda = \frac{b_0}{a_0}$ and c/a is actually $\mu = \frac{c_0}{a_0}$.)

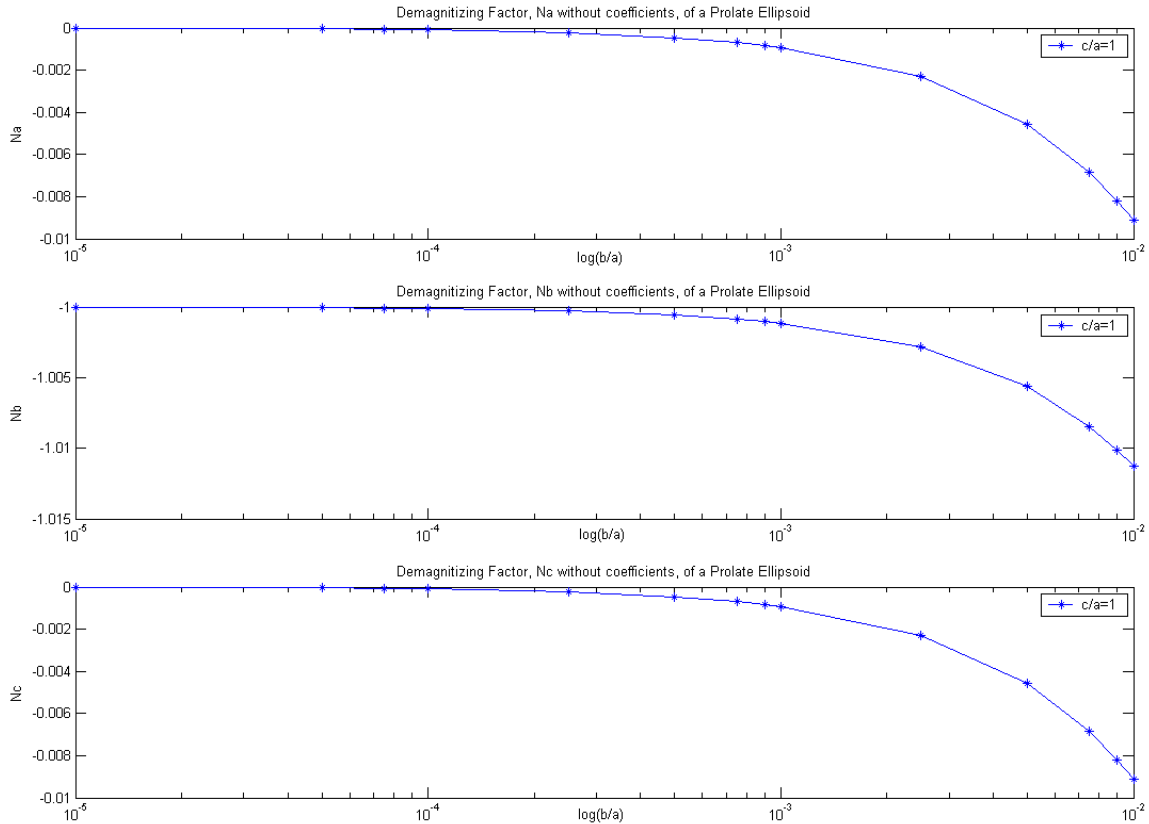


Figure 8

From the top graph, which corresponds to $N_a(\lambda, 1) = C\lambda^\alpha$, we can see that $\alpha \rightarrow 0$ as $\lambda \rightarrow 0$. This means that we should expect the integrand of N_a to converge to some constant $C \in \mathbb{R}$ and its integral will exist. We should expect the same result for N_c as well since $\alpha \rightarrow 0$ as $\lambda \rightarrow 0$. However, for N_b we get that $\alpha \rightarrow -1$. This means that $N_b(\lambda, \mu) = \frac{C}{\lambda}$, which means the integrand of N_b goes to infinity as $\lambda \rightarrow 0$ and therefore the integral should not exist. It does not look good for N_b at this point, but maybe we can find a more desirable answer with some asymptotic analysis.

First, let's study $N_a = \frac{\lambda\mu}{2} \int_0^\infty (1+x)^{-\frac{3}{2}} (\lambda^2+x)^{-\frac{1}{2}} (\mu^2+x)^{-\frac{1}{2}} dx$ rigorously as $\lambda \rightarrow 0$ to confirm the numerical results. So fix $\mu > 0$, and let $f(x) = (1+x)^{-3/2} (\mu^2+x)^{-1/2}$. Since $f(x)$ is independent of λ , we need to only to study $N(\lambda) = \int_0^\infty \frac{f(x)}{\sqrt{\lambda+x}} dx$, where λ^2 was replaced with λ . We want to show that we can represent $N(\lambda)$ with a Taylor expansion of the form: $N(\lambda) = N(0) + a\sqrt{\lambda} + b\lambda + O(\lambda)$, where $O(\lambda)$ includes all terms that converge to zero faster than λ as $\lambda \rightarrow 0$.

Notice that $f(x)$ has infinitely many continuous derivatives, $f(0) = \mu^{-1/2}$, and that $f(x) \leq x^{-2}$ for all $x > 0$, which means that $f(x)$ is bounded for all x . We also have the following inequality $\frac{f(x)}{\sqrt{\lambda+x}} \leq \frac{f(x)}{\sqrt{x}}$ for all $x > 0$. Since $\int_0^1 \frac{1}{\sqrt{x}} dx = 2$ and $f(x)$ is bounded for $0 < x < 1$, we have $\int_0^1 \frac{f(x)}{\sqrt{x}} dx < \infty$. Furthermore, since $f(x) \leq x^{-2}$ for $x \geq 1$ and $\int_1^\infty \frac{1}{x^{5/2}} dx = \frac{2}{3}$, we finally get the inequality $\int_0^\infty \frac{f(x)}{\sqrt{x}} dx < \infty$. Therefore, $N(0) = \int_0^\infty \frac{f(x)}{\sqrt{x}} dx = \int_0^\infty \frac{1}{(1+x)^{3/2} \sqrt{\mu^2+x} \sqrt{x}} dx$ exists and can be numerically approximated once $\mu > 0$ is fixed.

For the next term in the asymptotic expansion of $N(\lambda)$, we need to split the integral in the following manner: $N(\lambda) = \int_0^1 \frac{f(x)}{\sqrt{\lambda+x}} dx + \int_1^\infty \frac{f(x)}{\sqrt{\lambda+x}} dx$. The integral on the right has no problems since the lower limit is 1 and does not create a singularity in the integrand. So let's turn our attention to the integral on the left. Since $f(x)$ has nice properties, we can get a good idea of what's happening by looking at the integral without

$f(x)$ in it: $\int_0^1 \frac{1}{\sqrt{\lambda+x}} dx = 2\sqrt{1+\lambda} - 2\sqrt{\lambda}$. The term $2\sqrt{1+\lambda}$ has infinitely many continuous derivatives so it doesn't cause any problems. However, the first derivative of $2\sqrt{\lambda}$ is singular when $\lambda=0$ and is responsible for the singularities of the integrand. So we are able to use the Taylor's expansion $N(\lambda) = N(0) + a\sqrt{\lambda} + O(\lambda)$ to represent $N(\lambda)$. Now we just need to show that a exists.

Notice that $a = \lim_{\lambda \rightarrow 0} \frac{N(\lambda) - N(0)}{\sqrt{\lambda}}$ and can be computed:

$$\begin{aligned} \frac{N(\lambda) - N(0)}{\sqrt{\lambda}} &= \frac{1}{\sqrt{\lambda}} \int_0^\infty f(x) \left(\frac{1}{\sqrt{\lambda+x}} - \frac{1}{\sqrt{x}} \right) dx = \frac{1}{\sqrt{\lambda}} \int_0^\infty f(x) \frac{\sqrt{x} - \sqrt{\lambda+x}}{\sqrt{\lambda+x}\sqrt{x}} dx \\ &= -\frac{1}{\sqrt{\lambda}} \int_0^\infty f(x) \frac{\lambda}{\sqrt{\lambda+x}\sqrt{x}(\sqrt{x} + \sqrt{\lambda+x})} dx \end{aligned}$$

Next, after doing the change of variables $x = \lambda s \Rightarrow dx = \lambda ds$, one gets:

$$\begin{aligned} \frac{N(\lambda) - N(0)}{\sqrt{\lambda}} &= -\frac{1}{\sqrt{\lambda}} \int_0^\infty f(\lambda s) \frac{\lambda^2}{\sqrt{\lambda + \lambda s}\sqrt{\lambda s}(\sqrt{\lambda s} + \sqrt{\lambda + \lambda s})} ds \\ &= -\int_0^\infty f(\lambda s) \frac{1}{\sqrt{1+s}\sqrt{s}(\sqrt{s} + \sqrt{1+s})} ds \end{aligned}$$

Using this, we can take the limit and solve for a :

$$a = \lim_{\lambda \rightarrow 0} \frac{N(\lambda) - N(0)}{\sqrt{\lambda}} = -\int_0^\infty f(0) \frac{1}{\sqrt{1+s}\sqrt{s}(\sqrt{s} + \sqrt{1+s})} ds$$

This integral exists and is finite since when we are close to the lower limit of zero the integrand diverges like $\frac{1}{\sqrt{s}}$, which is integrable and as we approach the upper limit of ∞ the integrand converges to zero like $\frac{1}{\sqrt{s^3}}$.

Replacing λ back with λ^2 , we get $N(\lambda) = N(0) + a\lambda + O(\lambda^2)$. So we have $N_a = \frac{\lambda\mu}{2} N(\lambda) = \frac{\lambda\mu}{2} [N(0) + a\lambda + o(\lambda^2)]$ and $\lim_{\lambda \rightarrow 0} N_a \rightarrow 0$, since $\lim_{\lambda \rightarrow 0} N(\lambda) \rightarrow N(0)$ which is just some constant. Therefore $\frac{\lambda\mu}{2} N(0) \rightarrow 0$ as $\lambda \rightarrow 0$, and hence $\lim_{\lambda \rightarrow 0} N_a \rightarrow 0$ just as we expected from the numerical computations.

Next, let's study $N_c = \frac{\lambda\mu}{2} \int_0^\infty (1+x)^{-\frac{1}{2}} (\lambda^2+x)^{-\frac{1}{2}} (\mu^2+x)^{-\frac{3}{2}} dx$ rigorously as $\lambda \rightarrow 0$. For some fixed value $\mu > 0$, let $g(x) = (1+x)^{-1/2} (\mu^2+x)^{-3/2}$. Since $g(x)$ is independent of λ , we need to only to study $N(\lambda) = \int_0^\infty \frac{g(x)}{\sqrt{\lambda+x}} dx$, where λ^2 was replaced with λ . Notice that this setup is the same what we had for N_a . Then $g(x)$ and $f(x)$ have the same properties, and the $N(\lambda)$ of N_c will have the exact same analysis and results as N_a . Therefore $\lim_{\lambda \rightarrow 0} N_c \rightarrow 0$ is the same as $\lim_{\lambda \rightarrow 0} N_a \rightarrow 0$.

Finally, let's see what is happening for N_b . Recall that we are already expecting its integrand to diverge to ∞ , and we could easily be stuck with this if we were to attempt the same analysis that was done for N_a . However, we can get an idea of its asymptotic behavior as $\lambda \rightarrow 0$ since we know the behavior of N_a and N_c as $\lambda \rightarrow 0$ and by using the property $N_a + N_b + N_c = 1$.

We know $N_a + N_b + N_c = 1 \quad \forall \lambda > 0$. This means that $\lim_{\lambda \rightarrow 0} (N_a + N_b + N_c) = 1$ too. But, as we can see from Figure 6, $\lim_{\lambda \rightarrow 0} (N_a + N_c) = 0$ implying that $\lim_{\lambda \rightarrow 0} N_b = 1$. But if we let $N(\lambda)$ represent N_b with out the leading λ coefficient, we'll have $\lim_{\lambda \rightarrow 0} \lambda N(\lambda) = 1$. This implies that we must have $\lim_{\lambda \rightarrow 0} N(\lambda) = \lim_{\lambda \rightarrow 0} \frac{1}{\lambda}$. So we have that the integral of N_b diverges like $\frac{1}{\lambda} \rightarrow \infty$ when $\lambda \rightarrow 0$.

Summary of Analysis

Now we have an understanding of the asymptotic behavior of N_a , N_b , and N_c for when $\lambda \rightarrow 0$ or $\mu \rightarrow 0$. This is useful for, and corresponds to, changing the Single Domain Particle from a 3-D ellipsoid to as near a 2-D ellipse as possible. Therefore it will be nearly impossible to have the magnetic direction point in the 3rd dimension, but still can change directions about the "easy axis" since it can travel in the other two dimensions. This is important because a computer hard drive manufacturer wants to have bits that are manipulated in the most restricted environment as possible to avoid accidental magnetic direction change of each individual bit. So an understanding of the asymptotic behavior of magnetization on ever shrinking bit size is crucial to understand for efficient writing and efficient corruption protection.