Calculus with Applications II

Math 3B, Fall 2004

Instructor: **Carlos J. García-Cervera** December 7th, 2004

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1. (10 points) Evaluate

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^7 \tag{1}$$

This is the definition of the definite integral using Riemann sums for the function $f(x)=x^7$ in the interval $[0,1]$, using $x_i=\frac{i}{n}$, $\Delta x=\frac{1}{n}$. Therefore,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^7=\int_0^1 x^7\,dx=\left.\frac{x^8}{8}\right|_0^1=\frac{1}{8} \tag{2}$$

2. (10 points) Find the derivative of

$$g(x)=\int_{\sqrt{x}}^{x^3}\sqrt{t}\sin(t)\,dt \tag{3}$$

We evaluate the derivative using the Fundamental Theorem of Calculus, which states that if

$$F(x)=\int_a^x f(t)\,dt \tag{4}$$

for a continuous function f, then $F'(x)=f(x)$. Now we define

$$F(x)=\int_0^x\sqrt{t}\sin(t)\,dt \tag{5}$$

Then,

$$g(x)=F(x^3)-F(\sqrt{x}), \tag{6}$$

so, using the chain rule:

$$g'(x)=3x^2F'(x^3)-\frac{1}{2\sqrt{x}}F'(\sqrt{x})=3x^2\sqrt{x^3}\sin(x^3)-\frac{1}{2\sqrt{x}}x^{1/4}\sin(\sqrt{x}). \tag{7}$$

3. (10 points) Find the volume of the solid obtained by rotating about the y-axis the region between $y=\sqrt{x}$ and $y=x^2$.

We can compute the volume using washers. The point of intersection of the two curves satisfies:

$$\sqrt{x}=x^2\iff x=x^4\iff x=0,\text{ or } x^3=1\iff x=0,1 \tag{8}$$

When $x=0$, we get $y=0$, and when $x=1$ we get $y=1$, so the volume is

$$V=\int_0^1 A(y)\,dy, \tag{9}$$

where $A(y)$ is the area of the cross-section for a fixed value y. In this case, the cross-section is a washer, whore area is

$$A(y) = \pi \left(R(y)^2 - r(y)^2\right) \tag{10}$$

The outer radius is $R(y) = \sqrt{y}$, and the inner radius is $r(y) = y^2$, so

$$V = \pi \int_0^1 (y - y^4)\, dy, = \pi \left(\frac{y^2}{2} - \frac{y^5}{5}\right)\Bigg|_0^1 = \pi \left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3\pi}{10} \tag{11}$$

4. (10 points) Find the indefinite integral

$$I = \int e^x \cos(x)\, dx \tag{12}$$

We use integration by parts:

$$\begin{array}{rcl} u = e^x & \rightarrow & du = e^x\, dx \\ dv = \cos(x)\, dx & \rightarrow & v = \sin(x) \end{array} \tag{13}$$

Therefore,

$$I = e^x \sin(x) - \int e^x \sin(x)\, dx \tag{14}$$

We use integration by parts again:

$$\begin{array}{rcl} u = e^x & \rightarrow & du = e^x\, dx \\ dv = \sin(x)\, dx & \rightarrow & v = -\cos(x) \end{array} \tag{15}$$

Then,

$$I = e^x \sin(x) + e^x \cos(x) - I \tag{16}$$

Solving for I:

$$I = \frac{e^x}{2} \left(\sin(x) + \cos(x)\right) + C \tag{17}$$

5. (10 points) Find the indefinite integral

$$\int \frac{5x^2 + x + 1}{(x+1)(x^2+4)}\, dx \tag{18}$$

We decompose the integrand using partial fractions:

$$\frac{5x^2 + x + 1}{(x+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx + C}{x^2 + 4} = \frac{A(x^2+4) + (Bx+C)(x+1)}{(x+1)(x^2+4)} \tag{19}$$

We choose A, B, and C so that

$$5x^2 + x + 1 = A(x^2 + 4) + (Bx + C)(x + 1) \quad \forall x \; in\mathbb{R} \tag{20}$$

For $x = -1$, we get that $5 = 5A$, so $A = 1$. Then we get

$$Bx^2 + (B + C)x + C = 4x^2 + x - 3, \tag{21}$$

so $C = -3$, and $B = 4$. Then,

$$\int \frac{5x^2 + x + 1}{(x + 1)(x^2 + 4)}\,dx = \int \frac{1}{x + 1}\,dx + \int \frac{4x - 3}{x^2 + 4}\,dx = \ln|x + 1| + \int \frac{4x}{x^2 + 4}\,dx - \int \frac{3}{x^2 + 4}\,dx \tag{22}$$

Using the substitution $u = x^2 + 4$, $du = 2x\,dx$, the first integral becomes:

$$\int \frac{4x}{x^2 + 4}\,dx = 2\int \frac{du}{u} = 2\ln|u| + C = 2\ln|x^2 + 4| + C \tag{23}$$

Using the substitution $x = 2v$, $dx = 2\,dv$, the second integral becomes

$$\int \frac{3}{x^2 + 4}\,dx = \frac{3}{2}\int \frac{dv}{v^2 + 1} = \frac{3}{2}\arctan v + C = \frac{3}{2}\arctan\left(\frac{x}{2}\right) + C \tag{24}$$

Putting all together,

$$\int \frac{5x^2 + x + 1}{(x + 1)(x^2 + 4)}\,dx = \ln|x + 1| + 2\ln|x^2 + 4| - \frac{3}{2}\arctan\left(\frac{x}{2}\right) + C \tag{25}$$

6. (10 points) Find the indefinite integral

$$\int \frac{dx}{\sqrt{x^2 - 4}} \tag{26}$$

You may use the fact that

$$\int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C.$$

Remember that the answer must be a function of x.

We do the trigonometric substitution $x = 2\sec(\theta)$, $dx = 2\sec(\theta)\tan(\theta)\,d\theta$. Note that

$$x^2 - 4 = 4\left(\sec^2\theta - 1\right) = 4\tan^2\theta \tag{27}$$

Then,

$$\int \frac{dx}{\sqrt{x^2 - 4}} = \int \frac{1}{2\tan(\theta)} 2\sec(\theta)\tan(\theta)\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$
$$= \ln\left|\frac{x}{2} + \sqrt{\frac{x^2}{4} - 1}\right| + C \tag{28}$$

7. (20 points) Use the Comparison Theorem to determine whether the following integrals are convergent or divergent.

   (a) (10 points)
$$\int_0^{\pi/2} \frac{dx}{\sqrt{x}\sin(x)}.$$
The problem in this integral is only at 0, since there the integrand diverges to ∞. For x small, we know that $\sin x \approx x$, so $\sqrt{x}\sin(x) \approx x^{3/2}$, and
$$\int_0^a \frac{1}{x^{3/2}}\, dx = \infty \quad \forall x > 0 \tag{29}$$
Therefore we expect the integral to be divergent. In order to use the comparison principle, we need to make this a bit more formal, using the fact that
$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \tag{30}$$
which implies that is a is small enough, we have $\sin x < 2x$ for all $0 \le x \le a$. Therefore,
$$\frac{1}{\sqrt{x}\sin(x)} \ge \frac{1}{2x^{3/2}} \quad \forall 0 < x \le a, \tag{31}$$
and the integral is **divergent**.

   (b) (10 points)
$$\int_0^{\infty} \frac{1}{\sqrt[3]{x}(1+x)}\, dx.$$
There are two problems here: one is the point $x = 0$, since the integrand diverges, and the other is the fact that the interval of integration is infinite. We begin by splitting the integral:
$$\int_0^{\infty} \frac{1}{\sqrt[3]{x}(1+x)}\, dx. = \int_0^{a} \frac{1}{\sqrt[3]{x}(1+x)}\, dx + \int_a^{\infty} \frac{1}{\sqrt[3]{x}(1+x)}\, dx, \tag{32}$$
for some $a > 0$.

(10 points) If $0 < x \le a$, we have that
$$0 \le \frac{1}{\sqrt[3]{x}(1+x)} \le \frac{1}{\sqrt[3]{x}} \tag{33}$$
Since
$$\int_0^a \frac{1}{\sqrt[3]{x}}\, dx \tag{34}$$
is convergent, the first part of the integral is convergent. For the second part, we note that is $a \le x < \infty$,
$$0 \le \frac{1}{\sqrt[3]{x}(1+x)} \le \frac{1}{x\sqrt[3]{x}} = \frac{1}{x^{4/3}} \tag{35}$$

The integral

$$\int_a^\infty \frac{1}{x^{4/3}}\,dx \tag{36}$$

is also convergent, so

$$\int_0^\infty \frac{1}{\sqrt[3]{x}(1+x)}\,dx \tag{37}$$

is **convergent**.

8. (10 points) Compute the length of the curve $y = \frac{1}{6}(x^2+4)^{3/2}$, $0 \le x \le 3$.

   The length of the curve is

$$L = \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \tag{38}$$

   Now,

$$\frac{dy}{dx} = \frac{x}{2}\left(x^2+4\right)^{1/2}, \tag{39}$$

   so,

$$\begin{aligned} L = \int_0^3 \sqrt{1 + \frac{x^2}{4}\left(x^2+4\right)}\,dx &= \frac{1}{2}\int_0^3 \sqrt{x^4+4x^2+4}\,dx = \frac{1}{2}\int_0^3 \sqrt{(x^2+2)^2}\,dx \\ &= \frac{1}{2}\int_0^3 (x^2+2)\,dx = \frac{1}{2}\left(\frac{x^3}{3}+2x\right)\Big|_0^3 = \frac{1}{2}(9+6) = \frac{15}{2} \end{aligned} \tag{40}$$

9. (10 points) Find the area of the surface obtained by rotating the curve $y = \sqrt{x}$, $5 \le x \le 9$, about the x-axis.

   The surface area is given by the formula

$$A = \int_5^9 2\pi y(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \tag{41}$$

   Now,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \tag{42}$$

   so,

$$\begin{aligned} A = 2\pi\int_5^9 \sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx &= 2\pi\int_5^9 \sqrt{x+\frac{1}{4}}\,dx = \frac{4\pi}{3}\left(x+\frac{1}{4}\right)^{3/2}\Big|_5^9 \\ &= \frac{4\pi}{3}\left(\left(9+\frac{1}{4}\right)^{3/2} - \left(5+\frac{1}{4}\right)^{3/2}\right) = \frac{\pi}{6}\left(37^{3/2} - 21^{3/2}\right) \end{aligned} \tag{43}$$