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Analytic description of layer undulations in smectic A liquid crystals

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Abstract

We investigate the layer undulations that appear in smectic A liquid crystals when a magnetic field is applied in the direction parallel to the smectic layers. In an earlier work [12], the authors characterized the critical field using the Landau-de Gennes model for smectic A liquid crystals. In this paper, we obtain an asymptotic expression of the unstable modes using Γ-convergence theory, and a sharp estimate of the critical field. Under the assumption that the layers are fixed at the boundaries, the maximum layer undulation occurs in the middle of the cell and the displacement amplitude decreases near the boundaries. Our estimate of the critical field is consistent with the Helfrich-Hurault theory. When natural boundary conditions are considered, the displacement amplitude does not diminish near the boundary, in sharp contrast with the Dirichlet case, and the critical field is reduced compared to the one calculated in the classical theory. This is consistent with the experiments carried out by Lavrentovich et al. [17]. Furthermore, we prove the existence and stability of the solution to the nonlinear system of Landau-de Gennes model using bifurcation theory. Numerical simulations are used to illustrate the predictions of the analysis.

1. Introduction

Liquid crystal phases form when a material has a degree of positional or orientational ordering yet stays in a liquid state. In the nematic state, molecules tend to align themselves along a preferred direction with no positional order of centers of mass. The unit vector field $\mathbf{n}$, nematic director, represents the average direction of molecular alignment. Moreover, if the liquid crystal is chiral, $\mathbf{n}$ follows a helical pattern, with temperature dependent pitch. Upon lowering the temperature, or increasing concentration,

according to whether the liquid crystal is thermotropic or lyotropic, the nematic liquid crystal experiences a transition to the smectic A phase with molecules arranged along equally spaced layers. The molecules tend to align themselves along the direction perpendicular to the layers.

We consider a smectic A liquid crystal confined between two flat plates and uniformly aligned in a way that the smectic layers are parallel to the bounding plates. If a magnetic field is applied in the direction parallel to the smectic layers, an instability occurs above a threshold magnetic field. When the magnetic field reaches this critical threshold, periodic layer undulations are observed. This phenomenon is called the Helfrich-Hurault effect (see [15] and [16]). We study this phenomenon analytically by considering the minimizer of the second variation of the Landau-de Gennes free energy ([8]) at the undeformed state and by carrying out a bifurcation analysis of the nontrivial solution curve. We also perform numerical simulations to illustrate the results of our analysis.

The Helfrich-Hurault effect in a lamellar system can be caused by either magnetic/electric fields ([9], [12], [17], [22], [24], [26]) or by mechanical tension ([9], [23]). In this paper, we study the magnetic field driven instabilities in smectic A liquid crystals. Helfrich and Hurault proposed the model that can explain the periodic perturbations in cholesteric liquid crystals under a magnetic field or an electric field applied parallel to the helical axis ([15], [16]) in an infinite sample. They equated the director and the layer normal and assumed that the layers were fixed at the cell boundaries, i.e., that the undulations vanished at the boundaries. Under these same assumptions, Stewart extended the classical Helfrich-Hurault theory to three dimensional finite samples of smectic A liquid crystals in [24]. More recently, Stewart performed a linear stability analysis of the model introduced in [25], in which the director and the smectic layer normal differ [26]. However, a specific form of an ansatz was made under the assumption that there are no disturbances to the layers on the boundary plates. Our main concern in this article is to describe the undulation phenomena without fixing the layers on the boundaries.

Experimental studies of undulations of two dimensional and three dimensional systems were presented in [17] and [22], respectively. They used cholesteric liquid crystals with a pitch $5\mu m$ and $50-70\mu m$ cell thickness for their optical study. The layer thickness of smectic A liquid crystals is in the nanometer range (nm), which makes it very hard to visualize the layer distortions in experiments. Their experiments showed a lower critical field, and larger layer displacement than the ones expected from the Helfrich-Hurault theory. Furthermore, the displacement of the layer undulations immediately adjacent to the bounding plates was not zero. Motivated by these experimental results, Lavrentovich *et al.* proposed a model with weak anchoring conditions so that undulations can appear near the boundaries. By making an ansatz of periodic undulations, they show that their model explains the experiment more accurately than the classic Helfrich-Hurault theory. The model presented in this paper with the natural boundary condition on the

layer function, but with the strong anchoring condition, also agrees with the experiment.

The notion of Γ-convergence has been introduced by De Giorgi (See [3], [7]) in the 1970s to study the limiting behavior of a family of functionals depending on a parameter, and of its minimizers. In [4], the authors used Γ-convergence theory to study the switching process in elongated thin film elements by micromagnetic model. They identified four qualitatively different regimes with the choice of material parameters and verified the periodic oscillation in the direction of long axis in one of the regimes. We will identify a small parameter while performing nondimensionalization of the energy. With this small parameter, we study the Γ-convergence theory to understand the minimizer of the one dimensional energy. This corresponds to the study of the undulations in the infinite sample as in the Helfrich-Hurault theory.

We study the Euler-Lagrange equations of de Gennes energy via bifurcation theory. We prove that there is a pitchfork bifurcating curve of nontrivial solutions above the critical magnetic field. The stability of the undeformed state is lost at the bifurcation point.

In sections 2 and 3, we present the model and the geometry for our problem and state the stability of the undeformed state of smectic A liquid crystals. We use a simplified model of constant layer density. Also see [12]. We define the critical field H_c above which the undeformed state is unstable and below which the undeformed state is stable. In [12], we derived the estimate of the threshold in terms of the cell thickness. More precisely, we proved that there exist universal constants $0 < c_1 < c_2$, such that, if $2d$ is the cell thickness, then the critical field H_c satisfies

$$\left(\frac{c_1 K}{\chi_a d\lambda}\right)^{\frac{1}{2}} \leq H_c \leq \left(\frac{c_2 K}{\chi_a d\lambda}\right)^{\frac{1}{2}}, \tag{1}$$

where K, χ_a, λ are material constants, which will be discussed in the section 2. This estimate is consistent with the result found in the classical Helfrich-Hurault theory (see p.363 of [9] and [16]). We should mention that the scaling of the critical field in this case is different from the threshold for Fredericks transition of nematic liquid crystals, where the critical field is proportional to $1/d$ [28].

In section 4, we study the minimizer of the energy using Γ-convergence theory. In order to study the limiting problem with the help of Γ-convergence theory, we take the small parameter $\varepsilon = h^{-1}$ where h is the ratio of the layer thickness to the cell thickness. In [9], the parameter h is taken by 0.5×10^6. We study two cases: minimizer of the energy with Dirichlet boundary condition and with natural boundary condition on the layer variable. We recover from Γ-convergence a sharp estimate of the critical field. In the first case,

$$H_c \approx \left(\frac{\pi K}{\chi_a d\lambda}\right)^{\frac{1}{2}}. \tag{2}$$

Furthermore, we prove that the maximum undulation occurs in the middle of the cell and the displacement amplitude decreases as approaching the boundary by a cosine function. These are consistent with the result found in classic Helfrich-Hurault theory in an infinite sample. The theory also provides the period of the oscillation in the x axis.

For the general case, we eliminate the Dirichlet boundary condition for the layer function. The formulation of Γ-limit has been motivated from the work in [21] where they studied the Allen-Cahn functional with a Dirichlet boundary condition. Instead of the periodic profile as in the Dirichlet case, the director has boundary layers at both end points ($y = \pm d$). The frequency of the oscillation is no longer proportional to $d^{-1/2}$, in fact, it shows that the frequency is of smaller order than $d^{-1/2}$. Also, the Γ-convergence theory gives a sharp estimate of the critical field, which is lower than H_c in (2) by a factor of $1/\pi$. The result in [22] also shows that a finite surface anchoring strength leads to the lower threshold of undulation instability. It also shows that the amplitude of oscillation does not depend on the vertical y-axis in the limit. One can notice that the Γ-limit in this case is identical to the energy studied in [22] where they equated the director and layer normal, but included the Rapini-Popular surface energy.

In section 5, we find a first eigenvalue and corresponding eigenfunctions in order to consider a discrete spectrum of the linearized operator. We show that the kernel of the linearized operator at the first eigenvalue is one-dimensional for both Dirichlet and natural boundary conditions. This allows us to analyze the undulation phenomena as a bifurcation from the uniform solution. We prove that for $\kappa > \kappa_c$, the undeformed state loses its stability and bifurcation occurs generating a nontrivial solution curve which is stable.

In the general case, we proceed with asymptotic analysis to obtain the lower order term of the mode in the Fourier series so that we can determine the frequency of the oscillation. According to the experimental results, the vertical dependence of the layer undulation becomes zero as the strength of the applied field increases well above the critical field when they are using the cholesteric liquid crystals. However, our analysis and numerical simulation indicate that the undulation pattern is independent of the vertical coordinate at the onset of the undulation. This does not contradict our analysis since ε is about 0.2 in their experiment, i.e., ε is not sufficiently small.

In section 6, we illustrate the layer undulation patterns numerically by solving the gradient flow equations with the two different boundary conditions discussed. The simulations show that the threshold field and the frequency of the undulation pattern are lower than those in the classical theory, when natural boundary conditions on the layer function are considered. Furthermore we have a good agreement with the estimate of the critical fields and the frequencies of layer undulation in both cases. Also, the layer perturbation does not depend on the vertical axis as expected from the analysis. This is also consistent with the experimental results presented in [17] and [22].

In section 7, we use the original de Gennes-Landau model without the assumption of constant layer density and show that its linearized model can be transformed to the linear functional studied in the sections 4 and 5. Thus we perform the bifurcation analysis at a simple eigenvalue with both Dirichlet and natural boundary conditions on the layer function. We prove that the undeformed state at the critical field is a bifurcation point at which a nontrivial stable solution bifurcates. With this complex model one may simulate the layer profile well above the critical field, which will appear in a future publication.

Throughout this article, we shall use the letter C to indicate a generic positive constant which may have a different value at each of its occurrences.

2. Formulation of the problem

2.1. The free energy and the geometry

The total free energy density of smectic A liquid crystals consists of a nematic f_n and a smectic f_s contribution. The nematic contribution is described by the Oseen-Frank energy density, and is given by [9]

$$\begin{aligned} f_n &= K_1(\nabla\cdot\mathbf{n})^2 + K_2(\mathbf{n}\cdot\nabla\times\mathbf{n})^2 + K_3|\mathbf{n}\times(\nabla\times\mathbf{n})|^2 \\ &\quad + (K_2+K_4)(tr(\nabla\mathbf{n})^2 - (\nabla\cdot\mathbf{n})^2), \end{aligned}$$

where K_1, K_2 and K_3 are the splay, twist, and bend elastic constants, respectively. The last term in f_n is a null-Lagrangian since its integral only depends on the boundary values of $\mathbf{n}$. We consider the energy with one constant approximation case, $K_1 = K_2 = K_3 = \frac{K}{2} > 0$ and $K_4 = 0$. Then the nematic energy density reduces to

$$\frac{K}{2}|\nabla\mathbf{n}|^2.$$

In order to associate a smectic and a nematic structure with a state $(\mathbf{n}, \Psi)$ we introduce the complex order parameter

$$\Psi(\mathbf{x}) = \rho(\mathbf{x})e^{iq\omega(\mathbf{x})}.$$

Then the molecular mass density is defined by

$$\delta(\mathbf{x}) = \rho_0(\mathbf{x}) + \frac{1}{2}(\Psi(\mathbf{x}) + \Psi^*(\mathbf{x})) = \rho_0(\mathbf{x}) + \rho(\mathbf{x})\cos q\omega(\mathbf{x}),$$

where ρ_0 is a locally uniform mass density, $\rho(\mathbf{x})$ is the mass density of the smectic layers, and ω parametrizes the layers so that $\nabla\omega$ is the direction of the layer normal. Also, q is the wave number and $2\pi/q$ is the layer thickness.

The Landau-de Gennes energy density for smectic A is given by

$$f_s = \frac{C}{2}|\nabla\Psi - iq\mathbf{n}\Psi|^2,$$

where C is a positive constant. The energy density f_n+f_s with the additional term

$$r|\Psi|^2 + \frac{g}{2}|\Psi|^4,$$

was used in [1] to study the phase transition from nematic to smectic A, and the stability of the equilibrium states. Since we investigate smectic structure far from the nematic–smectic transition, we may assume that the magnitude of the smectic order parameter is a constant, i.e., ρ is a constant. Then f_s becomes

$$f_s = \frac{Cq^2\rho^2}{2}|\nabla\omega - \mathbf{n}|^2.$$

This energy density vanishes when $\nabla\omega = \mathbf{n}$, which describes the configuration of smectic A liquid crystals. The general case, without the assumption of the constant layer density, will be considered in the section 7.

The magnetic free energy density is given by [9],

$$f_m = -\frac{\chi_a}{2}(\mathbf{n}\cdot\mathbf{H})^2 = -\frac{\chi_a}{2}\sigma^2(\mathbf{n}\cdot\mathbf{h})^2,$$

where χ_a is the magnetic anisotropy, $\mathbf{H} = \sigma\mathbf{h}$, and $\sigma = |\mathbf{H}|$. We assume that $\chi_a > 0$. As a consequence, the director has a prefered orientation parallel to the direction of the applied magnetic field.

Collecting all contributions to the free energy, the free energy density for the one-constant approximation model becomes

$$f = \frac{K}{2}|\nabla\mathbf{n}|^2 + \frac{B}{2}|\nabla\omega - \mathbf{n}|^2 - \frac{\chi_a}{2}\sigma^2(\mathbf{h}\cdot\mathbf{n})^2, \tag{3}$$

where $B = Cq^2\rho^2$ is called the de Gennes compressibility constant. We consider a rectangular domain

$$\Omega = (-L, L) \times (-d, d),$$

where $L = cd$ for some constant $c > 0$. We also assume that $\mathbf{h} = (1, 0)$ so that the magnetic field tends to make the director orient along the x direction. We impose periodic boundary conditions for ϕ and $\mathbf{n}$ in the x direction, while we assume strong anchoring conditions for $\mathbf{n}$ on the boundary plates:

$$\mathbf{n}(x, \pm d) = (0, 1) \qquad \text{for all } x \in [-L, L]. \tag{4}$$

3. Stability of undeformed states

We write the model in dimensionless form by introducing new variables

$$(\widetilde{x}, \widetilde{y}) = \left(\frac{x}{\lambda}, \frac{y}{\lambda}\right) \qquad \text{and} \qquad \omega = \lambda\,\phi,$$

where $\lambda = \sqrt{\frac{K}{B}}$ is of the order of the smectic layer thickness. Then the free energy (3) becomes

$$\frac{B\lambda^2}{2}\int_{\widetilde{\Omega}}\left(|\widetilde{\nabla}\mathbf{n}|^2 + (\widetilde{\nabla}\phi - \mathbf{n})^2 - \kappa(\mathbf{n}\cdot\mathbf{h})^2\right)d\widetilde{\mathbf{x}}, \tag{5}$$

where the dimensionless parameters are given by

$$\kappa = \frac{\chi_a\sigma^2}{B}, \quad \widetilde{\Omega} = (-\widetilde{L}, \widetilde{L}) \times (-h, h), \quad h = \frac{d}{\lambda}, \quad \widetilde{L} = \frac{L}{\lambda}.$$

Since h is the ratio of the cell thickness to the layer thickness, we may assume that $h \gg 1$. In fact, the values $d = 1mm$ and $\lambda = 20\mathring{A}$ are employed in [9]. Then $h = 5 \times 10^5$.

From the fact $|\mathbf{n}| = 1$, we can introduce the scalar variable θ, such that

$$\mathbf{n} = (\sin\theta, \cos\theta).$$

Then the free energy (5) becomes

$$\mathfrak{F} = \int_{\widetilde{\Omega}}(\phi_{\widetilde{x}} - \sin\theta)^2 + (\phi_{\widetilde{y}} - \cos\theta)^2 + |\widetilde{\nabla}\theta|^2 - \kappa\sin^2\theta)\,d\widetilde{x}\,d\widetilde{y} \tag{6}$$

and the corresponding boundary condition on θ is the homogeneous Dirichlet boundary condition on the top and the bottom of the plate. This energy (6) has a trivial critical point, $\theta = 0, \phi = \widetilde{y}$, which describes the undeformed state where the layers are parallel to the bounding plates and the directors are aligned in the $\widetilde{y}$ direction. The second variation of the energy at the undeformed state, $\theta_0 = 0$ and $\phi_0 = \widetilde{y}$, gives

$$\begin{aligned}\frac{1}{2}D^2\mathfrak{F}(\theta_0 + t\theta, \phi_0 + t\phi) &:= \frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0}\mathfrak{F}(\theta_0 + t\theta, \phi_0 + t\phi)\\ &= \int_{\widetilde{\Omega}}((\phi_{\widetilde{x}} - \theta)^2 + \phi_{\widetilde{y}}^2 + |\widetilde{\nabla}\theta|^2 - \kappa|\theta|^2)\,d\widetilde{x}d\widetilde{y}.\end{aligned} \tag{7}$$

The undeformed state, (θ_0, ϕ_0), is stable if the second variation is nonnegative at (θ_0, ϕ_0). We set

$$\mathcal{G}(\theta, \phi) := \int_{\widetilde{\Omega}}((\phi_{\widetilde{x}} - \theta)^2 + \phi_{\widetilde{y}}^2 + |\widetilde{\nabla}\theta|^2)\,d\widetilde{x}d\widetilde{y}, \tag{8}$$

and consider two admissible sets, $\mathcal{A}$, and $\mathcal{A}_0$, defined by

$$\mathcal{A} = \{(\theta, \phi) \in H^1(U) \times H^1(U) : \|\theta\|_2 = 1, \theta(\widetilde{x}, \pm h) = 0 \text{ for all } \widetilde{x}\},$$

and

$$\mathcal{A}_0 = \{(\theta, \phi) \in \mathcal{A} : \phi(\widetilde{x}, \pm h) = 0 \text{ for all } \widetilde{x}\},$$

respectively, where $U = \mathbb{R}/(-\widetilde{L} + 2\widetilde{L}\mathbb{Z}) \times (-h, h)$. Note that since ϕ is the layer perturbation, the set $\mathcal{A}_0$ corresponds to the setting in the classic Helfrich-Hurault theory, where the layers are fixed at the cell boundaries.

Therefore, we use the admissible set $\mathcal{A}_0$ in the study of Dirichlet boundary conditions, and $\mathcal{A}$ in the study of Neumann boundary conditions.

One can see from (7) that the critical field in the Neumann case, κ_c, is defined by

$$\kappa_c = \inf_{(\theta,\phi)\in\mathcal{A}} \mathcal{G}(\theta,\phi). \tag{9}$$

Thus, the undeformed state, (θ_0, ϕ_0), is stable if $\kappa < \kappa_c$ and unstable if $\kappa > \kappa_c$. In addition, we show that when $\kappa = \kappa_c$, a stable bifurcation is possible.

In the Dirichlet case, the critical field κ_0 is defined by

$$\kappa_c^0 = \inf_{(\theta,\phi)\in\mathcal{A}_0} \mathcal{G}(\theta,\phi). \tag{10}$$

3.1. Factorization

The periodic boundary conditions allow us to use the Fourier series representation.

$$\theta(\widetilde{x},\widetilde{y}) = \sum_{n=-\infty}^{\infty} \theta_n(\widetilde{y}) e^{i\mu_n \widetilde{x}} \quad \text{and} \quad \phi(\widetilde{x},\widetilde{y}) = \sum_{n=-\infty}^{\infty} \phi_n(\widetilde{y}) e^{i\mu_n \widetilde{x}}, \tag{11}$$

where $\mu_n = \pi n/\widetilde{L}$. Then (8) becomes

$$\mathcal{G}(\theta,\phi) = 2\widetilde{L}\int_{-h}^{h}\left(\sum_{n=-\infty}^{\infty}(|\theta_n'|^2 + \mu_n^2|\theta_n|^2 + |\theta_n - i\mu_n\phi_n|^2 + |\phi_n'|^2)\right) d\widetilde{y} \tag{12}$$

where $'$ denotes differentiation with respect to $\widetilde{y}$. Setting $\psi_n = i\phi_n$, we define

$$\begin{aligned} \kappa_n &= \inf_{(\theta,\psi)\in\mathcal{B}} \int_{-h}^{h} (|\theta'|^2 + \mu_n^2|\theta|^2 + |\theta - \mu_n\psi|^2 + |\psi'|^2)\, d\widetilde{y} \\ &:= \inf_{(\theta,\psi)\in\mathcal{B}} E(\theta,\psi,n), \end{aligned} \tag{13}$$

where

$$\mathcal{B} = \{(\theta,\psi) \in W_0^{1,2}(-h,h) \times W^{1,2}(-h,h) : \int_{-h}^{h} |\theta(\widetilde{y})|^2\, d\widetilde{y} = 1\}.$$

Then, as in [2] and in [12], we have

$$\kappa_c = \inf_n \kappa_n. \tag{14}$$

Using (14), we showed in [12] that if $\widetilde{L} \geq h \geq 1$, then there exist universal constants $c_1 \geq 0$ and $c_2 \geq 0$ such that

$$\frac{c_1}{h} \leq \kappa_c \leq \frac{c_2}{h}, \tag{15}$$

which is equivalent to (1), when the estimate is expressed in terms of the real parameters.

Similarly, we define the critical field κ_n^0 by

$$\kappa_n^0 = \inf_{(\theta,\psi)\in\mathcal{B}_0} E(\theta,\psi,n),$$

where

$$\mathcal{B}_0 = \{(\theta,\psi)\in\mathcal{B} : \psi\in W_0^{1,2}(-h,h)\},$$

and it follows

$$\kappa_c^0 = \inf_n \kappa_n^0. \tag{16}$$

Note that the critical field κ_c^0 has the same scaling (15) as κ_c, since the construction (θ,ψ) for the upper bound from [12] belongs to $\mathcal{B}_0$.

Since the suggested value for h in [9] is $h = 0.5\times 10^6$, we may set the small parameter $\varepsilon = 1/h$. From (15), the energy is $\mathcal{O}(\varepsilon)$. Introducing the transformations, $x = \widetilde{x}/h, y = \widetilde{y}/h$, and $u = \phi/h$, the energy in (6) becomes

$$\varepsilon\,\mathfrak{F}(\theta,u) = \int_D \Big(\frac{(u_x-\sin\theta)^2}{\varepsilon} + \frac{(u_y-\cos\theta)^2}{\varepsilon} + \varepsilon|\nabla\theta|^2 - \sigma\sin^2\theta\Big)\,dx\,dy \tag{17}$$

where $D = (-c,c)\times(-1,1)$, $c = \widetilde{L}/h$, and $\sigma = h\kappa$. As in (7), one can see that the critical field σ_c is defined by

$$\sigma_c = \inf_{(\theta,u)\in\mathcal{A}} \mathcal{J}(\theta,u), \tag{18}$$

where

$$\mathcal{J}(\theta,u) := \int_D \Big(\frac{1}{\varepsilon}(u_x-\theta)^2 + \frac{1}{\varepsilon}u_y^2 + \varepsilon|\nabla\theta|^2\Big)\,dxdy. \tag{19}$$

It follows from (15) that there exist universal constants c_1 and c_2 such that

$$c_1 \le \sigma_c \le c_2. \tag{20}$$

Using notations $\delta = \delta(\varepsilon) = \varepsilon\mu^2$, $\mu = \mu_n = \frac{\pi n}{c}$, and $I = (-1,1)$, the corresponding one dimensional functional is

$$2c\int_I \Big(\frac{1}{\varepsilon}|i\mu u-\theta|^2 + \frac{1}{\varepsilon}(u')^2 + \varepsilon\mu^2\theta^2 + \varepsilon(\theta')^2\Big)\,dy. \tag{21}$$

Since we are interested in n such that $\sigma_c = \sigma_n$ and since $\sigma_0 \ge 1$, we may assume that $\mu_n \ne 0$ and define

$$\varphi = i\mu u \tag{22}$$

to obtain

$$\begin{aligned}\sigma_n &= \inf_{(\theta,\varphi)\in\tilde{\mathcal{B}}} \int_I (\varepsilon|\theta'|^2 + \delta|\theta|^2 + \frac{1}{\varepsilon}|\theta-\varphi|^2 + \frac{1}{\delta}|\varphi'|^2)\,dy \\ &:= \inf_{(\theta,\varphi)\in\tilde{\mathcal{B}}} F_\varepsilon(\theta,\varphi,\delta),\end{aligned} \tag{23}$$

where

$$\widetilde{\mathcal{B}} = \{(\theta, \varphi) \in W_0^{1,2}(I) \times W^{1,2}(I) : \int_{-1}^{1} |\theta(y)|^2 \, dy = 1\}.$$

From (20), we can see that the infimum of F is of order 1 in ε.

Similarly, we define the critical field σ_n^0 by

$$\sigma_n^0 = \inf_{(\theta,\varphi)\in\tilde{\mathcal{B}}_0} F_\varepsilon(\theta, \varphi, \delta),$$

where

$$\widetilde{\mathcal{B}_0} = \{(\theta, \psi) \in \widetilde{\mathcal{B}} : \psi \in W_0^{1,2}(-1, 1)\},$$

and it follows

$$\sigma_c^0 = \inf_n \sigma_n^0. \tag{24}$$

Lemma 1. *For a fixed $\delta > 0$, and under the notation above, there exists an absolute minimizer of F_ε, $(\theta, \varphi) \in \mathcal{B}$ ($(\theta, \varphi) \in \mathcal{B}_0$), such that both θ and φ are either positive or negative in $(-1, 1)$.*

Proof. For any pair $(\theta, \varphi) \in \mathcal{B}$ (or in $\mathcal{B}_0$),

$$F_\varepsilon(|\theta|, |\varphi|, \delta) \leq F_\varepsilon(\theta, \varphi, \delta).$$

Thus, if (θ, φ) is a global minimizer, so is $(|\theta|, |\varphi|)$, and $(|\theta|, |\varphi|)$ satisfies the same Euler-Lagrange equation, which is a system of ODEs. As a consequence, $(|\theta|, |\varphi|)$ are smooth functions. We can assume therefore that θ and φ are nonnegative, and we claim that $\theta, \varphi > 0$. Suppose, on the contrary, that there exists $x^* \in (-1, 1)$ such that $\theta(x^*) = 0$. Then $\theta'(x^*) = 0$, since x^* is a global minimum. Since (θ, φ) satisfies the system

$$\begin{aligned} -\varepsilon\theta'' + (\varepsilon\mu^2 + \frac{1}{\varepsilon})\theta - \frac{1}{\varepsilon}\varphi &= \sigma\theta \\ -\varphi'' + \mu^2\varphi - \mu^2\theta &= 0, \end{aligned}$$

we have $\varepsilon\theta''(x^*) = -\frac{1}{\varepsilon}\varphi(x^*)$. If $\varphi(x^*) > 0$ then $\theta''(x^*) < 0$ and hence θ has a local maximum at x^*, and therefore $\theta \equiv 0$. This contradicts the fact that $\|\theta\|_2 = 1$. Thus we have $\varphi(x^*) = 0$ and this implies $\varphi'(x^*) = 0$ as well. Therefore, (θ, φ) are the solution to the Cauchy problem for a second order ODE with initial conditions $\theta(x^*) = \theta'(x^*) = \varphi(x^*) = \varphi'(x^*) = 0$. By Picard's uniqueness theorem , we have that $\theta = \varphi = 0$ everywhere, which again is a contradiction. The same conclusion follows if we start by assuming the existence of a point x^* where $\varphi(x^*) = 0$.

Remark 1. The lemma can be alternatively proved as follows: If we denote by G_λ the Green's function associated to the operator

$$-\frac{d^2}{dx^2} + \lambda I,$$

with Dirichlet boundary conditions, then the solution to the Euler-Lagrange equations satisfies the integral equations:

$$\varphi = \mu^2 G_{\mu^2} * \theta, \tag{25}$$

$$\theta = \frac{1}{\epsilon^2} G_{\mu^2 + \frac{1}{\varepsilon^2} - \frac{\sigma}{\varepsilon}} * \varphi. \tag{26}$$

Since the Green's function defines a positive operator in this case, if $\theta \geq 0$ and $\varphi \geq 0$, then $\varphi > 0$, and $\theta > 0$.

In virtue of this Lemma 1, we can restrict our search to the spaces

$$X_n = \{(\theta, \varphi) \in H_0^1(I) \times H^1(I) : \|\theta\|_{L^2(I)} = 1, [\theta] \geq 0\}, \tag{27}$$

$$X_d = \{(\theta, \varphi) \in X_n : \varphi \in H_0^1(I)\}, \tag{28}$$

where $[\theta]$ is the average of θ on I.

4. Γ-Convergence

We study the configuration of a minimizer of F_ε in X_d and X_n by using Γ-convergence theory [14]. For the study of the Γ-limit, we use the following characterization [14]:

Theorem 1. *Let $(X, \mathcal{T})$ be a topological space, and let F_h a family of functionals parameterized by h. A functional F_0 is the Γ-limit of F_h as $h \to 0$ in $\mathcal{T}$ iff the two following conditions are satisfied:*

(i) If $u_h \to u_0$ in $\mathcal{T}$, then $\liminf_{h\to 0} F_h(u_h) \geq F_0(u_0)$.
(ii) For all $u_0 \in X$, there exists a sequence $u_h \in X$ such that $u_h \to u_0$ in $\mathcal{T}$, and $\lim_{h\to 0} F_h(u_h) = F_0(u_0)$.

Step (i) is related to lower-semicontinuity, while for step (ii) a specific construction for the converging sequence is typically required.

4.1. Fixed layers on the boundary

We consider first the case where we impose homogeneous Dirichlet boundary conditions for φ on the parallel plates. This corresponds to the classic Helfrich-Hurault theory, where the smectic layers are fixed at the boundary of the cell.

Let

$$F_\varepsilon(\theta, \varphi, \delta) := \begin{cases} \int_I (\varepsilon\theta'^2 + \frac{1}{\delta}\varphi'^2 + \delta\theta^2 + \frac{1}{\varepsilon}(\theta - \varphi)^2)\, dz & \text{if } (\theta, \varphi, \delta) \in X_d \times \mathbb{R}^+, \\ +\infty & \text{else,} \end{cases} \tag{29}$$

and

$$F_0(\theta, \varphi, \delta) := \begin{cases} \int_I (\delta\theta^2 + \frac{1}{\delta}\theta'^2)\, dz & \text{if } \theta = \varphi \in W_0^{1,2}(I), \\ +\infty & \text{else.} \end{cases} \tag{30}$$

We look for a minimizer of F_ε with the constraint

$$\int_I |\theta(z)|^2\, dz = 1. \tag{31}$$

Proposition 1. *(Compactness) Let the sequences $\{\varepsilon_j\}_{j\uparrow\infty} \subset (0,\infty)$, $\{\theta_j\}_{j\uparrow\infty} \subset W_0^{1,2}(I), \{\varphi_j\}_{j\uparrow\infty} \subset W_0^{1,2}(I)$ and $\{\delta_j\}_{j\uparrow\infty} \subset (0,\infty)$ be such that*

$$\varepsilon_j \to 0, \qquad \int_I |\theta_j|^2\, dy = 1, \text{ and} \qquad \{F_{\varepsilon_j}(\theta_j,\varphi_j,\delta_j)\}_{j\to\infty} \qquad \text{is bounded.}$$

Then

$$\{\theta_j,\varphi_j,\delta_j\}_{j\uparrow\infty} \quad \text{is relatively compact in } L^2(I)\times L^2(I)\times\mathbb{R}.$$

Moreover, the limit $(\theta,\varphi) \in W_0^{1,2}(I)\times W_0^{1,2}(I)$.

Proof. Assume that $\exists C > 0$ such that

$$F_{\varepsilon_j}(\theta_j,\varphi_j,\delta_j) \le C, \quad \forall j \in \mathbb{N}.$$

Since $\|\theta_j\|_2 = 1$, there exists $\theta \in L^2(I)$, and a subsequence, still labeled $\{\theta_j\}$, that

$$\theta_j \rightharpoonup \theta \quad \text{in } L^2(I).$$

From the definition of F_ε, it follows that $\delta_j \le C$, and thus $\{\delta_j\}_{j\uparrow\infty}$ is relatively compact in $\mathbb{R}$. Note also that $\|\varphi_j\|_{W^{1,2}(I)}$ is also uniformly bounded:

$$\int_I |\varphi_j|^2\, dz \le 2\int_I ((\varphi_j-\theta_j)^2+\theta_j^2)\, dz \le 2\varepsilon_j F_{\varepsilon_j}(\theta_j,\varphi_j,\delta_j) + 2 \le C$$

and

$$\int_I |\varphi_j'|^2\, dz \le \delta_j F_{\varepsilon_j}(\theta_j,\varphi_j,\delta_j) \le C.$$

Thus, there exists $\varphi \in W^{1,2}(I)$, and a subsequence, not relabeled, such that

$$\varphi_j \rightharpoonup \varphi \quad \text{in } W^{1,2}(I) \qquad \text{and} \qquad \varphi_j \to \varphi \quad \text{in } L^2(I).$$

It follows from the lower semicontinuity of the norm that

$$\int_I (\theta-\varphi)^2\, dy \le \liminf_{j\to\infty}\int_I (\theta_j-\varphi_j)^2\, dz \le \lim_{j\to\infty}(C\varepsilon_j) = 0,$$

so that $\theta = \varphi$ and $\theta \in W^{1,2}(I)$. Now we prove that $\lim_{j\to\infty}\int_I |\varphi_j|^2\, dz = 1$:

$$\begin{aligned}\left|\int_I \varphi_j^2\, dz - 1\right| &= \left|\int_I |\varphi_j|^2\, dz - \int_I |\theta_j|^2\, dz\right| \le \int_I |\varphi_j^2-\theta_j^2|\, dz \\ &\le \|\varphi_j-\theta_j\|_{L^2(I)}\|\varphi_j+\theta_j\|_{L^2(I)} \le C\|\varphi_j-\theta_j\|_{L^2(I)} \\ &\le C\varepsilon_j^{\frac{1}{2}}.\end{aligned}$$

Thus we have $\int_I \varphi^2\,dz = 1$ and hence $\int_I \theta^2\,dz = 1$. Since weak convergence of the sequence together with convergence of the norms imply strong convergence, we have

$$\theta_j \to \theta \quad \text{in } L^2(I).$$

Finally, in order to show that $(\theta,\varphi) \in W_0^{1,2}(I) \times W_0^{1,2}(I)$, we use the following estimate for $\varphi_j - \varphi$ (see Theorem 1.5.1.10 in [13]):

$$\int_{\partial\Omega} |u|^2\,d\sigma \le C(\Omega)\{\|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\} \tag{32}$$

for all $u \in W^{1,2}(\Omega)$.

Proposition 2. *The variational problem in $(\theta,\varphi,\delta) \in X_d \times \mathbb{R}^+$ of minimizing $F_\varepsilon(\theta,\varphi,\delta)$ Γ-converges in $[L^2(I)]^2 \times \mathbb{R}$ as $\varepsilon \to 0$, to the variational problem in $(\theta,\varphi,\delta) \in X_d \times \mathbb{R}^+$ of minimizing $F_0(\theta,\varphi,\delta)$, where F_0 is defined in (30).*

The proof of Proposition 2 consists of the two following lemmas.

Lemma 2. *(Lower semi-continuity) For every $(\theta,\varphi,\delta) \in [L^2(I)]^2 \times \mathbb{R}$ and every sequence $(\theta_j,\varphi_j,\delta_j) \in X_d \times \mathbb{R}^+$ such that $(\theta_j,\varphi_j,\delta_j)$ converges to (θ,φ,δ) in $[L^2(I)]^2 \times \mathbb{R}$ there holds*

$$\liminf_{j\to\infty} F_{\varepsilon_j}(\theta_j,\varphi_j,\delta_j) \ge F_0(\theta,\varphi,\delta),$$

and

$$\theta \in W_0^{1,2}(I), \qquad \textit{and} \qquad \varphi \in W_0^{1,2}(I).$$

Proof. We assume that $F_\varepsilon(\theta_j,\varphi_j,\delta_j) \le C$ for some constant C. As in the proof of Proposition 1, we have

$$\theta = \varphi, \qquad \varphi_j \rightharpoonup \varphi \quad \text{in } W^{1,2}(I), \qquad \text{and} \qquad \theta \in W_0^{1,2}(I). \tag{33}$$

Now we claim that $\delta \ne 0$. If not, we have

$$\int_I (\varphi')^2 \le \liminf_{j\to\infty} \int_I \varphi_j^2 \le \lim_{j\to\infty} C\delta_j = 0.$$

and hence $\int_I (\theta')^2 = 0$ from (33). A contradiction follows from the Poincaré inequality, the constraint (31), and $\theta \in W_0^{1,2}(I)$.

Now that $\delta \ne 0$, we have, from the weak lower semicontinuity and (33),

$$\frac{1}{\delta}\int_I (\theta')^2 \le \liminf_{j\to\infty} \frac{1}{\delta_j}\int_I (\varphi_j')^2$$

and hence

$$\delta\int_I \theta^2 + \frac{1}{\delta}\int_I (\theta')^2 \le \liminf_{j\to\infty} F_\varepsilon(\theta_j,\varphi_j,\delta_j).$$

Lemma 3. *(Construction) For any $(\theta, \varphi, \delta) \in X_d \times \mathbb{R}^+$ with $\theta = \varphi$ and $\int_I |\theta|^2\, dz = 1$, there exists a sequence $(\theta_j, \varphi_j, \delta_j) \in X_d \times \mathbb{R}^+$ with $\int_I |\theta_j|^2\, dz = 1$, converging in $[L^2(I)]^2 \times \mathbb{R}$ as $j \to \infty$, to $(\theta, \varphi, \delta)$, and such that*

$$\limsup_{j\to\infty} F_{\varepsilon_j}(\theta_j, \varphi_j, \delta_j) = F_0(\theta, \varphi, \delta).$$

Proof. We take $\varphi_j = \varphi$, $\theta_j = \theta$, and $\delta_j = \delta$ for all $j \geq 1$.

Lemma 4. *Let (θ, δ) be a minimizer of*

$$F_0 = \int_I (\delta\theta^2 + \frac{1}{\delta}|\theta'|^2)\, dz \quad \textit{constrained by} \int_I |\theta|^2\, dz = 1$$

with $[\theta] \geq 0$ and the boundary condition $\theta(\pm 1) = 0$. Then

$$\delta = \frac{\pi}{2} \qquad \textit{and} \qquad \theta(z) = \cos\frac{\pi}{2}z. \tag{34}$$

The minimum value of the functional is π.

Proof. Since $\theta \in W_0^{1,2}(I)$ and $\|\theta\|_2 = 1$, we have

$$\int_I |\theta'|^2\, dz \geq \frac{\pi^2}{4}.$$

and the equality holds if and only if θ is of the form (34). Then we see that

$$F_0 \geq \delta + \frac{\pi^2}{4\delta} \geq \pi$$

and the equality holds if $\delta = \pi/2$.

If the Γ-limit F_0 has a unique minimizer, then it follows from the Γ-convergence theory that the sequence of minimizers of F_ε converges to the minimizer of F_0 [7]. Thus, it follows the following

Theorem 2. *Let $(\theta_\varepsilon, \varphi_\varepsilon, \delta) \in X_d \times \mathbb{R}^+$ be a minimizer of F_ε. For $\varepsilon = 1/h \ll 1$, we have*

$$\varepsilon\mu^2 \approx \frac{\pi}{2}, \qquad \textit{and} \qquad F_\varepsilon(\theta_\varepsilon, \varphi_\varepsilon, \delta) \approx \pi \tag{35}$$

and

$$\int_I \left|\varphi_\varepsilon(z) - \cos\frac{\pi}{2}z\right|^2 dz \ll 1. \tag{36}$$

Remark 2. One can see from (16) and (35) that the critical field is given by

$$\kappa_c^0 \approx \frac{\pi}{h},$$

which is, in terms of real parameters,

$$H_c \approx \left(\frac{\pi K}{\chi_a d\lambda}\right)^{\frac{1}{2}}.$$

This estimate is consistent with the result found in the classic Helfrich-Hurault theory. (See p363 of [9] and [16]).

From (35) and (36) we infer that

$$\theta(x,y) = \varphi(x,y) = \cos\frac{\pi}{2}y \sin\sqrt{\frac{\pi}{2h}}x$$

and hence

$$\omega(x,y) = \sqrt{\frac{2\lambda d}{\pi}} \cos\left(\frac{\pi}{2d}y\right) \sin\left(\sqrt{\frac{\pi}{2\lambda d}}x\right),$$

which is again consistent with the result in [9].

4.2. Natural boundary condition on the layers

In this section, we do not impose the Dirichlet boundary condition on layers. Since the compactness property in this case follows from Proposition 1, we need to prove the Γ-convergence of F_ε for $\varphi \in W^{1,2}(I)$. In order to obtain the Γ-limit in this case, we follow an approach similar to that in [21].

Proposition 3. *The variational problem in $(\theta,\varphi,\delta) \in X_n \times \mathbb{R}^+$ of minimizing $F_\varepsilon(\theta,\varphi,\delta)$ Γ-converges in $[L^2(I)]^2 \times \mathbb{R}$ as $\varepsilon \to 0$, to the variational problem in $(\theta,\varphi,\delta) \in \bar{X}_n$ of minimizing $F(\theta,\varphi,\delta)$ where*

$$F(\theta,\varphi,\delta) = \begin{cases} \int_I (\delta|\theta|^2 + \frac{1}{\delta}|\theta'|^2)\,dz + \theta(1)^2 + \theta(-1)^2 & \text{if } \delta \neq 0,\ \theta = \varphi \\ \theta(1)^2 + \theta(-1)^2 = 1 & \text{if } \delta = 0,\ \theta = \varphi = \text{constant} \\ \infty & \text{else} \end{cases} \tag{37}$$

and

$$\bar{X}_n = \{(\theta,\varphi,\delta) \in H^1(I) \times H^1(I) \times \mathbb{R}^+ \cup \{0\} : \|\theta\|_{L^2(I)} = 1, [\theta] > 0\}.$$

Lemma 5. *(Lower semi-continuity) For every $(\theta,\varphi,\delta) \in [L^2(I)]^2 \times \mathbb{R}$ and every sequence $(\theta_j,\varphi_j,\delta_j) \in X_n \times \mathbb{R}^+$ such that $(\theta_j,\varphi_j,\delta_j)$ converges to (θ,φ,δ) in $[L^2(I)]^2 \times \mathbb{R}$ there holds*

$$\liminf_{j\to\infty} F_{\varepsilon_j}(\theta_j,\varphi_j,\delta_j) \geq F(\theta,\varphi,\delta),$$

$$\theta \in W^{1,2}(I), \qquad \text{and} \qquad \varphi \in W^{1,2}(I).$$

Proof. We may assume that $F_\varepsilon(\theta_j,\varphi_j,\delta_j) \leq C$ for some constant C. Then we have $\theta = \varphi$ as in the proof of Proposition 1. We define an auxiliary function

$$\Phi(\theta_j,\varphi_j) = (\theta_j - \varphi_j)^2.$$

Differentiating Φ with respect to z, we obtain

$$\partial_z\Phi(\theta_j,\varphi_j) = 2(\theta_j - \varphi_j)(\theta_j' - \varphi_j').$$

Then

$$\int_I (\varepsilon_j(\theta_j')^2 + \frac{1}{\varepsilon_j}(\theta_j - \varphi_j)^2)\,dz \geq \int_I 2|\theta_j'||\theta_j - \varphi_j|\,dz \tag{38}$$

$$= \int_I |\partial_z \Phi(\theta_j, \varphi_j) + 2\varphi_j'(\theta_j - \varphi_j)|\,dz$$

$$\geq \int_I |\partial_z \Phi(\theta_j, \varphi_j)|\,dz - 2\int_I |\varphi_j'(\theta_j - \varphi_j)|\,dz$$

$$\geq \int_I |\partial_z \Phi(\theta_j, \varphi_j)|\,dz - 2\|\varphi_j'\|_{L^2(I)}\|\theta_j - \varphi_j\|_{L^2(I)}. \tag{39}$$

We extend θ_j and θ continuously by setting them equal to zero outside I. Let $0 < \eta \ll 1$, $I_\eta = (-1-\eta, 1+\eta)$, and define

$$\widehat{\theta_j} = \begin{cases} \theta_j & \text{in } I \\ 0 & \text{in } I_\eta - \bar{I}, \end{cases}$$

and

$$\widehat{\theta} = \begin{cases} \theta & \text{in } I \\ 0 & \text{in } I_\eta - \bar{I}. \end{cases}$$

We also extend φ_j and φ continuously by setting their boundary values outside of I:

$$\widehat{\varphi_j} = \begin{cases} \varphi_j & \text{in } I \\ \varphi_j(-1) & \text{in } (-1-\eta, -1) \\ \varphi_j(1) & \text{in } (1, 1+\eta), \end{cases}$$

and

$$\widehat{\varphi} = \begin{cases} \varphi & \text{in } I \\ \varphi(-1) & \text{in } (-1-\eta, -1) \\ \varphi(1) & \text{in } (1, 1+\eta). \end{cases}$$

Since $\widehat{\theta_j}$ and $\widehat{\varphi_j}$ are extended continuously and they are constant outside I, we get

$$\int_I |\partial_z \Phi(\theta_j, \varphi_j)| = \int_{I_\eta} |\partial_z \Phi(\widehat{\theta_j}, \widehat{\varphi_j})|.$$

For $\delta \neq 0$, since $\Phi(\theta_j, \varphi_j) \to \Phi(\theta, \varphi)$ in $L^1(I)$, it follows from (39) and the lower semicontinuity of the BV norm that

$$\begin{aligned} &\liminf_{j\to\infty} F_{\varepsilon_j}(\theta_j, \varphi_j, \delta_j) \\ &\geq \liminf_{j\to\infty} \Big[\int_{I_\eta} |\partial_z \Phi(\widehat{\theta_j}, \widehat{\varphi_j})|\,dz + \int_I (\delta_j \theta_j^2 + \frac{1}{\delta_j}(\varphi_j')^2)\,dz \\ &\qquad -2\|\varphi_j'\|_{L^2(I)}\|\theta_j - \varphi_j\|_{L^2(I)} \Big] \\ &\geq \int_{I_\eta} |\partial_z \Phi(\widehat{\theta}, \widehat{\varphi})|\,dz + \int_I (\delta\theta^2 + \frac{1}{\delta}(\varphi')^2)\,dz. \end{aligned} \tag{40}$$

From the definition of $\widehat{\theta}$ and $\widehat{\varphi}$, the definition of Φ, and the fact that $\theta = \varphi$, we have

$$\begin{aligned}\int_{I_\eta} |\partial_z \Phi(\widehat{\theta}, \widehat{\varphi})|\, dz &= \int_I |\partial_z \Phi(\widehat{\theta}, \widehat{\varphi})|\, dz + \int_{\partial I} |\Phi(\theta, \varphi) - \Phi(0, \varphi)| \\ &= (\theta(-1))^2 + (\theta(1))^2.\end{aligned} \tag{41}$$

Together with (40), we have

$$\liminf_{j\to\infty} F_{\varepsilon_j}(\theta_j, \varphi_j, \delta_j) \geq F(\theta, \varphi, \delta).$$

For $\delta = 0$, it is obvious that φ must be a constant. In fact,

$$\int_I (\varphi_j')^2 \leq C\delta_j \to 0$$

as $j \to \infty$. Therefore, $\theta = \varphi$ must be constants. The lower semicontinuity in this case follows from (40) and (41).

Lemma 6. *(Construction) For any $(\theta, \varphi, \delta) \in \bar{X}_n$ with $\int_I |\theta|^2\, dz = 1$ and $\theta = \varphi$, there exists a sequence $(\theta_j, \varphi_j, \delta_j) \in W_0^{1,2}(I) \times W^{1,2}(I) \times \mathbb{R}$ with $\int_I |\theta_j|^2\, dz = 1$, converging in $[L^2(I)]^2 \times \mathbb{R}$ as $j \to \infty$, to $(\theta, \varphi, \delta)$, and such that*

$$\limsup_{j\to\infty} F_{\varepsilon_j}(\theta_j, \varphi_j, \delta_j) = F(\theta, \varphi, \delta).$$

Proof. Given $(\theta, \varphi, \delta)$, where $\theta = \varphi$, we take $\varphi_\varepsilon = \varphi, \delta_\varepsilon = \delta$ for all ε. Then $\|\varphi\|_{L^2(I)} = 1$. We define the boundary layer construction for θ_ε by

$$\theta_\varepsilon(z) = \frac{\rho_\varepsilon(z)}{\|\rho_\varepsilon\|_{L^2(I)}},$$

where

$$\rho_\varepsilon(z) = \begin{cases} \theta(z) - \theta(-1)e^{-\frac{z+1}{\varepsilon}} & z \in I_1 \equiv (-1, -1+\sqrt{\varepsilon}), \\ \theta(z) + \frac{\theta(-1)e^{-\frac{1}{\sqrt{\varepsilon}}}}{\sqrt{\varepsilon}}(1 + z - 2\sqrt{\varepsilon}) & z \in I_2 \equiv (-1+\sqrt{\varepsilon}, -1+2\sqrt{\varepsilon}), \\ \theta(z) & z \in I_3 \equiv (-1+2\sqrt{\varepsilon}, 1-2\sqrt{\varepsilon}), \\ \theta(z) + \frac{\theta(1)e^{-\frac{1}{\sqrt{\varepsilon}}}}{\sqrt{\varepsilon}}(1 - z - 2\sqrt{\varepsilon}) & z \in I_4 \equiv (1-2\sqrt{\varepsilon}, 1-\sqrt{\varepsilon}), \\ \theta(z) - \theta(1)e^{-\frac{1-z}{\varepsilon}} & z \in I_5 \equiv (1-\sqrt{\varepsilon}, 1). \end{cases} \tag{42}$$

Then we have $\theta_\varepsilon \in W_0^{1,2}(I)$ and $\|\theta_\varepsilon\|_{L^2(I)} = 1$. Since $\|\rho_\varepsilon\|_{L^2(I)} = 1+\mathcal{O}(\sqrt{\varepsilon})$, a direct computation gives

$$\begin{aligned}
&\int_{I_1} [\frac{1}{\varepsilon}(\theta_\varepsilon - \varphi_\varepsilon)^2 + \varepsilon|\theta_\varepsilon'|^2]\, dz \\
&\quad = \frac{1}{\varepsilon\|\rho_\varepsilon\|_{L^2}} \int_{I_1} \left(\theta(z) - \theta(-1)e^{-\frac{1+z}{\varepsilon}} - \|\rho_\varepsilon\|_2\theta(z)\right)^2 dz \\
&\quad + \frac{\varepsilon}{\|\rho_\varepsilon\|^2_{L^2(I)}} \int_{I_1} \left(\theta'(z) + \frac{\theta(-1)}{\varepsilon} e^{-\frac{1+z}{\varepsilon}}\right)^2 dz \\
&\quad = \frac{\theta(-1)^2}{2} + \frac{\theta(-1)^2}{2} + \mathcal{O}(\sqrt{\varepsilon}) = \varphi(-1)^2 + \mathcal{O}(\sqrt{\varepsilon}).
\end{aligned}$$

Also, we get

$$\int_{I_2} [\frac{1}{\varepsilon}(\theta_\varepsilon - \varphi_\varepsilon)^2 + \varepsilon|\theta_\varepsilon'|^2]\, dz = \mathcal{O}(\sqrt{\varepsilon}).$$

Similarly, we have

$$\begin{aligned}
&\int_{I_4} [\frac{1}{\varepsilon}(\theta_\varepsilon - \varphi_\varepsilon)^2 + \varepsilon|\theta_\varepsilon'|^2]\, dz = \mathcal{O}(\sqrt{\varepsilon}) \\
&\int_{I_5} [\frac{1}{\varepsilon}(\theta_\varepsilon - \varphi_\varepsilon)^2 + \varepsilon|\theta_\varepsilon'|^2]\, dz = \theta(1)^2 + \mathcal{O}(\sqrt{\varepsilon}).
\end{aligned}$$

Hence, we have

$$\limsup_{\varepsilon\to 0} \int_I [\frac{1}{\varepsilon}(\theta_\varepsilon - \varphi_\varepsilon)^2 + \varepsilon|\theta_\varepsilon'|^2]\, d\widetilde{y} = \theta(1)^2 + \theta(-1)^2,$$

and therefore

$$\limsup_{\varepsilon\to 0} F_\varepsilon(\theta_\varepsilon, \varphi_\varepsilon, \delta_\varepsilon) = \theta(1)^2 + \theta(-1)^2 + \delta + \frac{1}{\delta}\int_I |\theta'|^2\, dz.$$

For the case $\delta = 0$, since φ is constant, the same construction gives the desired upper bound.

In Figure 1 we depict the minimizer $(\theta_\varepsilon, \varphi_\varepsilon)$ of F_ε when $\varepsilon = 0.05$ and $\varepsilon = 0.01$. The value of δ was chosen such that $\delta = 10\varepsilon$. Although this choice was not the consequence of the Γ-convergence theory, this is in fact an approximating value for a minimizer δ from the analysis in section 5.2.1. The minimizers were obtained using a Truncated Newton minimization algorithm [20]. The results suggest that θ_ε has a boundary layer and φ_ε tends to a constant for small ε. Also, the computation of the energy suggests that the minimum value of F_ε is close to 1 for $\varepsilon \ll 1$. In fact, the following theorem shows that the minimum value of the Γ-limit F defined in (37) is indeed 1.

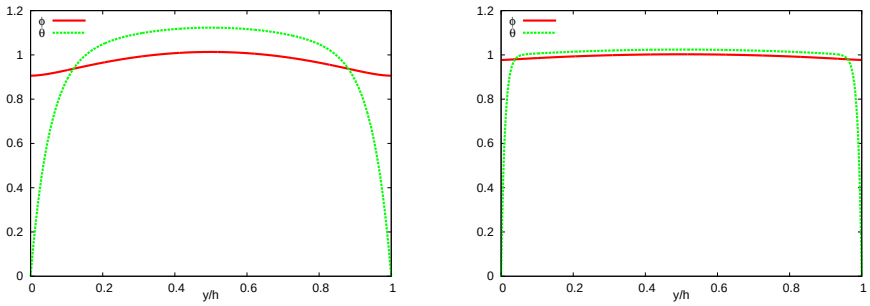

Fig. 1. Minimizers of F_ε with $\varepsilon = 0.05$ and $\varepsilon = 0.01$, both with the choice of $\delta = 10\varepsilon$.

Lemma 7. *Consider the functional F defined in (37), and let $(\theta, \theta, \delta) \in \bar{X}_n$ be a minimizer of F constrained by $\int_I |\theta|^2\, dz = 1$. Then*

$$\delta = 0 \qquad \textit{and} \qquad \theta(z) = \frac{1}{\sqrt{2}}.$$

Moreover, the minimum value of the functional is 1.

Proof. If $\delta = 0$, then θ must be a constant and hence $F = \theta(1)^2 + \theta(-1)^2$. Since $\|\theta\|_{L^2(I)} = 1$, it follows that $\theta = 1/\sqrt{2}$, and $F(\theta, \theta, \delta) = 1$. Now we consider the case $\delta \neq 0$ and find the corresponding Euler-Lagrange equation

$$\begin{aligned} &\frac{1}{\delta}\theta'' + \lambda\theta = 0 && \text{in } I \\ &\frac{1}{\delta}\frac{\partial\theta}{\partial\nu} + \theta = 0 && \text{on } \partial I, \end{aligned} \tag{43}$$

where $\lambda = \sigma - \delta$. The general solution to (43) is given by

$$\theta = c_1 \cos\sqrt{\delta\lambda}z + c_2 \sin\sqrt{\delta\lambda}z.$$

By adding and subtracting the equations at the boundaries we obtain

$$\begin{aligned} c_1\left(-\sqrt{\lambda}\sin\sqrt{\lambda\delta} + \sqrt{\delta}\cos\sqrt{\lambda\delta}\right) &= 0, \\ c_2\left(\sqrt{\lambda}\cos\sqrt{\lambda\delta} + \sqrt{\delta}\sin\sqrt{\lambda\delta}\right) &= 0. \end{aligned}$$

Since c_1 and c_2 cannot be simultaneously equal to zero, this implies

$$c_1 = 0 \quad \text{and} \quad \tan\sqrt{\delta\lambda} = -\sqrt{\frac{\lambda}{\delta}} \tag{44}$$

$$\text{or} \qquad c_2 = 0 \quad \text{and} \quad \tan\sqrt{\delta\lambda} = \sqrt{\frac{\delta}{\lambda}}. \tag{45}$$

In the case (44), one must have $\sqrt{\delta\lambda} > \frac{\pi}{2}$. This gives

$$\sigma > \delta + \frac{\pi^2}{4\delta} \geq 2\sqrt{\delta}\frac{\pi}{2\sqrt{\delta}} = \pi > 1.$$

Now we assume that $c_2 = 0$ and c_1 is chosen such that $\|\theta\|_{L^2(I)} = 1$. By considering λ as a function of δ, one can see that F is minimized when $\frac{\partial\sigma(\delta)}{\partial\delta} = 0$. We write (δ_0, σ_0) as a critical point. Differentiating (44) with respect to δ using $\frac{\partial\sigma(\delta)}{\partial\delta} = 0$ and evaluating it at (δ_0, σ_0), we have

$$(\sigma_0 - 2\delta_0 - 1)\left(\sqrt{\delta_0}\sin\sqrt{\delta_0(\sigma_0 - \delta_0)} + \sqrt{\sigma_0 - \delta_0}\cos\sqrt{\delta_0(\sigma_0 - \delta_0)}\right) = 0,$$

which leads to

$$\sigma_0 = 2\delta_0 + 1 \qquad \text{or} \qquad \tan\sqrt{\delta_0(\sigma_0 - \delta_0)} = -\sqrt{\frac{\sigma_0 - \delta_0}{\delta_0}}.$$

From (45), however, the latter is not possible. It follows that $\sigma_0 = 2\delta_0 + 1$ from (45). Thus

$$F(\theta, \theta, \delta) = \sigma \geq 1 + 2\delta_0 > 1$$

and hence the minimum occurs when $\delta = 0$.

The proof of the following theorem is analogous to that of Theorem 2, and we therefore skip it:

Theorem 3. *Let $(\theta, \varphi, \delta)$ be a minimizer of F_ε constrained by $\int_I |\theta|^2\, dz = 1$. For $\varepsilon = 1/h \ll 1$, we have*

$$\varepsilon\mu^2 \approx 0, \qquad \text{and} \qquad F_\varepsilon \approx 1. \tag{46}$$

and

$$\int_I |\varphi(z) - \frac{1}{\sqrt{2}}|^2\, dz \ll 1. \tag{47}$$

Remark 3. One can see that the critical field is given by

$$\kappa_c \approx \frac{1}{h},$$

which is, in terms of real parameters,

$$H_c \sim \left(\frac{K}{\chi_a d\lambda}\right)^{\frac{1}{2}}.$$

It follows from Remarks 2 and 3 that the undulation may occur at a lower critical field than in the classic Helfrich-Hurault theory. Also, from (46) we infer that the undulation frequency is lower than it is expected from the classic theory. The estimate (47) suggests that the amplitude of the undulation may not decrease in the proximity of the cell boundaries. Instead one may expect the undulations to have the same amplitude throughout the cell.

Remark 4. In [17] and [22], the authors considered director and layer normal vector to be equaled, and added a weak anchoring condition in the form of a Rapini-Popular surface energy

$$W \int_{\partial\Omega} (1 - (\mathbf{n} \cdot \mathbf{q})^2)\, ds,$$

where $\mathbf{q}$ is the unit normal to the surface, which is $\mathbf{e}_2$ in this case. After the linearization around the undeformed state, this surface term is precisely the boundary terms $\theta(1)^2 + \theta(-1)^2$ that is recovered in the Γ-limit.

5. Bifurcation analysis

To understand the onset of undulations above the critical field, we carry out a bifurcation analysis of the Euler-Lagrange equations. We consider the two cases: 1) Fixed layers on the boundary, and 2) Natural boundary conditions. We show that the dimension of the kernel and the codimension of the range of the linearized operator around the uniform state is one-dimensional in both cases, and find a basis for the kernel. We can therefore describe the set of critical points in the vicinity of the critical field using the theory of Crandall and Rabinowitz [5]. Moreover, we determine the stability of the new branch of solutions.

5.1. Fixed layers on the boundary

5.1.1. Linearized functional In virtue of the decomposition (11), it suffices to consider the one-dimensional problem (23). A solution for the two-dimensional domain can easily be constructed. We will therefore consider the functional

$$F_\varepsilon(\theta, \varphi, \delta) := \begin{cases} \int_I (\varepsilon\theta'^2 + \frac{1}{\delta}\varphi'^2 + \delta\theta^2 + \frac{1}{\varepsilon}(\theta - \varphi)^2)\, dz & \text{if } (\theta, \varphi, \delta) \in X_d \times \mathbb{R}^+, \\ +\infty & \text{else.} \end{cases} \tag{48}$$

We know that a minimizer always exists in X_d, and proceed to determine the structure of the minimizers.

Theorem 4. *Let $(\theta_n, \varphi_n) \in X_d$ be a global minimizer of F_ε in X_d. Then (θ_n, φ_n) satisfies the Euler-Lagrange equations*

$$\begin{aligned} &-\varepsilon\theta_n'' + (\delta_n + \frac{1}{\varepsilon})\theta_n - \frac{1}{\varepsilon}\varphi_n = \sigma_n\theta_n, \\ &-\frac{1}{\delta_n}\varphi_n'' + \frac{1}{\varepsilon}\varphi_n = \frac{1}{\varepsilon}\theta_n, \end{aligned} \tag{49}$$

with boundary conditions

$$\theta_n(\pm 1) = 0, \quad \textit{and} \quad \varphi_n(\pm 1) = 0. \tag{50}$$

There exists a discrete set $\mathcal{H} \subset \mathbb{R}$ such that for any $\varepsilon \in \mathbb{R} \setminus \mathcal{H}$ and $\sigma_n^0 = \min F_\varepsilon(\theta_n, \varphi_n)$ in $\tilde{\mathcal{B}}_0$, the space of solutions to the system of equations (49) is one-dimensional, and is spanned by the functions

$$\theta(z) = c\cos\frac{\pi}{2}z, \qquad \varphi(z) = \frac{4\delta_m}{4\delta_m + \varepsilon\pi^2}\theta(z),$$

for some m. Furthermore, $\sigma_c^0 = \inf_n \sigma_n^0 = \sigma_m^0$.

Proof. Consider the linear operator

$$L\theta = -\varepsilon\theta'' + \delta\theta,$$

defined in $\mathcal{D}(L) = \{\theta \in H^2(I) | \theta(\pm 1) = 0\}$. and let $T = L^{-1}$. It is well known that T is compact. The system (49) can be written as

$$\theta = \left(\sigma - \frac{1}{\varepsilon}\right) T\theta + \frac{1}{\varepsilon}T\varphi, \tag{51}$$

$$\varphi = \delta T\theta. \tag{52}$$

Substituting φ in (51) using (52) we obtain

$$\theta = \left(\left(\sigma - \frac{1}{\varepsilon}\right) T + \frac{\delta}{\varepsilon}T^2\right)\theta = p(T)\theta, \tag{53}$$

where we have defined the polynomial

$$p(z) = \left(\sigma - \frac{1}{\varepsilon}\right) z + \frac{\delta}{\varepsilon}z^2.$$

Therefore $1 \in \sigma(p(T))$.

Let κ be an eigenvalue of T, i.e., $T\psi = \kappa\psi$, then ψ satisfies the equation $-\varepsilon\psi'' + \delta\psi = \frac{1}{\kappa}\psi$ whose solution is $\psi(x) = c_1 \sinh(i\sqrt{\alpha}(z-1)) + c_2 \sinh(i\sqrt{\alpha}(z+1))$, where $\alpha = (1/\kappa - \delta)/\varepsilon$. From the Dirichlet boundary condition, we have $\sinh(2i\sqrt{\alpha}) = 0$, which implies $\alpha = \left(\frac{m\pi}{2}\right)^2$ for some integer $m \neq 0$. Then $\kappa = \frac{4}{4\delta + \varepsilon m^2\pi^2}$. Since T is compact, from (53) and the fact that $\sigma(p(T)) = p(\sigma(T))$ where $\sigma(T)$ is a spectrum of T and p is a polynomial, we have $1 \in p(\sigma(T))$. This can be written as

$$1 = \frac{\delta\kappa^2}{\varepsilon} + \left(\sigma - \frac{1}{\varepsilon}\right)\kappa,$$

and this gives

$$\sigma = \sigma(\kappa) = \frac{1}{\kappa} - \frac{\delta\kappa}{\varepsilon} + \frac{1}{\varepsilon}.$$

From $\sigma'(\kappa) = -\frac{1}{\kappa^2} - \frac{\delta}{\varepsilon} < 0$, σ is decreasing in κ. Since $\kappa = \frac{4}{4\delta + \varepsilon m^2\pi^2}$ has a maximum at $m = \pm 1$, we have

$$\begin{aligned}\sigma &= \frac{4\delta + \varepsilon\pi^2}{4} - \frac{4\delta}{\varepsilon(4\delta + \varepsilon\pi^2)} + \frac{1}{\varepsilon} \\ &= \frac{\varepsilon^2\pi^4 + 8\varepsilon\delta\pi^2 + 16\delta^2 + 4\pi^2}{4(4\delta + \varepsilon\pi^2)}.\end{aligned}$$

From the direct computation with $\alpha = \left(\frac{\pi}{2}\right)^2$,

$$\begin{aligned}
\theta(z) &= c_1 \sinh(i\frac{\pi}{2}(z-1)) + c_2 \sinh(i\frac{\pi}{2}(z+1)) = i(c_2 - c_1)\cos\frac{\pi}{2}z \\
\varphi(z) &= \frac{4\delta}{4\delta + \varepsilon\pi^2}\theta(z) \\
\sigma(\delta) &= \frac{\varepsilon^2\pi^4 + 4\pi^2 + 8\varepsilon\delta\pi^2 + 16\delta^2}{4(4\delta + \varepsilon\pi^2)}
\end{aligned}$$

where $\delta = \varepsilon\mu^2$. Differentiating $\sigma(\delta)$, one can see that the minimum value of σ is obtained when $\delta = \frac{\pi}{2} - \frac{\pi^2}{4}\varepsilon$ and the minimum value of σ is π for $\delta \in \mathbb{R}^+$. The leading terms of these values are consistent with those found in Theorem 2. However, note that $\mu \equiv \mu_n = \pi n/c$ for some integer n and $c = \widetilde{L}/h$. We let n^+ and n^- be integers closest to $n_0 \equiv \frac{c}{2\sqrt{\varepsilon}}\sqrt{\frac{2}{\pi} - \varepsilon}$ for $n \geq n_0$ and $n \leq n_0$, respectively. Since σ is a convex function in $\delta = \varepsilon\mu_n^2 > 0$, the minimum of σ can be taken by either n^+ or n^-. If $\sigma(n^+) = \sigma(n^-)$, then $n^+ = n^- + 1$. The equation $\sigma(n) = \sigma(n+1)$ gives a specific expression for h in terms of n and c. Thus, for all h except for a discrete set, the eigenvalue is simple.

Remark 5. From Theorem 4, the solution (θ_m, φ_m) to the system (49, 50) is unique under the normalization $(\theta_m, \varphi_m) \in X_d$. Under this normalization, (θ_m, φ_m) is unique positive minimizer for F_ε and thus

$$\sigma_c^0 = F_\varepsilon(\theta_m, \varphi_m, \varepsilon\mu_m^2).$$

5.1.2. Bifurcation analysis Setting $\varphi = u - y$ in the energy (17), $\varepsilon\mathfrak{F}$ becomes, with $D = (-c, c) \times (-1, 1)$,

$$\mathcal{F}(\theta, \varphi, \sigma) := \int_D \left(\varepsilon|\nabla\theta|^2 + \frac{1}{\varepsilon}(\varphi_x - \sin\theta)^2 + \frac{1}{\varepsilon}(\varphi_y + 1 - \cos\theta)^2 - \sigma\sin^2\theta\right) dydx \tag{54}$$

and $(\theta, \varphi) = (0, 0)$ corresponds to the flat layer state. We define the spaces

$$\begin{aligned}
&\mathbf{Z} = \{\mathbf{z} = (\theta, \varphi) : \theta, \varphi \in C^2(\bar{U}), \theta(x, \pm 1) = \varphi(x, \pm 1) = 0 \text{ for all } x\}, \\
&\mathbf{Y} = C^0(\bar{U}) \times C^0(\bar{U})
\end{aligned}$$

with the graph topology, where $U = \mathbb{R}/(-c + 2c\mathbb{Z}) \times (-1, 1)$, having the periodic boundary condition in the x direction. The Euler-Lagrange equations associated with the energy (54) are, for $\mathbf{z} = (\theta, \varphi)$,

$$F_1(\mathbf{z}, \sigma) := \varepsilon\Delta\theta + \frac{1}{\varepsilon}\begin{pmatrix}\cos\theta \\ -\sin\theta\end{pmatrix}\nabla\varphi - \frac{1}{\varepsilon}\sin\theta + \frac{\sigma}{2}\sin 2\theta = 0, \tag{55}$$

$$F_2(\mathbf{z}, \sigma) := \Delta\varphi - \begin{pmatrix}\cos\theta \\ -\sin\theta\end{pmatrix}\nabla\theta = 0. \tag{56}$$

We let $\mathbf{F} = (F_1, F_2) : \mathbf{Z} \times \mathbb{R} \to \mathbf{Y}$. Recall that σ_c^0 is the minimum of the linearized functional at the undeformed state under the Dirichlet boundary

conditions for both θ and φ. From Theorem 4, we have for $\sigma = \sigma_c^0$ and except for a discrete set for ε, the linearized problem

$$\begin{aligned} &\varepsilon \Delta\theta + \frac{1}{\varepsilon}\varphi_x - \frac{1}{\varepsilon}\theta + \sigma\theta = 0, \\ &\Delta\varphi - \theta_x = 0, \end{aligned}$$

has a solution set spanned by

$$\mathbf{z}_1 = (\theta_1, \varphi_1) := (\theta_m(y)\cos\mu_m x, \varphi_m(y)\sin\mu_m x) \tag{57}$$

for some m. Reformulating these equations in Fourier space and using (22) and Theorem 4, one can find that

$$\begin{aligned} \theta_1 &= \cos\frac{\pi}{2}y\cos\mu_m x, \\ \varphi_1 &= \frac{4\mu_m^2}{\mu_m(4\mu_m^2+\pi^2)}\cos\frac{\pi}{2}y\sin\mu_m x. \end{aligned} \tag{58}$$

Then σ_c^0 is a simple eigenvalue and the structure of the local solution curve can be understood using the Crandall-Rabinowitz theory for bifurcations at simple eigenvalues ([5]):

Theorem 5. *At $\sigma = \sigma_c^0$, there is an $r > 0$ and bifurcation curve of solutions to the system (55, 56) for $s \in (-r, r)$, satisfying*

1. $\mathbf{F}(\mathbf{z}(s), \sigma(s)) = 0$,
2. $(\mathbf{z}(0), \sigma(0)) = (0, \sigma_c^0)$,
3. $\mathbf{z}'(0) = (\theta_1, \varphi_1)$,
4. $\theta(s) = s\theta_1 + \mathcal{O}(s^2), \varphi(s) = s\varphi_1 + \mathcal{O}(s^2)$ *and* $\sigma(s) = \sigma_c^0 + \mathcal{O}(s^2)$.

Furthermore, the system (55, 56) has only two solutions $\mathbf{z} = 0$ and $(\mathbf{z}(s), \sigma(s))$ and the nontrivial solution is stable in a sufficiently small neighborhood of $(0, \sigma_c^0)$

Proof. The existence of such a bifurcation curve will follow from the Crandall-Rabinowitz theory, once we prove the following conditions:

1. $\mathbf{F}(0, \sigma) = 0$,
2. Ker $\mathbf{F}_{\mathbf{z}}(0, \sigma_c^0) = \text{span}\{\theta_1, \varphi_1\}$,
3. codim $\mathcal{R}\mathbf{F}_{\mathbf{z}}(0, \sigma_c^0) = 1$,
4. $\mathbf{F}_{\sigma\mathbf{z}}(0, \sigma_c^0)(\mathbf{z}_1) \notin \mathcal{R}\mathbf{F}_z(0, \sigma_c^0)$.

From the definition of $\mathbf{F}$, the kernel of $L = (L_1, L_2) := \mathbf{F}_{\mathbf{z}}(0, 0, \sigma_c^0)$ is spanned by one element $\mathbf{z}_1 = (\theta_1, \varphi_1)$. The third condition follows from the fact that L is self-adjoint with the inner product $\langle\, , \rangle$,

$$\langle Lu, v\rangle = (L_1 u, v) + \frac{1}{\varepsilon}(L_2 u, v)$$

where $(,)$ is the usual L^2 inner product. Since $\mathbf{F}_{\sigma\mathbf{z}}(0, 0, \sigma_c^0)(\theta_1, \varphi_1) = (\theta_1, 0)$, we find $\mathbf{F}_{\sigma\mathbf{z}}(0, 0, \sigma_c^0)(\theta_1, \varphi_1) \notin \mathcal{R}\mathbf{F}_z(0, 0, \sigma_c^0)$. Therefore $(0, \sigma_c^0)$ is a bifurcation point and there is a one parameter family of nontrivial solutions $(\theta(s), \varphi(s), \sigma(s)) = (s\theta_1 + \theta_2(s), s\varphi_1 + \varphi_2(s), \sigma_c^0 + s\sigma_1)$.

We proceed to study the stability of the nontrivial solution curve. From Theorem 1.16 in [6] it follows that there are eigenvalues $\mu(s)$ and $\gamma(\sigma)$ with eigenvectors $\omega(s)$ and $u(\sigma)$, respectively, such that

$$\mathbf{F}_{\mathbf{z}}(\mathbf{z}(s), \sigma(s))\omega(s) = \mu(s)\omega(s), \tag{59}$$

$$\mathbf{F}_{\mathbf{z}}(0, 0, \sigma)u(\sigma) = \gamma(\sigma)u(\sigma), \tag{60}$$

with $\mu(0) = \gamma(\sigma_c^0) = 0$ and $\omega(0) = (\theta_1, \varphi_1) = u(\sigma_c^0)$. In order to show $\mu(s) < 0$, which means stability of the nontrivial curve, we need to prove [6] that

$$\gamma'(\sigma_c^0) \neq 0 \qquad \text{and} \qquad \lim_{s\to 0} \frac{-s\sigma'(s)\gamma'(\sigma_c^0)}{\mu(s)} = 1, \tag{61}$$

if $\mu(s) \neq 0$ for $s \neq 0$. Since the codimension of the range of $(\mathbf{F}_{\mathbf{z}}(0, 0, \sigma_c^0))$ is 1, there exists $l \in Y^*$ such that $N(l) = \mathcal{R}(\mathbf{F}_{\mathbf{z}}(0, 0, \sigma_c^0))$. Differentiating (60), we obtain that

$$\langle l, \mathbf{F}_{\mathbf{z}\sigma}(0, 0, \sigma_c^0)\mathbf{z}_1\rangle = \gamma'(\sigma_c^0)\langle l, \mathbf{z}_1\rangle. \tag{62}$$

A direct computation gives

$$\mathbf{F}_{\mathbf{z}\sigma}(0, 0, \sigma_c^0)(\theta_1, \varphi_1) = (\theta_1, 0) \tag{63}$$

and hence

$$\gamma'(\sigma_c^0) > 0. \tag{64}$$

It is not surprising to see that $\sigma'(0) = 0$ since the linearized operator is invariant with respect to $(\theta, \varphi) \to (-\theta, -\varphi)$ transformation. The proof is below. Then $\sigma'(s) = \sigma^{''}(0)s + \mathcal{O}(s^2)$ and hence the formula (61) gives $\mu(s) < 0$ if $\sigma^{''}(0) > 0$. Thus, Theorem 5 follows from the following lemma and this is a supercritical pitchfork bifurcation.

Lemma 8. $\sigma'(0) = 0$ *and* $\sigma^{''}(0) > 0$.

Proof. The derivative of σ can be evaluated using the formula [18]

$$\sigma'(0) = -\frac{1}{2}\frac{\langle \mathbf{F}_{\mathbf{zz}}(0, 0, \sigma_c^0)(\mathbf{z}_1, \mathbf{z}_1), (\mathbf{z}_1)\rangle}{\langle \mathbf{F}_{\mathbf{z}\sigma}(0, 0, \sigma_c^0)\mathbf{z}_1, \mathbf{z}_1\rangle}. \tag{65}$$

Calculating the derivatives of $\mathbf{F}$, we get

$$\mathbf{F}_{\mathbf{zz}}(0, 0, \sigma_c^0)(\theta_1, \varphi_1) = (-\frac{2}{\varepsilon}\theta_1(\varphi_1)_y, 2\theta_1(\theta_1)_y).$$

Combining this with (65) and (63), we have

$$\begin{aligned}
\sigma'(0) &= -\frac{1}{2}\int_D \left(-\frac{2}{\varepsilon}\theta_1(\varphi_1)_y\theta_1 + \frac{2}{\varepsilon}\theta_1(\theta_1)_y\varphi_1\right) dy\, dx \\
&= -\frac{1}{\varepsilon}\int_D \left(-\theta_1^2(\varphi_1)_y - \frac{1}{2}\theta_1^2(\varphi_1)_y\right) dxdy \\
&= \frac{3}{2\varepsilon}\int_D \theta_1^2(\varphi_1)_y\, dxdy
\end{aligned} \tag{66}$$

Using (58), one can find that $\sigma'(0) = 0$. Then the formula for $\sigma''(0)$ is given by [18]

$$\sigma''(0) = -\frac{1}{3}\frac{\langle \mathbf{F}_{\mathbf{zzz}}(0,0,\sigma_c^0)(\mathbf{z}_1,\mathbf{z}_1,\mathbf{z}_1),\mathbf{z}_1\rangle}{\langle \mathbf{F}_{\mathbf{z}\sigma}(0,0,\sigma_c^0)\mathbf{z}_1,\mathbf{z}_1\rangle}. \tag{67}$$

From a direct computation, we have

$$\mathbf{F}_{\mathbf{zzz}}(0,0,\sigma_c)(\mathbf{z}_1,\mathbf{z}_1,\mathbf{z}_1) = (\frac{1}{\varepsilon}(-\theta_1^2(\varphi_1)_x + \theta_1^3) - 2\sigma_c^0, 3\theta_1^2(\theta_1)_x).$$

Using this in (67), we obtain

$$\begin{aligned}\sigma''(0) &= -\frac{1}{3}\int_D \left(\frac{1}{\varepsilon}(-3\theta_1^2(\varphi_1)_x + \theta_1^3)\theta_1 - 2\sigma_c^0\theta_1^4 + \frac{3}{\varepsilon}\theta_1^2(\theta_1)_x\varphi\right) dxdy \\ &= -\frac{1}{3}\int_D \left[\frac{1}{\varepsilon}\Big(-3\theta_1^3(\varphi_1)_x + \theta_1^4 + 3\theta_1^2(\theta_1)_x\varphi\Big) - 2\sigma_c^0\theta_1^4\right] dxdy \\ &= -\frac{1}{3}\int_D \left(\frac{1}{\varepsilon}\theta_1^4 - \frac{4}{\varepsilon}\theta_1^3(\varphi_1)_x - 2\sigma_c^0\theta_1^4\right) dxdy, \end{aligned} \tag{68}$$

if we integrate by parts. Using (58), this becomes

$$\sigma''(0) = -\frac{1}{3}\int_D \left(\frac{1}{\varepsilon} - 2\sigma_c^0 - \frac{4}{\varepsilon}\frac{4\mu_m^2}{4\mu_m^2+\pi^2}\right)\theta_1^4\, dxdy.$$

Since $\varepsilon \ll 1$ and $\mu_m^2 = \mathcal{O}(\frac{1}{\varepsilon})$, it is easy to see that $\sigma''(0) > 0$ as claimed.

5.2. Natural boundary condition on layers

5.2.1. Linearized functional We use a similar procedure to find an eigenfunction and an eigenvalue for the Euler-Lagrange equation of the functional F_ε and use formal asymptotic expansions to solve the algebraic equation that occurs in the process. Furthermore, we show that the kernel of the linearized operator at the first eigenvalue is one dimensional. We also find the first eigenvalue and the corresponding eigenfunction explicitly for bifurcation study.

Theorem 6. *A minimizer of F_ε in X_n is taken by real valued functions (θ_n, φ_n), which satisfy the system (49) with the boundary condition*

$$\theta_n(\pm 1) = 0 \quad \textit{and} \quad \varphi_n'(\pm 1) = 0. \tag{69}$$

Also, the minimizer of F_ε is obtained when $\delta_n = \delta_1 = \varepsilon(\pi/c)^2 + \mathcal{O}(\varepsilon^2)$ and the minimum value of the functional is $\lambda_n = \lambda_1 = 1 + (0.5 + 2/3(\pi/c)^2)\varepsilon + \mathcal{O}(\varepsilon^2)$. For $\sigma_n = min\, F_\varepsilon(\theta_n, \varphi_n)$ in $\tilde{\mathcal{B}}$, the solution to the system is spanned by the one element,

$$\begin{aligned}\theta_n(y) &= -\cos\sqrt{\gamma}\cosh\sqrt{\alpha}y + \cosh\sqrt{\alpha}\cos\sqrt{\gamma}y, \\ \varphi_n(y) &= -A\cos\sqrt{\gamma}\cosh\sqrt{\alpha}y + B\cosh\sqrt{\alpha}\cos\sqrt{\gamma}y,\end{aligned} \tag{70}$$

where α, γ, A, and B are given in the proof below. Furthermore, any solution of (49, 69) is of the form $c(\theta_n, \varphi_n)$ for some constant c.

Proof. The system of Euler-Lagrange equations (49) can be written as

$$(-\varepsilon D^2+\delta)\theta-\frac{1}{\varepsilon}\varphi+(\frac{1}{\varepsilon}-\lambda)\theta=0$$
$$(-\varepsilon D^2+\delta)\varphi=\delta\theta$$

where $D=\frac{d}{dy}$ denotes the derivative. Applying $-\varepsilon D^2+\delta$, we obtain a fourth order differential equation for θ,

$$\varepsilon^2\theta^{''''}+(-2\varepsilon\delta+\lambda\varepsilon-1)\theta^{''}+\delta(\delta-\lambda)\theta=0. \tag{71}$$

One can check that the following functions are solutions to this equation that are linearly independent:

$$\theta_1(y)=\sinh(\sqrt{\alpha}(y-1)) \qquad \theta_2(y)=\sinh(\sqrt{\alpha}(y+1))$$
$$\theta_3(y)=\sin(\sqrt{\gamma}(y-1)) \qquad \theta_4(y)=\sin(\sqrt{\gamma}(y+1)),$$

where

$$\alpha=\frac{2\varepsilon\delta+a+\sqrt{a^2+4\varepsilon\delta}}{2\varepsilon^2},\quad \gamma=-\frac{2\varepsilon\delta+a-\sqrt{a^2+4\varepsilon\delta}}{2\varepsilon^2},\quad \text{and}\quad a=1-\lambda\varepsilon.$$

Therefore the general solution to (71) is $\theta(x)=\sum_{i=1}^4 c_i\theta_i(x)$.

From the definition of λ, one can see that $\delta<\lambda$ and it is equivalent to $\gamma>0$. From the second equation of (49), we have

$$\varphi=c_1A\theta_1+c_2A\theta_2+c_3B\theta_3+c_4B\theta_4 \tag{72}$$

where

$$A=-\varepsilon^2\alpha+\varepsilon\delta+a=\frac{a-\sqrt{a^2+4\delta\varepsilon}}{2}<0,$$
$$B=\varepsilon^2\gamma+\varepsilon\delta+a=\frac{a+\sqrt{a^2+4\delta\varepsilon}}{2}>a.$$

Using the boundary condition (69),we have

$$\begin{vmatrix} \sinh(2\sqrt{\alpha}) & 0 & \sin(2\sqrt{\gamma}) & 0 \\ 0 & \sinh(2\sqrt{\alpha}) & 0 & \sin(2\sqrt{\gamma}) \\ A\sqrt{\alpha}\cosh(2\sqrt{\alpha}) & A\sqrt{\alpha} & B\sqrt{\gamma}\cos(2\sqrt{\gamma}) & B\sqrt{\gamma} \\ A\sqrt{\alpha} & A\sqrt{\alpha}\cosh(2\sqrt{\alpha}) & B\sqrt{\gamma} & B\sqrt{\gamma}\cos(2\sqrt{\gamma}) \end{vmatrix}=0. \tag{73}$$

Using the notation

$$s_1=\sinh(2\sqrt{\alpha}),\ s_2=\sin(2\sqrt{\gamma}),\ c_1=\cosh(2\sqrt{\alpha}),\ c_2=\cos(2\sqrt{\gamma}),$$

and setting $K=\sqrt{\gamma}B/(\sqrt{\alpha}A)<0$, this matrix is equivalent to

$$\begin{pmatrix} s_1 & 0 & s_2 & 0 \\ 0 & s_1 & 0 & s_2 \\ c_1 & 1 & Kc_2 & K \\ 1 & c_1 & K & Kc_2 \end{pmatrix}\sim\begin{pmatrix} 1 & c_1 & K & Kc_2 \\ 0 & 1 & 0 & s_2/s_1 \\ 0 & 0 & 1 & \frac{s_1s_2+K(1-c_1c_2)}{K(c_2-c_1)} \\ 0 & 0 & 0 & b \end{pmatrix}, \tag{74}$$

where $b = s_2c_1 - Kc_2s_1 + (Ks_1 - s_2)\frac{s_1s_2+K(1-c_1c_2)}{K(c_2-c_1)}$. Then a direct computation for $b = 0$ gives

$$\begin{aligned} 0 &= (K^2(s_1 - c_2^2 s_1) + Ks_2(2c_1c_2 - c_1^2 - s_1^2 - 1) - s_1s_2^2 \\ &= s_2\big((K^2-1)s_1s_2 - 2K(1-c_1c_2)\big). \end{aligned}$$

Thus we have either $s_2 = \sin(2\sqrt{\gamma}) = 0$ or

$$(K^2-1)s_1s_2 - 2K(1-c_1c_2) = 0. \tag{75}$$

Note that the first equation is identical to the formula with Dirichlet boundary condition. In order to simplify the second equation, we let $p_1 = \sinh\sqrt{\alpha}, p_2 = \sin\sqrt{\gamma}, q_1 = \cosh\sqrt{\alpha}$, and $q_2 = \cos\sqrt{\gamma}$. Then, using hyperbolic identities, (75) becomes

$$2p_1p_2q_1q_2(1-K^2) = -2K(1-(2p_1^2+1)(1-2p_2^2)),$$

that is,

$$p_1p_2q_1q_2K^2 + (p_1^2q_2^2 - p_2^2q_1^2)K - p_1p_2q_1q_2 = 0.$$

By solving this equation for K and using $K < 0$, we get

$$K = -\frac{p_1q_2}{p_2q_1}. \tag{76}$$

This can be written as

$$\tanh(\sqrt{\alpha}) = -K\tan(\sqrt{\gamma}). \tag{77}$$

Now we find an eigenfunction from (74). Using (76), the matrix (74) becomes

$$\begin{pmatrix} 1 & 0 & 0 & -s_2/s_1 \\ 0 & 1 & 0 & s_2/s_1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and hence we have

$$\begin{aligned} \theta(y) &= c[\sin(2\sqrt{\gamma})(\theta_1-\theta_2) - \sinh(2\sqrt{\alpha})(\theta_3-\theta_4)] \\ &= c[\sin(2\sqrt{\gamma})(-2\cosh(\sqrt{\alpha}y)\sinh\sqrt{\alpha}) - \sinh(2\sqrt{\alpha})(-2\cos(\sqrt{\gamma}y)\sin\sqrt{\gamma})] \\ &= c[\cos\sqrt{\gamma}\cosh\sqrt{\alpha}y - \cosh\sqrt{\alpha}\cos\sqrt{\gamma}y] \end{aligned}$$

for some positive constant c, which may differ from line to line. Thus we can see that a solution to the system (49) with the boundary condition (69) is given by (70). The graphs of these eigenfunctions are given in Figure 2 with $\varepsilon = 0.1$ and $\varepsilon = 0.02$. Note that these are similar to Figure 1 with the appropriate choice of c.

Now we need to solve (77) to have an estimate of the first eigenvalue. From the Γ-convergence theory in section 4.2, for $\varepsilon \ll 1$, we have

$$\lambda \sim 1 \qquad \text{and} \qquad \delta \sim 0. \tag{78}$$

From $\alpha > a/\varepsilon^2$, we have $\tanh(\sqrt{\alpha}) \sim 1$. Since

$$\begin{aligned}\gamma &= \frac{1}{2\varepsilon^2}\left(-2\varepsilon\delta - a + \sqrt{a^2+4\varepsilon\delta}\right)\\ &= \frac{1}{2\varepsilon^2}\left(-2\varepsilon\delta - a + a + \frac{2\varepsilon\delta}{a} - \frac{2\varepsilon^2\delta^2}{a^3} + \mathcal{O}(\varepsilon^3\delta^3)\right)\\ &= \frac{1}{2\varepsilon^2}\left(2\varepsilon\delta\left(\frac{1}{a}-1\right) - \frac{2\varepsilon^2\delta^2}{a^3} + \mathcal{O}(\varepsilon^3\delta^3)\right)\\ &= \frac{\lambda\delta}{a} - \frac{\delta^2}{a^3} + \mathcal{O}(\varepsilon\delta^3),\end{aligned} \tag{79}$$

and from (78), we see $\gamma \sim 0$. A direct calculation of K gives

$$K = \frac{-2\delta + \lambda(a+D)}{-2\sqrt{\delta}\sqrt{\lambda-\delta}} \tag{80}$$

where $D = \sqrt{a^2+4\varepsilon\delta}$. For (77), we proceed with a formal asymptotic expansion for $\varepsilon \ll 1$. Assume that

$$\lambda = 1 + b_1\varepsilon + \mathcal{O}(\varepsilon^2), \qquad \delta = c_1\varepsilon + \mathcal{O}(\varepsilon^2). \tag{81}$$

Since $\tanh(\sqrt{\alpha}) \to 1$ as $\varepsilon \to 0$ exponentially and $\gamma \to 0$, after squaring both sides of the equation, we solve

$$1 = K^2\left(\gamma + \frac{2}{3}\gamma^2 + \mathcal{O}(\varepsilon^2)\right)$$

for (77). Using (80), it becomes

$$4\delta(\lambda-\delta) = (-2\delta + \lambda(a+D))^2\left(\gamma + \frac{2}{3}\gamma^2 + \mathcal{O}(\varepsilon^2)\right). \tag{82}$$

From (79) and direct calculations, we write

$$\begin{aligned}\gamma^2 &= \frac{\delta^2}{a^2}(\lambda^2 + \mathcal{O}(\varepsilon)),\\ (-2\delta + \lambda(a+D))^2 &= 4(\lambda^2a^2 - 2\lambda a\delta + \mathcal{O}(\varepsilon^2)).\end{aligned}$$

The equation (82) with (79) gives

$$a(\lambda-\delta) = (\lambda^2a^2 - 2\lambda a\delta + \mathcal{O}(\varepsilon^2))\left(\lambda - \frac{\delta}{a^2} + \frac{2\delta}{3a}\lambda^2 + \mathcal{O}(\varepsilon^2)\right).$$

Using (81), it becomes

$$1 + (b_1 - c_1 - 1)\varepsilon = 1 + \left(3b_1 - 2 - \frac{7}{3}c_1\right)\varepsilon + \mathcal{O}(\varepsilon^2).$$

Then we have

$$b_1 = \frac{1}{2} + \frac{2}{3}c_1.$$

Since $\delta = \varepsilon\mu_n^2 = \varepsilon(\pi n/c)^2 > 0$, we have $c_1 > 0$ and hence b_1 is minimized when $n = 1$, i.e., $c_1 = (\pi/c)^2$.

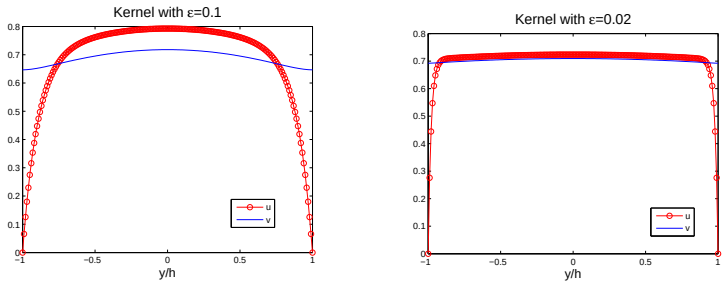

Fig. 2. plots of eigenfunctions of the system (49) with (69) when $\varepsilon = 0.1$ and $\varepsilon = 0.02$.

5.2.2. Bifurcation analysis As in the section 5.1.2, we consider the energy (54) so that $(\theta, \varphi) = (0, 0)$ corresponds to the flat layer state. We define the spaces

$$\mathbf{Z} = \{\mathbf{z} = (\theta, \varphi) : \theta, \varphi \in C^2(\bar{U}), \theta(x, \pm 1) = 0, \varphi'(x, \pm 1) = 0 \text{ for all } x\},$$
$$\mathbf{Y} = C^0(\bar{U}) \times C^0(\bar{U})$$

where $U = \mathbb{R}/(-c+2c\mathbb{Z}) \times (-1, 1)$. The Euler-Lagrange equations associated with the energy (54) are (55, 56). Note that σ_c is the minimum of the linearized functional at the undeformed state under the Natural boundary conditions for φ assuming that $\|\theta\|_2 = 1$. From Theorem 4, we have for $\sigma = \sigma_c$, the linearized problem

$$\varepsilon \Delta\theta + \frac{1}{\varepsilon}\varphi_x - \frac{1}{\varepsilon}\theta + \sigma\theta = 0$$
$$\Delta\varphi - \theta_x = 0$$

has a solution set spanned by

$$\mathbf{z}_1 := (\theta_1, \varphi_1) = (u(y) \cos \mu_1 x, \frac{v(y)}{\mu_1} \sin \mu_1 x). \tag{83}$$

Reformulating these equations in Fourier space and using Theorem 4, one can find that from (70),

$$\begin{aligned} u(y) &= c(-\cos\sqrt{\gamma}\cosh\sqrt{\alpha}y + \cosh\sqrt{\alpha}\cos\sqrt{\gamma}y), \\ v(y) &= c\left(-A\cos\sqrt{\gamma}\cosh\sqrt{\alpha}y + B\cosh\sqrt{\alpha}\cos\sqrt{\gamma}y\right), \end{aligned} \tag{84}$$

where $c > 0$ is chosen so that $(u, v) \in X_n$. The graphs of u and v are shown in Figure 2. Thus σ_c is a simple eigenvalue and therefore, as in section 5.1.2, we may use the Crandall-Rabinowitz theory to study the local behavior of the nontrivial solution curve.

Theorem 7. *At $\sigma = \sigma_c$, there is an $r > 0$ and bifurcation curve of solutions to the system (55, 56) for $s \in (-r, r)$, satisfying*

1. $\mathbf{F}(\mathbf{z}(s), \sigma(s)) = 0$,

2. $(\mathbf{z}(0), \sigma(0)) = (0, \sigma_c)$,
3. $\mathbf{z}'(0) = (\theta_1, \varphi_1)$,
4. $\theta(s) = s\theta_1 + \mathcal{O}(s^2), \varphi(s) = s\varphi_1 + \mathcal{O}(s^2)$ *and* $\sigma(s) = \sigma_c + \mathcal{O}(s^2)$.

Furthermore, the system (55, 56) with condition (69) has only two solutions $\mathbf{z} = 0$ *and* $(\mathbf{z}(s), \sigma(s))$ *and the nontrivial solution is stable in a sufficiently small neighborhood of* $(0, \sigma_c)$.

Proof. The proof for this theorem proceeds in exactly the same way as the proof of Theorem 5. One can easily see that from (66) and (83),

$$\sigma'(0) = \frac{3}{2\varepsilon} \int_D \theta_1^2 (\varphi_1)_y \, dxdy = 0.$$

Also, we derive from(68),

$$\sigma''(0) = -\frac{1}{3} \int_D \left(\frac{1}{\varepsilon}\theta_1^4 - \frac{4}{\varepsilon}\theta_1^3 (\varphi_1)_x - 2\sigma_c \theta_1^4 \right) dxdy.$$

Then one can see that $\sigma''(0) > 0$, which gives a supercritical bifurcation. Moreover,

$$\begin{aligned}
\theta_1^4 - 4\theta_1^3(\varphi_1)_x =& u^3 \Big((-1 + 4A) \cos\sqrt{\gamma} \cosh(\sqrt{\alpha} y) \\
&+ (1 - 4B) \cosh\sqrt{\alpha} \cos(\sqrt{\gamma} y) \Big) \cos^4 \mu_1 x, \\
=& u^3 \cos^4 \mu_1 x \Big((1 - 4B) u(y) + 4(B - A)(-\cos\sqrt{\gamma} \cosh(\sqrt{\alpha} y)) \Big) \\
<& 0,
\end{aligned}$$

which follows from the estimates $A = \mathcal{O}(\varepsilon), B \sim 1, \gamma \sim 0$, and $\alpha = \mathcal{O}(\frac{1}{\varepsilon^2})$. From (61) and the same analysis as in section 5.1.2, one concludes that there exists a stable bifurcation curve of nontrivial solutions.

6. Numerical Simulations

We consider the gradient flow (in L^2) of the energy

$$F(\mathbf{n}, \phi) = \frac{1}{2} \int_\Omega \left\{ |\nabla \mathbf{n}|^2 + |\nabla \phi - \mathbf{n}|^2 - \kappa (\mathbf{n} \cdot \mathbf{h})^2 \right\},$$

and study the behavior of the solutions with both Dirichlet and Neumann conditions, as described in the previous sections. The gradient flow equations are

$$\begin{aligned}
\frac{\partial \phi}{\partial t} &= \Delta \phi - \nabla \cdot \mathbf{n}, \\
\frac{\partial \mathbf{n}}{\partial t} &= \Pi_{\mathbf{n}} \left(\Delta \mathbf{n} + \nabla \phi - \mathbf{n} + \kappa (\mathbf{n} \cdot \mathbf{h}) \mathbf{h} \right),
\end{aligned} \tag{85}$$

where we have defined, for a given vector $f \in \mathbb{R}^3$, the orthogonal projection onto the plane orthogonal to the vector $\mathbf{n}$ as

$$\Pi_n(f) = f - (\mathbf{n} \cdot f)\mathbf{n}.$$

This projection appears as a result of the constraint $|\mathbf{n}| = 1$. For numerical purposes, it is more convenient to write this term as

$$\frac{\partial \mathbf{n}}{\partial t} = -\mathbf{n} \times (\mathbf{n} \times (\Delta \mathbf{n} + \nabla \phi - \mathbf{n} + \kappa(\mathbf{n} \cdot \mathbf{h})\mathbf{h})). \tag{86}$$

Written in this way, the equation resembles the Landau-Lifshitz equation of micromagnetics in the high damping limit [10], and the heat-flow of harmonic maps [27].

As initial condition, we consider a small perturbation from the undeformed state. More precisely, for all $(x, y) \in \Omega$,

$$\begin{aligned} \mathbf{n}(x, y, 0) &= \frac{(\epsilon u_1, 1 + \epsilon u_2)}{|(\epsilon u_1, 1 + \epsilon u_2)|}, \\ \phi(x, y, 0) &= y + \epsilon \phi_0, \end{aligned}$$

where a small number $\epsilon = 0.01$ and u_1, u_2, and ϕ_0 are arbitrarily chosen. As described in section 2, we impose strong anchoring condition for the director field, (4), and either a homogeneous Dirichlet boundary condition or a natural boundary condition on ϕ at the top and the bottom plates. The natural boundary condition on ϕ is

$$\frac{\partial \phi}{\partial \nu}\Big|_{y=\pm h} = \mathbf{n} \cdot \nu\Big|_{y=\pm h}.$$

Periodic boundary conditions are imposed for both $\mathbf{n}$ and ϕ in the x direction.

We use a Fourier spectral discretization in the x direction, and second order finite differences in the y direction. The fast Fourier transform is computed using the FFTW libraries [11]. For the temporal discretization, we combine a projection method for the variable $\mathbf{n}$ [10], with a semi-implicit scheme for ϕ: Given $(\phi^k, \mathbf{n}^k)$, we solve

$$\frac{\phi^{k+1} - \phi^k}{\Delta t} = \Delta \phi^{k+1} - \nabla \cdot \mathbf{n}^*, \tag{87}$$

$$\frac{\mathbf{n}^* - \mathbf{n}^k}{\Delta t} = \Delta \mathbf{n}^* + \nabla \phi^{k+1} - \mathbf{n}^k + \kappa(\mathbf{n}^k \cdot \mathbf{h})\mathbf{h}, \tag{88}$$

$$\mathbf{n}^{k+1} = \frac{\mathbf{n}^*}{|\mathbf{n}^*|}. \tag{89}$$

The last step ensures that $|\mathbf{n}^{k+1}| = 1$ at each grid point. The method is first order accurate in time and second order accurate in space. To solve the implicit system, we perform a discrete Fourier transform in the x direction. The resulting tridiagonal systems in the y direction are solved using Gauss

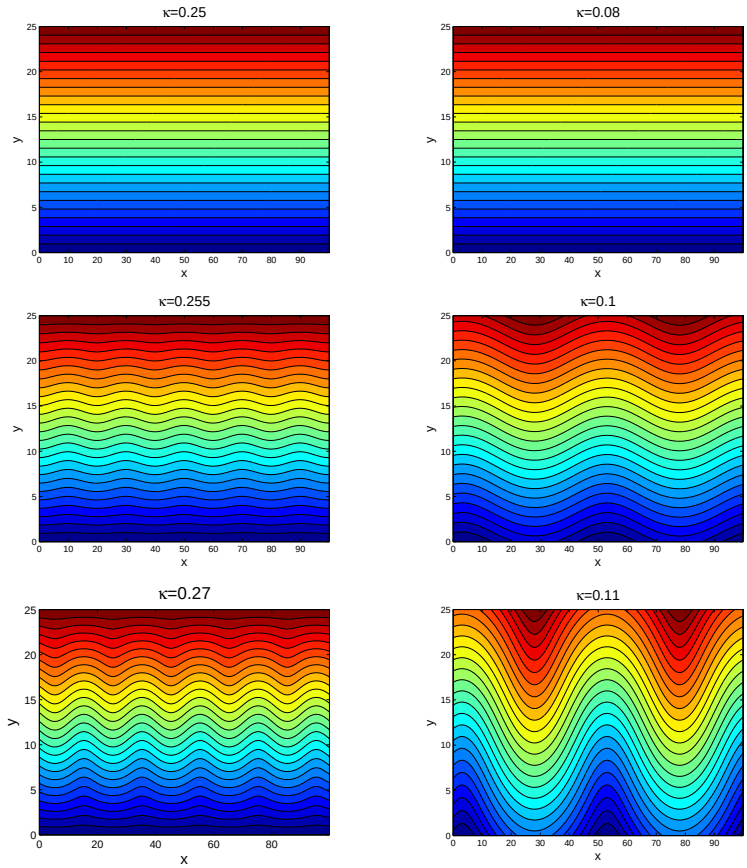

Fig. 3. Contour plots of ϕ, the solution of the system (85). The first and the second column depict the layer when the homogeneous Dirichlet and natural boundary conditions are imposed on ϕ, respectively.

elimination. The complexity of the method is comparable to that of solving the heat equation with constant coefficients using the Backward Euler method.

We take the domain size $L = 50$ and $h = 12.5$, then $c = L/h = 4$. A more physically relevant value for h in smectic A liquid crystals is 5×10^5. However, the layer undulations can already be observed for moderate values of h. In fact, the undulations in cholesteric liquid crystals occur with $h \approx 5$ ([22]). The numbers of grid points in the x and y directions are 1024 and 512, respectively.

In Figure 3 we show the layer structures in response to the various magnetic field strengths κ. In the figures we show the contour maps of ϕ since the level sets of ϕ represent the smectic layers. One can see that the undeformed state is stable for values of κ below the critical field κ_c. If κ increases and reaches κ_c, the layer undulations occur.

We analyzed the undulations when either homogeneous Dirichlet boundary conditions or natural boundary conditions are imposed on the layer function ϕ. The first column of Figure 3 depicts the layers when the homogeneous Dirichlet boundary condition is applied while in the second column the natural boundary condition is used for the layer function. Our analysis gives estimates on the critical field (Remark 2 and Remark 3) and the wave numbers ((35), (46)). In the case of fixed layers, we have an estimate on the critical field, $\kappa \approx \pi/h = \pi/12.5 \approx 0.251$. The numerical simulation shows that the critical field is between 0.25 and 0.255. Also from (35), we see that $\mu^2 = \pi/2 - \pi^2/(4h)$. After changing a variable, the period of the undulations in the x-direction is given by $2\pi h(\pi h/2 - \pi^2/4)^{-1/2} \approx 19$ and hence we expect about 5.2 wave numbers. The numerical simulations show that the same results in the first column of Figure 3.

For the general case, the critical field is given by $\kappa \approx 1/h = 0.08$ from (46). Also, according to our analysis, the frequency in the x direction are lower when the natural boundary condition is applied on the layer function. The critical field and the frequency are higher in the first column in Fig. 3, which confirms these results.

In the first column of the Figure 3 the maximum undulation is seen to occur in the middle of the cell ($y = 0$) and the displacement amplitude decreases gradually as approaching the boundary ($y = \pm h$), which was expected from the estimate (36) and Theorem 4. This was the same result from the classic Helfrich-Hurault theory, in which the director and the gradient of the layer function are considered to be equal. In this case, and in the case of strong anchoring conditions, the layers are fixed at the boundary. Thus it is natural that the Dirichlet boundary condition on the layers leads the same result as in the Helfrich-Hurault theory. However, the second column of Figure 3 indicates that the displacement amplitude does not diminish when approaching the boundary and it seems that the amplitude does not depend on the y coordinate. This agrees with the estimate (47). As κ increases beyond κ_c, the displacement amplitude increases as in the third row of Figure 3. In the case well above the critical field, however, one must study the model with nonconstant layer density for more appropriate undulation phenomena. According to the experiment [22], when ε is not sufficiently small such as $\varepsilon \sim 1/5$, the displacement amplitude weakly depends on the vertical coordinate and loses the dependence completely as the magnetic field increases. However, (47) and more precisely (70) indicate that if ε is sufficiently small, then the amplitude is almost the same throughout the cell at the onset of the undulations.

7. Complex de Gennes Energy

In this section, we consider the original de Gennes functional without the assumption that the smectic layer density is constant. The model is

given by

$$\mathcal{G}(\psi,\mathbf{n}) = \int_\Omega \left(C|\nabla\psi - iq\mathbf{n}\psi|^2 + K|\nabla\mathbf{n}|^2 + \frac{g}{2}\big(|\psi|^2 - \frac{r}{g}\big)^2 - \lambda(\mathbf{n}\cdot\mathbf{H})^2 \right) dxdy. \tag{90}$$

Here we assume that material constants C, K, g, and temperature r are fixed positive constants. By scaling ψ, we may assume that $\frac{r}{g} = 1$. With this model we will investigate bifurcation analysis for the onset of the layer undulation phenomena in this section. Assuming $\mathbf{n} = (\sin\theta, \cos\theta)$, $(\psi_0, \theta_0) \equiv (\tilde{c}e^{iqy}, 0)$ is a trivial critical point of $\mathcal{G}$ where $\tilde{c} \in \mathbb{C}$ such that $|\tilde{c}| = 1$. Now we find the second variation of $\mathcal{G}$ at the undeformed state, $\psi_0 = \tilde{c}e^{iqy}$, $\theta_0 = 0$,

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}\mathcal{G}(\psi_0 + t\psi, t\theta)\Big|_{t=0} = \int_\Omega (&C|\psi_x - iq\theta\psi_0|^2 + C|\psi_y - iq\psi|^2 \\ &+K|\nabla\theta|^2 + 2g[Re(\psi_0\bar{\psi})]^2 - \lambda|\theta|^2)\, dxdy.\end{aligned} \tag{91}$$

The undeformed state, (θ_0, ϕ_0), is stable if the second variation is nonnegative. As in [19], we write

$$\psi = iq\psi_0\varphi. \tag{92}$$

Then the second variation above becomes

$$\int_\Omega (Cq^2|\varphi_x - \theta|^2 + Cq^2|\varphi_y|^2 + K|\nabla\theta|^2 + 2gq^2[Im(\varphi)]^2 - \lambda|\theta|^2)\, dxdy. \tag{93}$$

The critical field σ_c is defined by

$$\sigma_c = \inf_{(\varphi,\theta)\in\mathcal{A}} \mathcal{L}(\varphi,\theta), \tag{94}$$

where

$$\mathcal{L}(\varphi,\theta) = \int_\Omega (Cq^2|\varphi_x - \theta|^2 + Cq^2|\varphi_y|^2 + K|\nabla\theta|^2)\, dxdy$$

and the admissible set $\mathcal{A}$ is given by

$$\mathcal{A} = \{(\varphi,\theta) \in H^1(U) \times H^1(U) : \|\theta\|_2 = 1, \theta(x, \pm d) = 0 \text{ for all } x\},$$

where $U = \mathbb{R}/(-L+2L\mathbb{Z}) \times (-d, d)$. From this formulation, one may assume that φ is a real-valued function since given a minimizer $\varphi = \Re\{\varphi\} + i\Im\{\varphi\}$, the function $\widetilde{\varphi} = \Re\{\varphi\}$ is admissible, and has lower energy.

In the Dirichlet case, the critical field σ_c^0 is defined by

$$\sigma_c^0 = \inf_{(\varphi,\theta)\in\mathcal{A}_0} \mathcal{L}(\varphi,\theta),$$

where

$$\mathcal{A}_0 = \{(\varphi,\theta) \in \mathcal{A} : \varphi(x, \pm d) = 0 \text{ for all } x\},$$

This linearized functional was analyzed in the previous theorems 4 and 6 in detail. Here we make a change of variables as before. Setting

$$\widetilde{x} = \frac{1}{d}x, \qquad \widetilde{y} = \frac{1}{d}y, \qquad \varphi(x,y) = d\,\phi(\widetilde{x},\widetilde{y}), \tag{95}$$

we write (91), recalling (19),

$$\frac{K}{\varepsilon}\int_{\widetilde{\Omega}}\left(\frac{1}{\varepsilon}(\phi_x-\theta)^2+\frac{1}{\varepsilon}\phi_y^2+\varepsilon|\nabla\theta|^2-\sigma\theta^2\right)d\widetilde{x}d\widetilde{y} = \frac{K}{\varepsilon}\mathcal{J}(\theta,\phi), \tag{96}$$

where

$$\varepsilon = \frac{\sqrt{K}}{\sqrt{C}qd} \qquad \text{and} \qquad \sigma = \frac{d\lambda}{q\sqrt{CK}}.$$

With both Dirichlet and Natural boundary conditions on φ, we will show the existence and stability of the bifurcation curves at the undeformed state. The natural boundary condition on ψ is

$$\frac{\partial\psi}{\partial\nu} = iq\mathbf{n}\psi\cdot\nu.$$

We define the spaces

$$\begin{aligned}
\mathbf{Z}_d &= \{\mathbf{z}=(u,v,\theta): u,v,\theta\in C^2(\bar{U}), \theta(x,\pm d)=0,\\
&\qquad u(x,\pm d)=\Re\{\tilde{c}e^{\pm iqd}\}, v(x,\pm d)=\Im\{\tilde{c}e^{\pm iqd}\},\ \text{for all } x\},\\
\mathbf{Z}_n &= \{\mathbf{z}=(u,v,\theta): u,v,\theta\in C^2(\bar{U}), \theta(x,\pm d)=0,\\
&\qquad u_y(x,\pm d)=-qv(x,\pm d), v_y(x,\pm d)=qu(x,\pm d),\ \text{for all } x\},\\
\mathbf{Y} &= C^0(\bar{U})\times C^0(\bar{U})
\end{aligned}$$

where $U=\mathbb{R}/(-L+2L\mathbb{Z})\times(-d,d)$, having $\mathbf{z}$ periodic in the x direction. The Euler-Lagrange equations associated with the energy (90) are, for $\mathbf{z}=(u,v,\theta)$,

$$\begin{aligned}
F_1(\mathbf{z},\sigma) &:= C[\Delta u+2q(\sin\theta v_x+\cos\theta v_y)+q(\cos\theta\theta_x-\sin\theta\theta_y)v-q^2u]\\
&\quad +g(1-u^2-v^2)u=0,
\end{aligned} \tag{97}$$

$$\begin{aligned}
F_2(\mathbf{z},\sigma) &:= C[\Delta v-2q(\sin\theta u_x+\cos\theta u_y)-q(\cos\theta\theta_x-\sin\theta\theta_y)u-q^2v]\\
&\quad +g(1-u^2-v^2)v=0,
\end{aligned} \tag{98}$$

$$\begin{aligned}
F_3(\mathbf{z},\sigma) &:= Cq[\cos\theta(uv_x-vu_x)-\sin\theta(uv_y-vu_y)]+K\Delta\theta\\
&\quad +\sigma\sin\theta\cos\theta=0.
\end{aligned} \tag{99}$$

We let $\mathbf{F}=(F_1,F_2,F_3):\mathbf{Z}_i\times\mathbb{R}\to\mathbf{Y}$, where $i=d$ for Dirichlet boundary condition or $i=n$ for natural boundary condition on u and v. We find

$$\begin{aligned}
D_{\mathbf{z}}F_1(\mathbf{z}_0,\sigma)[\mathbf{z}] &= C(\Delta u+2qv_y+qv_0\theta_x-q^2u)-2g(u_0u+v_0v)u_0,\\
D_{\mathbf{z}}F_2(\mathbf{z}_0,\sigma)[\mathbf{z}] &= C(\Delta v-2qu_y-qu_0\theta_x-q^2v)-2g(u_0u+v_0v)v_0,\\
D_{\mathbf{z}}F_3(\mathbf{z}_0,\sigma)[\mathbf{z}] &= Cq(u_0v_x-v_0u_x-q\theta)+K\Delta\theta+\sigma\theta.
\end{aligned} \tag{100}$$

From sections 5.1.1 and 5.2.1, the Kernel of $F_{\mathbf{z}}$ is one dimensional at the critical fields σ_c^0 and σ_c, for Dirichlet and Natural boundary conditions, respectively. We denote by $\mathbf{z}_1 = (u_1, v_1, \theta_1)$ the generator of the Kernel of $F_{\mathbf{z}}$ at the critical field. Then we have

Theorem 8. *Let $F : \mathbf{Z}_i \times \mathbb{R} \to \mathbf{Y}$, where $i = d$ or n. At $\sigma = \sigma_c$, there is an $r > 0$ and bifurcation curve of solutions to the system (97, 98, 99) for $s \in (-r, r)$, satisfying*

i) $\mathbf{F}(\mathbf{z}(s), \sigma(s)) = 0$,
ii) $(\mathbf{z}(0), \sigma(0)) = (0, \sigma_c^0)$,
iii) $\mathbf{z}'(0) = (u_1, v_1, \theta_1)$,
iv) $u(s) = u_0 + su_1 + \mathcal{O}(s^2)$, $v(s) = v_0 + sv_1 + \mathcal{O}(s^2)$, $\theta(s) = s\theta_1 + \mathcal{O}(s^2)$, *and* $\sigma(s) = \sigma_c + \mathcal{O}(s^2)$.

Furthermore, the system (97, 98, 99) has only two solutions $\mathbf{z} = \mathbf{z}_0$ and $(\mathbf{z}(s), \sigma(s))$ and the nontrivial solution is stable in a sufficiently small neighborhood of $(0, \sigma_c)$.

Proof. The existence of the bifurcation curve can be proved in exactly the same way as in Theorem 5. Note that $u'(0) = u_1, v'(0) = v_1, \theta'(0) = \theta_1$. A direct computation gives, using $u_0 = \Re\{\tilde{c}e^{iqy}\}$ and $v_0 = \Im\{\tilde{c}e^{iqy}\}$, differentiating (100),

$$D_{\mathbf{zz}}F_1(\mathbf{z}_0, \sigma)[\mathbf{z}, \mathbf{z}] = 2(Cq\theta(v_x + v_0\theta_y) - g(u^2 + v^2)u_0), \tag{101}$$

$$D_{\mathbf{zz}}F_2(\mathbf{z}_0, \sigma)[\mathbf{z}, \mathbf{z}] = 2(-Cq\theta(u_x + u_0\theta_y) - g(u^2 + v^2)v_0), \tag{102}$$

$$\begin{aligned} D_{\mathbf{zz}}F_3(\mathbf{z}_0, \sigma)[\mathbf{z}, \mathbf{z}] = 2Cq[&(uv_x - vu_x) \\ &+(-u(v_0)_y - u_0v_y + v(u_0)_y + v_0u_y)\theta]. \end{aligned} \tag{103}$$

From (65), we have

$$\begin{aligned} &\langle D_{\mathbf{zz}}\mathbf{F}(\mathbf{z}_0, \sigma)[\mathbf{z}_1, \mathbf{z}_1], \mathbf{z}_1\rangle \\ &\quad = 2Cq \int_\Omega \left[\theta_1 \left(u_1 \frac{\partial v_1}{\partial x} - v_1 \frac{\partial u_1}{\partial x}\right) + \frac{\partial \theta_1}{\partial y}(v_0u_1 - u_0v_1)\right] dxdy \\ &\quad -2g \int_\Omega (u_1^2 + v_1^2)(u_0u_1 + v_0v_1)\, dxdy \\ &\quad +2Cq \int_\Omega \left(-u_1 \frac{\partial v_0}{\partial y} - u_0 \frac{\partial v_1}{\partial y} + v_1 \frac{\partial u_0}{\partial y} + v_0 \frac{\partial u_1}{\partial y}\right) \theta_1^2\, dxdy. \end{aligned} \tag{104}$$

Recalling that $\psi_1 = iq\psi_0\varphi_1 = u_1 + iv_1$ and $i\psi_0 = iu_0 - v_0$, we have

$$\begin{aligned} &u_1 \frac{\partial v_1}{\partial x} - v_1 \frac{\partial u_1}{\partial x} = 0, \\ &v_0u_1 - u_0v_1 = -q\varphi_1(v_0^2 + u_0^2) = -q\varphi_1, \\ &u_0u_1 + v_0v_1 = 0, \\ &-u_1 \frac{\partial v_0}{\partial y} + v_1 \frac{\partial u_0}{\partial y} = 0, \\ &-u_0 \frac{\partial v_1}{\partial y} + v_0 \frac{\partial u_1}{\partial y} = -q \frac{\partial \varphi_1}{\partial y}. \end{aligned}$$

Thus (104) becomes

$$\langle D_{\mathbf{zz}}\mathbf{F}(\mathbf{z}_0,\sigma)[\mathbf{z}_1,\mathbf{z}_1],\mathbf{z}_1\rangle = -2Cq^2\int_\Omega\left(\varphi_1\frac{\partial\theta_1}{\partial y}+\theta_1^2\frac{\partial\varphi_1}{\partial y}\right)dxdy. \tag{105}$$

First we consider Dirichlet boundary condition on ψ. Then from section 5.1.1, we find an eigenfunction of $\mathcal{J}$ with respect to $\sigma=\sigma_c^0$,

$$(\theta_1,\phi_1) = (\cos\frac{\pi\widetilde{y}}{2}\cos\widetilde{\mu_m}\widetilde{x},\ \frac{A}{\widetilde{\mu_m}}\cos\frac{\pi\widetilde{y}}{2}\sin\widetilde{\mu_m}\widetilde{x})$$

where $\widetilde{\mu_m}=\frac{\pi m}{c}$ and $A=\frac{4\widetilde{\mu_m}^2}{4\widetilde{\mu_m}^2+\pi^2}$. Returning to the original variables from (95), recalling (92), we have

$$\mathbf{z}_1 = (\Re\psi_1,\Im\psi_1,\theta_1),$$

where

$$\psi_1 = iq\psi_0\varphi_1,\ \varphi_1=\frac{A}{\mu_m}\cos\frac{\pi y}{2d}\sin\mu_m x,\ \theta_1=\cos\frac{\pi y}{2d}\cos\mu_m x,\ \mu_m=\frac{\pi m}{L}.$$

Using these eigenfunctions in (105), we see that $\langle D_{\mathbf{zz}}\mathbf{F}(\mathbf{z}_0,\sigma)[\mathbf{z}_1,\mathbf{z}_1],\mathbf{z}_1\rangle=0$ and hence $\sigma'(0)=0$ from (65).

For stability of the bifurcation curve, one needs to compute the third order Fréchet derivatives of F. We find

$$D_{\mathbf{zzz}}F_1(\mathbf{z}_0,\sigma)[\mathbf{z},\mathbf{z},\mathbf{z}] = -3Cq\theta(2\theta v_y+\theta\theta_x v_0+2\theta_y v), \tag{106}$$

$$D_{\mathbf{zzz}}F_2(\mathbf{z}_0,\sigma)[\mathbf{z},\mathbf{z},\mathbf{z}] = 3Cq\theta(2\theta u_y+\theta\theta_x u_0+2\theta_y u), \tag{107}$$

$$\begin{aligned}D_{\mathbf{zzz}}F_3(\mathbf{z}_0,\sigma)[\mathbf{z},\mathbf{z},\mathbf{z}] &= Cq\theta[3\theta(-u_0v_x+v_0u_x)+q\theta^2-6(uv_y-vu_y)]\\ &\quad -4\sigma\theta^3.\end{aligned} \tag{108}$$

This gives

$$\begin{aligned}&\langle F_{\mathbf{zzz}}(\mathbf{z}_0,\sigma)[\mathbf{z}_1,\mathbf{z}_1,\mathbf{z}_1],\mathbf{z}_1\rangle\\ &\quad= Cq\int\Big[6\theta_1^2(\frac{\partial u_1}{\partial y}v_1-\frac{\partial v_1}{\partial y}u_1)+3\theta_1^2\frac{\partial\theta_1}{\partial x}(u_0v_1-v_0u_1)\\ &\qquad+3\theta_1^3(-u_0\frac{\partial v_1}{\partial x}+v_0\frac{\partial u_1}{\partial x})+\theta_1^4(u_0\frac{\partial v_0}{\partial y}-v_0\frac{\partial u_0}{\partial y})\\ &\qquad+6\theta_1^2(v_1\frac{\partial u_1}{\partial y}-u_1\frac{\partial v_1}{\partial y})\Big]dxdy-4\sigma\int\theta_1^4dxdy.\end{aligned}$$

Recalling that $\psi_1=iqd\psi_0\phi_1=u_1+iv_1$ and $i\psi_0=iu_0-v_0$,

$$\begin{aligned}u_0\frac{\partial v_1}{\partial x}-v_0\frac{\partial u_1}{\partial x} &= u_0qd\frac{\partial\phi_1}{\partial x}\Im(i\phi_0)-v_0qd\frac{\partial\phi_1}{\partial x}\Re(i\phi_0)\\ &= qd\frac{\partial\phi_1}{\partial x}(u_0u_0+v_0v_0)=qd\frac{\partial\phi_1}{\partial x}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&u_0 v_1 - v_0 u_1 = q\varphi_1,\\
&v_1 \frac{\partial u_1}{\partial y} - u_1 \frac{\partial v_1}{\partial y} = q^2 d^2 \phi_1^2 (v_0 \frac{\partial u_0}{\partial y} - u_0 \frac{\partial v_0}{\partial y}) = -q^3 \varphi_1^2,\\
&u_0 \frac{\partial v_0}{\partial y} - v_0 \frac{\partial u_0}{\partial y} = q.
\end{aligned}$$

Then

$$\begin{aligned}
&\langle F_{\mathbf{zzz}}(\mathbf{z}_0, \sigma)[\mathbf{z}_1, \mathbf{z}_1, \mathbf{z}_1], \mathbf{z}_1\rangle\\
&= \int_\Omega Cq^2 \Big[-12q^2\varphi_1^2\theta_1^2 + 3\theta_1^2 \frac{\partial \theta_1}{\partial x}\varphi_1 - 3q\theta_1^3 \frac{\partial \varphi_1}{\partial x} + \theta_1^4 \Big] dxdy - 4\sigma \int_\Omega \theta_1^4 \, dxdy.
\end{aligned} \tag{109}$$

Also, we get

$$\langle F_{\mathbf{z}\sigma}(\mathbf{z}_0, \sigma)\mathbf{z}_1, \mathbf{z}_1\rangle = \sigma \int_\Omega \theta_1^2 = \sigma > 0. \tag{110}$$

Since we have

$$\frac{\partial \theta_1}{\partial x} = -\mu_m \frac{\mu_m}{A}\varphi_1 \qquad \text{and} \qquad \frac{\partial \varphi_1}{\partial x} = \mu_m \frac{A}{\mu_m}\theta_1,$$

(109) becomes

$$\begin{aligned}
&\langle F_{\mathbf{zzz}}(\mathbf{z}_0, \sigma)[\mathbf{z}_1, \mathbf{z}_1, \mathbf{z}_1], \mathbf{z}_1\rangle\\
&= -Cq^2 \int \Big(12q^2\theta_1^2\varphi_1^2 + 3\frac{\mu_m^2}{A}\theta_1^2\varphi_1^2 + 3A\theta_1^4 - \theta_1^4 \Big) - 4\sigma \int \theta_1^4.
\end{aligned}$$

Since $A = \mathcal{O}(1)$ from $\mu_m = \mathcal{O}(\frac{1}{\varepsilon})$ as $\varepsilon \to 0$ given in Theorem 4, one can see that

$$\langle F_{\mathbf{zzz}}(\mathbf{z}_0, \sigma)[\mathbf{z}_1, \mathbf{z}_1, \mathbf{z}_1], \mathbf{z}_1\rangle < 0.$$

From the formula (67) and (110), we have $\sigma''(0) > 0$, which implies a supercritical bifurcation. As before, we use (61) to determine the sign of an eigenvalue of the derivative of the Euler-Lagrange operator at the nontrivial solution curve. Using (62), we obtain again $\gamma'(\sigma_c^0) > 0$ and (61) gives the negative sign for this eigenvalue. Thus we have a stable supercritical bifurcation curve for Dirichlet condition.

Now we prove the stability and pitchfork type of bifurcation with natural boundary condition. From Theorem 6, after rescaling, we have

$$\mathbf{z}_1 = (\Re\psi_1, \Im\psi_1, \theta_1),$$

where

$$\psi_1 = iq\psi_0\varphi_1, \quad \varphi_1 = \frac{1}{\mu_1} f(y) \sin \mu_1 x, \quad \theta_1 = g(y) \cos \mu_1 x,$$

and

$$f(y) = -A\cos\sqrt{\gamma}\cosh\sqrt{\alpha}\frac{y}{d} + B\cosh\alpha\cos\sqrt{\gamma}\frac{y}{d},$$
$$g(y) = -\cos\sqrt{\gamma}\cosh\sqrt{\alpha}\frac{y}{d} + \cosh\alpha\cos\sqrt{\gamma}\frac{y}{d}.$$

Here constants A, B, α, and γ are explicitly given in Theorem 6. Note that $f(y), g(y) > 0$. First $\sigma'(0) = 0$ follows easily from (105) with above eigenfunctions. We only need to determine the sign of $\sigma''(0)$. From

$$\frac{\partial\varphi_1}{\partial x} = f(y)\cos\mu_1 x, \quad \text{and} \quad \frac{\partial\theta_1}{\partial x} = -\mu_1 g(y)\sin\mu_1 x,$$

(109) becomes

$$\begin{aligned}
&\langle F_{\mathbf{zzz}}(\mathbf{z}_0,\sigma)[\mathbf{z}_1,\mathbf{z}_1,\mathbf{z}_1],\mathbf{z}_1\rangle \\
&\quad = -Cq^2\int\Big(12q^2\theta_1^2\varphi_1^2 + 3g(y)^3 f(y)\cos^4\mu_1 x \\
&\qquad +3g(y)^3 f(y)\cos^2\mu_1 x\sin^2\mu_1 x - g(y)^4\cos^4\mu_1 x\Big) - 4\sigma\int\theta_1^4.
\end{aligned}$$

Using the estimates $A = \mathcal{O}(\varepsilon), B \sim 1, \gamma \sim 0$, and $\alpha = \mathcal{O}(\frac{1}{\varepsilon^2})$, we have

$$\begin{aligned}
&3f(y) - g(y) \\
&\quad = (3A-1)\left(-\cos\sqrt{\gamma}\cosh\left(\sqrt{\alpha}\frac{y}{d}\right)\right) + (3B-1)\cosh\sqrt{\alpha}\cos\left(\sqrt{\gamma}\frac{y}{d}\right) \\
&\quad > 0.
\end{aligned}$$

Thus $\langle F_{\mathbf{zzz}}(\mathbf{z}_0,\sigma)[\mathbf{z}_1,\mathbf{z}_1,\mathbf{z}_1],\mathbf{z}_1\rangle$ is negative and gives a stable supercritical pitchfork bifurcation curve following the same procedure as that employed in the Dirichlet case.

8. Conclusion

In this work, we have investigated layer undulation phenomena, or Helfrich-Hurault effect in a lamellar system using the smectic A de Gennes energy. We performed bifurcation analysis on the nonlinear system to study the onset of layer undulation at the critical field with two boundary conditions on the layer function, Dirichlet and natural boundary condition. In the first case, results from our analysis and numerical simulations are consistent with the classic Helfrich-Hurault theory. Employing the natural boundary condition decreases the critical field and frequency of the undulation. Also, it makes layer tilt at the bounding plate possible and shows weakened dependence of layer shapes on the vertical axis.

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