ON THE NON-VANISHING OF GENERALIZED KATO CLASSES FOR ELLIPTIC CURVES OF RANK TWO

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Abstract. Let $E/\mathbf{Q}$ be an elliptic curve, $p>3$ a good ordinary prime for E, and assume that $L(E,1)=0$ with root number $+1$ (so $\mathrm{ord}_{s=1}L(E,s)\geqslant 2$). A construction of Darmon–Rotger attaches to E, and an auxiliary weight one cuspidal eigenform g such that $L(E,\mathrm{ad}^0(g),1)\neq 0$, a Selmer class $\kappa_p\in\mathrm{Sel}(\mathbf{Q},V_pE)$, and they conjectured the equivalence

$$\kappa_p\neq 0\quad\Longleftrightarrow\quad \dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q},V_pE)=2.$$

In this paper, we prove the first cases on Darmon–Rotger's conjecture when the auxiliary eigenform g has CM. In particular, this provides a new construction of non-trivial Selmer classes for elliptic curves of rank two.

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1. Introduction

Let E be an elliptic curve over $\mathbf{Q}$ (hence modular, [Wil95, TW95, BCDT01]) with associated L-function $L(E,s)$. In the late 1980s, a major advance towards the Birch and Swinnerton-Dyer conjecture was the proof, by Gross–Zagier and Kolyvagin, of the implication

$$\mathrm{ord}_{s=1}L(E,s)=1 \quad\Longrightarrow\quad \mathrm{rank}_{\mathbf{Z}}E(\mathbf{Q})=1 \text{ and } \#\text{Ш}(E/\mathbf{Q})<\infty. \tag{1.1}$$

In the proof of (1.1) an imaginary quadratic field $K/\mathbf{Q}$ is chosen such that $\mathrm{ord}_{s=1}L(E/K,s)=1$ and for which a Heegner point $y_K\in E(\mathbf{Q})$ can be constructed using the theory of complex multiplication and a modular parametrization of E. By the Gross–Zagier formula [GZ86], the non-vanishing of $L'(E/K,1)$ implies that y_K has infinite order, and the proof of (1.1) is reduced to the proof of the implication

$$y_K\notin E(\mathbf{Q})_{\mathrm{tors}} \quad\Longrightarrow\quad \mathrm{rank}_{\mathbf{Z}}E(\mathbf{Q})=1 \text{ and } \#\text{Ш}(E/\mathbf{Q})<\infty, \tag{1.2}$$

which is a celebrated theorem by Kolyvagin [Kol88].

A more recent major advance towards BSD arises from the works of Kato [Kat04], Skinner–Urban [SU14] and Xin Wan [Wan20] on the Iwasawa main conjectures for elliptic modular forms, which in particular yield a proof of a p-converse to (1.2):

$$\mathrm{rank}_{\mathbf{Z}}E(\mathbf{Q})=1 \text{ and } \#\text{Ш}(E/\mathbf{Q})[p^\infty]<\infty \quad\Longrightarrow\quad y_K\notin E(\mathbf{Q})_{\mathrm{tors}} \tag{1.3}$$

for certain primes p of good ordinary reduction for E, an implication first realized by Skinner [Ski20]. (A different proof of (1.3) was obtained independently by Wei Zhang [Zha14] as a consequence of his proof of Kolyvagin's conjecture.) Together with the Gross–Zagier formula, (1.3) yields a p-converse to the theorem (1.1) of Gross–Zagier and Kolyvagin.

It is natural to ask about the extension of these results to elliptic curves $E/\mathbf{Q}$ of rank $r>1$. As a first step in this direction, in this paper we prove certain analogues of (1.2) and (1.3) in rank two, with y_K replaced by a *generalized Kato class*

$$\kappa_p\in\mathrm{Sel}(\mathbf{Q},V_pE)$$

introduced by Darmon–Rotger. Here $\mathrm{Sel}(\mathbf{Q},V_pE)\subset\mathrm{H}^1(\mathbf{Q},V_pE)$ is the p-adic Selmer group fitting into the exact sequence

$$0\to E(\mathbf{Q})\otimes_{\mathbf{Z}}\mathbf{Q}_p\to\mathrm{Sel}(\mathbf{Q},V_pE)\to T_p\text{Ш}(E/\mathbf{Q})\otimes_{\mathbf{Z}_p}\mathbf{Q}_p\to 0,$$

where $T_p\text{Ш}(E/\mathbf{Q})$ is the p-adic Tate module of the Tate–Shafarevich group $\text{Ш}(E/\mathbf{Q})$.

1.1. The Darmon–Rotger conjecture.

We begin by briefly recalling the construction of κ_p by Darmon–Rotger [DR17, DR16]. One starts by associating a global cohomology class

$$\kappa_{\gamma,\delta}(f,g,h)\in\mathrm{H}^1(\mathbf{Q},V_{fgh}),$$

where $V_{fgh}=V_p(f)\otimes V_p(g)\otimes V_p(h)$ is the tensor product of the p-adic Galois representations associated to f, g and h to the data of:

- a triple of eigenforms $(f,g,h)\in S_2(\Gamma_0(N_f))\times S_1(\Gamma_0(N_g),\chi)\times S_1(\Gamma_0(N_h),\bar\chi)$ of weights $(2,1,1)$ and levels prime-to-p with
$$\gcd(N_f,N_gN_h)=1, \tag{1.4}$$
- a choice of roots $\gamma\in\{\alpha_g,\beta_g\}$ and $\delta\in\{\alpha_h,\beta_h\}$ of the Hecke polynomials of g and h at p, assumed to be *regular*, i.e. $\alpha_g\neq\beta_g$ and $\alpha_h\neq\beta_h$.

Letting $g^\flat$ and $h^\flat$ be the p-stabilizations of g and h with U_p-eigenvalue γ and δ, the class $\kappa_{\gamma,\delta}(f,g,h)$ is defined as the *p-adic limit*

$$\kappa_{\gamma,\delta}(f,g,h):=\lim_{\ell\to 1}\kappa(f,\boldsymbol{g}_\ell,\boldsymbol{h}_\ell), \tag{1.5}$$

where $(\boldsymbol{g}_\ell, \boldsymbol{h}_\ell)$ runs over the classical weight $\ell \geqslant 2$ specializations of Hida families $\boldsymbol{g}$ and $\boldsymbol{h}$ passing through $g^\flat$ and $h^\flat$ in weight 1, and $\kappa(f, \boldsymbol{h}_\ell, \boldsymbol{h}_\ell)$ is obtained from the p-adic étale Abel–Jacobi image of generalized Gross–Kudla–Schoen diagonal cycles, [GK92, GS95].

Remark 1.1. Under assumption (1.4), the sign in the functional equation for the triple product L-series $L(s, f \otimes \boldsymbol{g}_\ell \otimes \boldsymbol{h}_\ell)$ is -1 for all $\ell \geqslant 2$; in particular, $L(1, f \otimes \boldsymbol{g}_\ell \otimes \boldsymbol{h}_\ell) = 0$, and by the Gross–Zagier formula for diagonal cycles (proved in [YZZ12] for $\ell = 2$), the classes $\kappa(f, \boldsymbol{g}_\ell, \boldsymbol{h}_\ell)$ should be non-trivial precisely when $L'(1, f \otimes \boldsymbol{g}_\ell \otimes \boldsymbol{h}_\ell) \neq 0$. On the other hand, the global root number of $L(s, f \otimes g \otimes h)$ is $+1$, and it is precisely this *sign-change* phenomenon between weights $\ell \geqslant 2$ and $\ell = 1$ that makes it possible for the p-adic limit construction (1.5) to yield interesting cohomology classes in situations of *even* analytic rank; in fact, as we recall below, classes that are crystalline at p precisely when $\mathrm{ord}_{s=1} L(s, f \otimes g \otimes h) \geqslant 2$.

Under the hypothesis that $p > 3$ is a prime of *good ordinary* reduction for f, the explicit reciprocity law of [DR17] yields a formula of the form

$$\exp_p^*(\kappa_{\gamma,\delta}(f, g, h)) = L(1, f \otimes g \otimes h) \cdot (\text{nonzero constant}), \tag{1.6}$$

where $\exp_p^* : \mathrm{H}^1(\mathbf{Q}, V_{fgh}) \to \mathbf{Q}_p$ is the composition of the restriction map

$$\mathrm{Loc}_p : \mathrm{H}^1(\mathbf{Q}, V_{fgh}) \to \mathrm{H}^1(\mathbf{Q}_p, V_{fgh})$$

with the dual exponential map of Bloch–Kato [BK90] paired against a differential associated to (f, g, h). As a result, the class $\kappa_{\gamma,\delta}(f, g, h)$ is crystalline at p, and therefore lands in the Bloch–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, V_{fgh}) \subset \mathrm{H}^1(\mathbf{Q}, V_{fgh})$, precisely when $L(s, f \otimes g \otimes h)$ vanishes at $s = 1$. With the different choices for γ and δ, one thus obtains four—*a priori* distinct—classes $\kappa_{\gamma,\delta}(f, g, h) \in \mathrm{Sel}(\mathbf{Q}, V_{fgh})$ whenever $L(1, f \otimes g \otimes h) = 0$, and Darmon–Rotger conjectured (see [DR16, Conj. 3.2]) that the following are equivalent:

(i) the classes $\kappa_{\gamma,\delta}(f, g, h)$ span a non-trivial subspace of $\mathrm{Sel}(\mathbf{Q}, V_{fgh})$;
(ii) $\dim_{\mathbf{Q}_p} \mathrm{Sel}(\mathbf{Q}, V_{fgh}) = 2$,

assuming for simplicity that the Hecke fields of f, g and h embed into $\mathbf{Q}_p$.

The adjoint rank $(2, 0)$ *setting.* The construction of $\kappa_{\gamma,\delta}(f, g, h)$ yields classes with a bearing on the arithmetic of elliptic curves $E/\mathbf{Q}$ by taking f to be the newform associated to E, and $h = g^*$ to be the dual of g, so that the triple tensor product V_{fgg^*} decomposes as

$$V_{fgg^*} \simeq V_p E \oplus \big(V_p E \otimes \mathrm{ad}^0 V_p(g)\big), \tag{1.7}$$

where $\mathrm{ad}^0 V_p(g)$ is the three-dimensional $G_{\mathbf{Q}}$-representation on the space of trace zero endomorphisms of $V_p(g)$. Correspondingly, $L(s, f \otimes g \otimes g^*)$ factors as

$$L(s, f \otimes g \otimes g^*) = L(E, s) \cdot L(E, \mathrm{ad}^0(g), s).$$

In particular, the above construction yields the four *generalized Kato classes*

$$\kappa_{\alpha_g, \alpha_g^{-1}}(f, g, g^*), \quad \kappa_{\alpha_g, \beta_g^{-1}}(f, g, g^*), \quad \kappa_{\beta_g, \alpha_g^{-1}}(f, g, g^*), \quad \kappa_{\beta_g, \beta_g^{-1}}(f, g, g^*) \tag{1.8}$$

landing (thanks to the explicit reciprocity law (1.6)) in the Selmer group

$$\mathrm{Sel}(\mathbf{Q}, V_{fgg^*}) \simeq \mathrm{Sel}(\mathbf{Q}, V_p E) \oplus \mathrm{Sel}(\mathbf{Q}, V_p E \otimes \mathrm{ad}^0 V_p(g)).$$

whenever $L(E, 1) = 0$. Since one expects $L(E, \mathrm{ad}^0(g), 1) \neq 0 \Longleftrightarrow \mathrm{Sel}(\mathbf{Q}, V_p E \otimes \mathrm{ad}^0 V_p(g)) = 0$ by the Bloch–Kato conjecture, the non-vanishing criterion in [DR16, Conj. 3.2] leads to the following prediction (see the "adjoint rank $(2, 0)$ setting" discussed in [DR17, §4.5.3]).

Conjecture 1.2 (Darmon–Rotger)**.** *Suppose that $L(E, s)$ has sign $+1$ and vanishes at $s = 1$, and that $L(E, \mathrm{ad}^0(g), 1) \neq 0$. Then the following are equivalent:*

(i) *the four classes in* (1.8) *span a non-trivial subspace of* $\mathrm{Sel}(\mathbf{Q}, V_p E)$*;*
(ii) $\dim_{\mathbf{Q}_p} \mathrm{Sel}(\mathbf{Q}, V_p E) = 2$.

Remark 1.3. Of course, by the Birch and Swinnerton-Dyer conjecture, condition (ii) in Conjecture 1.2 should be equivalent to the condition $\mathrm{ord}_{s=1}L(E,s)=2$.

Remark 1.4. Note that Conjecture 1.2 does not predict that the four classes in (1.8) generate $\mathrm{Sel}(\mathbf{Q},V_pE)$. In fact, a strengthtening of the *elliptic Stark conjectures* in [DLR15] predicts that in the setting of Conjecture 1.2 only the classes $\kappa_{\alpha_g,\alpha_g^{-1}}(f,g,g^*)$ and $\kappa_{\beta_g,\beta_g^{-1}}(f,g,g^*)$ are nonzero, and that they are the same class up to a nonzero algebraic constant. Our results also confirm this prediction (see Remark 1.6 below and §5.7 for further details).

1.2. **Statement of the main results.** In this paper we prove Conjecture 1.2 in the case when g has CM, assuming $\#\text{Ш}(E/\mathbf{Q})[p^\infty]<\infty$ (in fact, a weaker condition suffices) for one of the implications.

As before, let $E/\mathbf{Q}$ be an elliptic curve with good ordinary reduction at $p>3$, and let $f\in S_2(\Gamma_0(N_f))$ be the associated newform. Let K be an imaginary quadratic field of discriminant prime of N_f in which $(p)=\mathfrak{p}\overline{\mathfrak{p}}$ splits, and let ψ be a ray class character of K of conductor prime to pN_f valued in a number field L. The weight one theta series $g=\theta_\psi$ then satisfies

$$L(E,\mathrm{ad}^0(g),s)=L(E^K,s)\cdot L(E/K,\chi,s),$$

where E^K is the twist of E by the quadratic character associated to K, and χ is the ring class character given by ψ/ψ^τ, for ψ^τ the composition of ψ with the action of complex conjugation τ. In this case, $\alpha_g=\psi(\overline{\mathfrak{p}})$ and $\beta_g=\psi(\mathfrak{p})$ are the roots of the Hecke polynomial of g and p, which we shall simply denote by α and β, respectively, and g^* is the theta series of ψ^{-1}. As in the formulation of the conjectures in [DR16], we assume that $\alpha_g\neq\beta_g$, i.e., $\chi(\overline{\mathfrak{p}})\neq 1$.

Let $\bar\rho_{E,p}:G_{\mathbf{Q}}\to\mathrm{Aut}_{\mathbf{F}_p}(E[p])$ be the mod p representation associated to E, and denote by N_f^- the largest factor of N_f divisible only by primes that are inert in K. Finally, let

$$\mathrm{Loc}_p:\mathrm{Sel}(\mathbf{Q},V_pE)\to\mathrm{H}^1(\mathbf{Q}_p,V_pE)$$

be the restriction map at p.

Theorem A. *Suppose that $L(E,s)$ has sign $+1$ and vanishes at $s=1$, and that the value*

$$L(E,\mathrm{ad}^0(g),1)=L(E^K,1)\cdot L(E/K,\chi,1)$$

is nonzero. Suppose also that:

- *$\bar\rho_{E,p}$ is irreducible,*
- *N_f^- is the squarefree of an odd number of primes,*
- *$\bar\rho_{E,p}$ is ramified at every prime $q|N_f^-$.*

Then $\kappa_{\alpha,\beta^{-1}}(f,g,g^*)=\kappa_{\beta,\alpha^{-1}}(f,g,g^*)=0$, and the following hold:

$$\kappa_{\alpha,\alpha^{-1}}(f,g,g^*)\neq 0\quad\Longrightarrow\quad \dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q},V_pE)=2, \tag{1.9}$$

and conversely,

$$\left.\begin{array}{r}\dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q},V_pE)=2\\ \mathrm{Sel}(\mathbf{Q},V_pE)\neq\ker(\mathrm{Loc}_p)\end{array}\right\}\quad\Longrightarrow\quad\kappa_{\alpha,\alpha^{-1}}(f,g,g^*)\neq 0. \tag{1.10}$$

In particular, if $\mathrm{Sel}(\mathbf{Q},V_p)\neq\ker(\mathrm{Loc}_p)$ then Conjecture 1.2 holds.

Remark 1.5. The condition $\mathrm{Sel}(\mathbf{Q},V_pE)\neq\ker(\mathrm{Loc}_p)$ should always hold when $\mathrm{Sel}(\mathbf{Q},V_pE)\neq 0$. Indeed, if $\mathrm{Sel}(\mathbf{Q},V_pE)$ equals $\ker(\mathrm{Loc}_p)$, then $E(\mathbf{Q})$ must be finite (since $E(\mathbf{Q})$ injects into $E(\mathbf{Q}_p)$), so if also $\mathrm{Sel}(\mathbf{Q},V_pE)\neq 0$ we would conclude that $\text{Ш}(E/\mathbf{Q})[p^\infty]$ is infinite.

Remark 1.6. It also follows from our results that, for $g=\theta_\psi$ as above, the classes $\kappa_{\alpha,\alpha^{-1}}(f,g,g^*)$ and $\kappa_{\beta,\beta^{-1}}(f,g,g^*)$ are the same up to a nonzero algebraic constant, and they span the *p-adic* line

$$\mathscr{L}_p:=\ker(\log_p)\subset\mathrm{Sel}(\mathbf{Q},V_pE),$$

where $\log_p : \mathrm{Sel}(\mathbf{Q}, V_pE) \to \mathbf{Q}_p$ is the composition of Loc_p with the formal group logarithm of E. When $\#Ш(E/\mathbf{Q})[p^\infty] < \infty$, it is suggestive to view $\mathscr{L}_p$ as the line spanned by the image of

$$P \wedge Q := P \otimes Q - Q \otimes P \in \bigwedge^2 (E(\mathbf{Q}) \otimes \mathbf{Q})$$

under the natural map

$$\mathrm{Log}_p : \bigwedge^2 (E(\mathbf{Q}) \otimes \mathbf{Q}) \to E(\mathbf{Q}) \otimes \mathbf{Q}_p$$

induced by $\log_p$. This is consistent with predictions by Darmon–Rotger (see [DR16, §4.5.3]), and suggests regarding $\kappa_{\alpha,\alpha^{-1}}(f, g, g^*)$ as a "p-adic shadow" of a rank two motivic regulator.

Remark 1.7. Note that the implications (1.9) and (1.10) in Theorem A are rank two analogues of the implications (1.3) and (1.2) by Skinner and Kolyvagin, respectively.

The key new ingredient in the proof of Theorem A is a *leading term formula* for an anticyclotomic p-adic L-function $\Theta_{f/K} \in \mathbf{Z}_p[\![T]\!]$ attached to E/K in terms of anticyclotomic *derived p-adic heights* (see Theorem 5.3). This formula applies in arbitrary order of vanishing of $\Theta_{f/K}$ at $T = 0$, and in particular it allows us to deduce the following p-adic analytic criterion for the non-vanishing of generalized Kato classes.

Theorem B. *Under the hypotheses of Theorem A, assume in addition that* $\mathrm{rank}_{\mathbf{Z}} E(\mathbf{Q}) > 0$. *Then the following implication holds:*

$$\mathrm{ord}_T(\Theta_{f/K}) = 2 \quad \Longrightarrow \quad \kappa_{\alpha,\alpha^{-1}}(f, g, g^*) \neq 0.$$

The same result holds with $\kappa_{\alpha,\alpha^{-1}}(f, g, g^*)$ *replaced by* $\kappa_{\beta,\beta^{-1}}(f, g, g^*)$.

Remark 1.8. If $\bar{\rho}_{E,p}$ is irreducible and ramified at some prime $q \neq p$ (e.g., if E is semistable and $p \geqslant 11$ is good ordinary for E, by [Rib90] and [Maz78]), the non-vanishing results of [BFH90] and [Vat03] assure the existence of infinitely many imaginary quadratic fields K and ring class characters χ such that

- q is inert in K,
- every prime factor of N_f/q splits in K,
- $L(E, \mathrm{ad}^0(g), 1) = L(E^K, 1) \cdot L(E/K, \chi, 1) \neq 0$.

Thus Theorem B suggests a general construction of non-trivial p-adic Selmer classes for rational elliptic curves of rank two.

Remark 1.9. In the Appendix to this paper, we apply Theorem B to numerically verify the non-vanishing of generalized Kato classes for specific rational elliptic curves of algebraic and analytic rank two, a task that was left as "an interesting challenge" by Darmon–Rotger (see [DR16, p. 31]).

Remark 1.10. Assume that $\mathrm{rank}_{\mathbf{Z}} E(\mathbf{Q}) = 2$ and $\#Ш(E/\mathbf{Q})[p^\infty] < \infty$. A refinement of Conjecture 1.2 predicting the *position* of $\kappa_{\gamma,\delta}(f, g, g^*)$ relative to the natural rational structure on $\mathrm{Sel}(\mathbf{Q}, V_pE) = E(\mathbf{Q}) \otimes \mathbf{Q}_p$ then leads to the expectation

$$\kappa_{\alpha,\alpha^{-1}}(f, g, g^*) \overset{?}{\sim}_{\overline{\mathbf{Q}}^\times} \mathrm{Log}_p(P \wedge Q) \overset{?}{\sim}_{\overline{\mathbf{Q}}^\times} \kappa_{\beta,\beta^{-1}}(f, g, g^*) \tag{1.11}$$

where (P, Q) is any basis for $E(\mathbf{Q}) \otimes \mathbf{Q}$ and $\sim_{\overline{\mathbf{Q}}^\times}$ denotes equality up to multiplication by a nonzero algebraic number. Our results confirm the predicted relation

$$\kappa_{\alpha,\alpha^{-1}}(f, g, g^*) \sim_{\overline{\mathbf{Q}}^\times} \kappa_{\beta,\beta^{-1}}(f, g, g^*)$$

and in Theorem 5.5 we show that

$$\kappa_{\alpha,\alpha^{-1}}(f, g, g^*) \sim_{\mathbf{Q}^\times} C \cdot \frac{1 - p^{-1}\alpha_p}{1 - \alpha_p^{-1}} \cdot \frac{\Theta_{f/K}^{(\mathfrak{r})}}{h_p^{(\mathfrak{r})}(P, Q)} \cdot \mathrm{Log}_p(P \wedge Q),$$

where C is a nonzero algebraic number, α_p is the p-adic unit root of $x^2 - a_p(E)x + p$ (with $a_p(E) = p + 1 - \#E(\mathbf{F}_p)$ as usual), $\Theta^{(\mathfrak{r})}_{f/K}$ is the leading term of $\Theta_{f/K}$ at $T = 0$, and $h^{(\mathfrak{r})}_p$ is the anticyclotomic $\mathfrak{r}$-th derived p-adic height pairing. In particular, this implies that the conjectured algebraicity in (1.11) follows from a p-adic Birch and Swinnerton-Dyer conjecture refining [BD96, Conjecture 4.3] (see §5.7).

1.3. **Relation to previous work.** Prior to this paper, the only general results (known to the authors) on the existence on nonzero Selmer classes for elliptic curves $E/\mathbf{Q}$ of rank $r > 1$ are in forthcoming work of Skinner–Urban (see [Urb13] for a report). Their methods, which extend those outlined in their ICM address [SU06] for cuspidal eigenforms of weight $k \geqslant 4$, are completely different form ours.

On the other hand, Darmon–Rotger [DR17] exhibited, under a certain *non-vanishing* hypothesis, the existence of two linearly independent classes in the Selmer groups $\mathrm{Sel}(\mathbf{Q}, V_pE \otimes \varrho)$ of elliptic curves $E/\mathbf{Q}$ twisted by degree four Artin representations ϱ. The required non-vanishing is that of a special value $\mathscr{L}_p^{g_\alpha}$ of a certain p-adic L-function. Both their works and ours exploit the construction of generalized Kato classes introduced in [DR17], but in the setting we have placed ourselves in, the special value $\mathscr{L}_p^{g_\alpha}$ *vanishes*. The proofs of our main results are based on anticyclotomic Iwasawa theory and derived p-adic heights, both of which make no appearance in [DR17].

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2. Triple products and theta elements

In this section we describe the triple product p-adic L-function for Hida families [Hsi21], and recall its relation with the square-root anticyclotomic p-adic L-functions of Bertolini–Darmon [BD96].

2.1. **Ordinary Λ-adic forms.** Fix a prime $p > 2$. Let $\mathbb{I}$ be a normal domain finite flat over $\Lambda := \mathcal{O}[\![1 + p\mathbf{Z}_p]\!]$, where $\mathcal{O}$ is the ring of integers of a finite extension $L/\mathbf{Q}_p$. We say that a point $x \in \mathrm{Spec}\,\mathbb{I}(\overline{\mathbf{Q}}_p)$ is *locally algebraic* if its restriction to $1+p\mathbf{Z}_p$ is given by $x(\gamma) = \gamma^{k_x}\epsilon_x(\gamma)$ for some integer k_x, called the *weight* of x, and some finite order character $\epsilon_x : 1+p\mathbf{Z}_p \to \mu_{p^\infty}$; we say that x is *arithmetic* if it has weight $k_x \geqslant 2$. Let $\mathfrak{X}^+_{\mathbb{I}}$ be the set of arithmetic points.

Fix a positive integer N prime to p, and let $\chi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathcal{O}^\times$ be a Dirichlet character modulo Np. Let $S^o(N, \chi, \mathbb{I})$ be the space of *ordinary* $\mathbb{I}$*-adic cusp forms* of tame level N and branch character χ, consisting of formal power series

$$\boldsymbol{f}(q) = \sum_{n=1}^{\infty} a_n(\boldsymbol{f})q^n \in \mathbb{I}[\![q]\!]$$

such that for every $x \in \mathfrak{X}^+_{\mathbb{I}}$ the specialization $\boldsymbol{f}_x(q)$ is the q-expansion of a p-ordinary cusp form $\boldsymbol{f}_x \in S_{k_x}(Np^{r_x+1}, \chi\omega^{2-k_x}\epsilon_x)$. Here r_x is such that $\epsilon_x(1 + p)$ has exact order p^{r_x}, and $\omega : (\mathbf{Z}/p\mathbf{Z})^\times \to \mu_{p-1}$ is the Teichmüller character.

We say that $\boldsymbol{f} \in S^o(N, \chi, \mathbb{I})$ is a *primitive Hida family* if for every $x \in \mathfrak{X}^+_{\mathbb{I}}$ we have that $\boldsymbol{f}_x$ is an ordinary p-stabilized newform (in the sense of [Hsi21, Def. 2.4]) of tame level N. Given a primitive Hida family $\boldsymbol{f} \in S^o(N, \chi, \mathbb{I})$, and writing $\chi = \chi'\chi_p$ with χ' (resp. χ_p) a Dirichlet modulo N (resp. p), there is a primitive Hida family $\boldsymbol{f}^\iota \in S^o(N, \chi_p\overline{\chi}', \mathbb{I})$ with Fourier coefficients

$$a_\ell(\boldsymbol{f}^\iota)=\begin{cases}\overline{\chi}'(\ell)a_\ell(\boldsymbol{f}) & \text{if } \ell\nmid N,\\ a_\ell(\boldsymbol{f})^{-1}\chi_p\omega^2(\ell)\langle\ell\rangle_{\mathbb{I}}\ell^{-1} & \text{if } \ell\mid N,\end{cases}$$

having the property that for every $x\in\mathfrak{X}_{\mathbb{I}}^{+}$ the specialization $\boldsymbol{f}_x^\iota$ is the p-stabilized newform attached to the character twist $\boldsymbol{f}_x\otimes\overline{\chi}'$.

By [Hid86] (*cf.* [Wil88, Thm. 2.2.1]), attached to every primitive Hida family $\boldsymbol{f}\in S^o(N,\chi,\mathbb{I})$ there is a continuous $\mathbb{I}$-adic representation $\rho_{\boldsymbol{f}}:G_{\mathbf{Q}}\to\mathrm{GL}_2(\mathrm{Frac}\,\mathbb{I})$ which is unramified outside Np and such that for every prime $\ell\nmid Np$,

$$\mathrm{tr}\,\rho_{\boldsymbol{f}}(\mathrm{Frob}_\ell)=a_\ell(\boldsymbol{f}),\quad \det\rho_{\boldsymbol{f}}(\mathrm{Frob}_\ell)=\chi\omega^2(\ell)\langle\ell\rangle_{\mathbb{I}}\ell^{-1},$$

where $\langle\ell\rangle_{\mathbb{I}}\in\mathbb{I}^\times$ is the image of $\langle\ell\rangle:=\ell\omega^{-1}(\ell)\in 1+p\mathbf{Z}_p$ under the natural map

$$1+p\mathbf{Z}_p\to\mathcal{O}[\![1+p\mathbf{Z}_p]\!]^\times=\Lambda^\times\to\mathbb{I}^\times.$$

In particular, letting $\langle\varepsilon_{\mathrm{cyc}}\rangle_{\mathbb{I}}:G_{\mathbf{Q}}\to\mathbb{I}^\times$ be defined by $\langle\varepsilon_{\mathrm{cyc}}\rangle_{\mathbb{I}}(\sigma)=\langle\varepsilon_{\mathrm{cyc}}(\sigma)\rangle_{\mathbb{I}}$, it follows that $\rho_{\boldsymbol{f}}$ has determinant $\chi_{\mathbb{I}}^{-1}\varepsilon_{\mathrm{cyc}}^{-1}$, where $\chi_{\mathbb{I}}:G_{\mathbf{Q}}\to\mathbb{I}^\times$ is given by $\chi_{\mathbb{I}}:=\sigma_\chi\langle\varepsilon_{\mathrm{cyc}}\rangle^{-2}\langle\varepsilon_{\mathrm{cyc}}\rangle_{\mathbb{I}}$, with σ_χ the Galois character sending $\mathrm{Frob}_\ell\mapsto\chi(\ell)^{-1}$. Moreover, by [Wil88, Thm. 2.2.2] the restriction of $\rho_{\boldsymbol{f}}$ to $G_{\mathbf{Q}_p}$ is given by

$$\rho_{\boldsymbol{f}}|_{G_{\mathbf{Q}_p}}\sim\begin{pmatrix}\psi_{\boldsymbol{f}} & *\\ 0 & \psi_{\boldsymbol{f}}^{-1}\chi_{\mathbb{I}}^{-1}\varepsilon_{\mathrm{cyc}}^{-1}\end{pmatrix}\tag{2.1}$$

where $\psi_{\boldsymbol{f}}:G_{\mathbf{Q}_p}\to\mathbb{I}^\times$ is the unramified character with $\psi_{\boldsymbol{f}}(\mathrm{Frob}_p)=a_p(\boldsymbol{f})$.

Let $T^o(N,\chi,\mathbb{I})$ be the $\mathbb{I}$-algebra generated by Hecke operators acting on $S^0(N,\chi,\mathbb{I})$ and let $\lambda_{\boldsymbol{f}}:T^o(N,\chi,\mathbb{I})\to\mathbb{I}$ be the $\mathbb{I}$-algebra homomorphism induced by $\boldsymbol{f}$. Let $C(\lambda_{\boldsymbol{f}})$ be the congruence module associated with $\lambda_{\boldsymbol{f}}$ (see [Hid88]). Under the following hypothesis:

$$\text{the residual representation } \bar\rho_{\boldsymbol{f}} \text{ is absolutely irreducible and } p\text{-distinguished,}\tag{CR}$$

it follows from results of Hida and Wiles that $C(\lambda_{\boldsymbol{f}})$ is isomorphic to $\mathbb{I}/(\eta_{\boldsymbol{f}})$ for some nonzero $\eta_{\boldsymbol{f}}\in\mathbb{I}$.

2.2. **Triple product p-adic L-function.** Let

$$(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})\in S^o(N_{\boldsymbol{f}},\chi_{\boldsymbol{f}},\mathbb{I}_{\boldsymbol{f}})\times S^o(N_{\boldsymbol{g}},\chi_{\boldsymbol{g}},\mathbb{I}_{\boldsymbol{g}})\times S^o(N_{\boldsymbol{h}},\chi_{\boldsymbol{h}},\mathbb{I}_{\boldsymbol{h}})$$

be a triple of primitive Hida families. Set

$$\mathcal{R}:=\mathbb{I}_{\boldsymbol{f}}\hat\otimes_{\mathcal{O}}\mathbb{I}_{\boldsymbol{g}}\hat\otimes_{\mathcal{O}}\mathbb{I}_{\boldsymbol{h}},$$

which is a finite extension of the three-variable Iwasawa algebra $\mathcal{R}_0:=\Lambda\hat\otimes_{\mathcal{O}}\Lambda\hat\otimes_{\mathcal{O}}\Lambda$, and define the weight space $\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$ for the triple $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ in the $\boldsymbol{f}$*-dominated unbalanced range* by

$$\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}:=\left\{(x,y,z)\in\mathfrak{X}_{\mathbb{I}_{\boldsymbol{f}}}^{+}\times\mathfrak{X}_{\mathbb{I}_{\boldsymbol{g}}}^{\mathrm{cls}}\times\mathfrak{X}_{\mathbb{I}_{\boldsymbol{h}}}^{\mathrm{cls}}:\ k_x\geqslant k_y+k_z\text{ and }k_x\equiv k_y+k_z\ (\mathrm{mod}\ 2)\right\},\tag{2.2}$$

where $\mathfrak{X}_{\mathbb{I}_{\boldsymbol{g}}}^{\mathrm{cls}}\supseteq\mathfrak{X}_{\mathbb{I}_{\boldsymbol{g}}}^{+}$ (and similarly $\mathfrak{X}_{\mathbb{I}_{\boldsymbol{h}}}^{\mathrm{cls}}$) is the set of locally algebraic points in $\mathrm{Spec}\,\mathbb{I}_{\boldsymbol{g}}(\overline{\mathbf{Q}}_p)$ for which $\boldsymbol{g}_x(q)$ is the q-expansion of a classical modular form.

For $\boldsymbol{\phi}\in\{\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}\}$ and a positive integer N prime to p and divisible by $N_{\boldsymbol{\phi}}$, define the space of Λ*-adic test vectors* $S^o(N,\chi_{\boldsymbol{\phi}},\mathbb{I}_{\boldsymbol{\phi}})[\boldsymbol{\phi}]$ of level N to be the $\mathbb{I}_{\boldsymbol{\phi}}$-submodule of $S^o(N,\chi_{\boldsymbol{\phi}},\mathbb{I}_{\boldsymbol{\phi}})$ generated by $\{\boldsymbol{\phi}(q^d)\}$, as d ranges over the positive divisors of $N/N_{\boldsymbol{\phi}}$.

For the next result, set $N:=\mathrm{lcm}(N_{\boldsymbol{f}},N_{\boldsymbol{g}},N_{\boldsymbol{h}})$, and consider the following hypothesis:

$$\text{for some } (x,y,z)\in\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}},\text{ we have } \varepsilon_q(\boldsymbol{f}_x^\circ,\boldsymbol{g}_y^\circ,\boldsymbol{h}_z^\circ)=+1 \text{ for all } q\mid N,\tag{$\Sigma^-=\emptyset$}$$

where $\varepsilon_q(\boldsymbol{f}_x^\circ,\boldsymbol{g}_y^\circ,\boldsymbol{h}_z^\circ)$ is the local root number at q of the Kummer self-dual twist of the tensor product of the p-adic Galois representations attached to the newforms $\boldsymbol{f}_x^\circ$, $\boldsymbol{g}_y^\circ$, and $\boldsymbol{h}_z^\circ$ corresponding to $\boldsymbol{f}_x$, $\boldsymbol{g}_y$, and $\boldsymbol{h}_z$. We shall say that a point $(x,y,z)\in\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$ is *crystalline* if the conductors of $\boldsymbol{f}_x^\circ$, $\boldsymbol{g}_y^\circ$, and $\boldsymbol{h}_z^\circ$ are all prime-to-p.

Theorem 2.1. *Assume that $\boldsymbol{f}$ satisfies hypothesis (CR) and that, in addition to hypothesis $(\Sigma^- = \emptyset)$, the triple $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ satisfies:*

(ev) $\chi_{\boldsymbol{f}}\chi_{\boldsymbol{g}}\chi_{\boldsymbol{h}} = \omega^{2a}$ *for some* $a \in \mathbf{Z}$,
(sq) $\gcd(N_{\boldsymbol{f}}, N_{\boldsymbol{g}}, N_{\boldsymbol{h}})$ *is squarefree.*

Fix a generator $\eta_{\boldsymbol{f}}$ of the congruence module of $\boldsymbol{f}$. Then there exist Λ-adic test vectors $(\underline{\breve{\boldsymbol{f}}}, \underline{\breve{\boldsymbol{g}}}, \underline{\breve{\boldsymbol{h}}})$ and an element $\mathscr{L}_p^f(\underline{\breve{\boldsymbol{f}}}, \underline{\breve{\boldsymbol{g}}}, \underline{\breve{\boldsymbol{h}}}) \in \mathcal{R}$ such that for all crystalline $(x, y, z) \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$ of weight (k, ℓ, m) we have

$$\mathscr{L}_p^f(\underline{\breve{\boldsymbol{f}}}, \underline{\breve{\boldsymbol{g}}}, \underline{\breve{\boldsymbol{h}}})(x,y,z)^2 = \Gamma(k,\ell,m) \cdot \mathcal{E}_p(\boldsymbol{f}_x, \boldsymbol{g}_y, \boldsymbol{h}_z)^2 \cdot \prod_{q|N} \tau_q^2 \cdot \frac{L(\boldsymbol{f}_x^\circ \otimes \boldsymbol{g}_y^\circ \otimes \boldsymbol{h}_z^\circ, c)}{(\sqrt{-1})^{2k} \cdot \Omega_{\boldsymbol{f}_x}^2},$$

where:

- $c = (k + \ell + m - 2)/2$,
- $\Gamma(k,\ell,m) = (c-1)! \cdot (c-m)! \cdot (c-\ell)! \cdot (c+1-\ell-m)!$,
- $\mathcal{E}_p(\boldsymbol{f}_x, \boldsymbol{g}_y, \boldsymbol{h}_z) = (1 - \frac{\beta_{\boldsymbol{f}_x}\alpha_{\boldsymbol{g}_y}\alpha_{\boldsymbol{h}_z}}{p^c})(1 - \frac{\beta_{\boldsymbol{f}_x}\beta_{\boldsymbol{g}_y}\alpha_{\boldsymbol{h}_z}}{p^c})(1 - \frac{\beta_{\boldsymbol{f}_x}\alpha_{\boldsymbol{g}_y}\beta_{\boldsymbol{h}_z}}{p^c})(1 - \frac{\beta_{\boldsymbol{f}_x}\beta_{\boldsymbol{g}_y}\beta_{\boldsymbol{h}_z}}{p^c})$,
- τ_q *is a nonzero constant (equal to either* 1 *or* $1 + q^{-1}$*),*
- $\Omega_{\boldsymbol{f}_x} \in \mathbf{C}^\times$ *is the* canonical period *in* [Hsi21, Def. 3.12] *computed with respect to* $\eta_{\boldsymbol{f}}$,

and $L(\boldsymbol{f}_x^\circ \otimes \boldsymbol{g}_y^\circ \otimes \boldsymbol{h}_z^\circ, c)$ is the central value of the triple product L-function.

Proof. This is a special case of Theorem A in [Hsi21]. The construction of $\mathscr{L}_p^f(\underline{\breve{\boldsymbol{f}}}, \underline{\breve{\boldsymbol{g}}}, \underline{\breve{\boldsymbol{h}}})$ under hypotheses (CR), (ev) and (sq) is given in [Hsi21, §3.6]; the proof of its interpolation property (for all points $(x, y, z) \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$, rather than just those that are crystalline) assuming hypothesis $(\Sigma^- = \emptyset)$ is given in [Hsi21, §7]. □

Remark 2.2. The construction of $\mathscr{L}_p^f(\underline{\breve{\boldsymbol{f}}}, \underline{\breve{\boldsymbol{g}}}, \underline{\breve{\boldsymbol{h}}})$ is based on Hida's p-adic Rankin–Selberg convolution [Hid85] and applies to any choice of test vectors for $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$. In the following, for any test vectors $(\breve{\boldsymbol{f}}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{h}})$ we use $\mathscr{L}_p^f(\breve{\boldsymbol{f}}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{h}})$ to denote the associated triple product p-adic L-function (but note that in the proof of our main results the specific choice $(\underline{\breve{\boldsymbol{f}}}, \underline{\breve{\boldsymbol{g}}}, \underline{\breve{\boldsymbol{h}}})$ will be critical).

2.3. **Triple tensor product of big Galois representations.** Let $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ be a triple of primitive Hida families with $\chi_{\boldsymbol{f}}\chi_{\boldsymbol{g}}\chi_{\boldsymbol{h}} = \omega^{2a}$ for some $a \in \mathbf{Z}$. For $\boldsymbol{\phi} \in \{\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}\}$, let $V_{\boldsymbol{\phi}}$ be the natural lattice in $(\mathrm{Frac}\,\mathbb{I}_{\boldsymbol{\phi}})^2$ realizing the Galois representation $\rho_{\boldsymbol{\phi}}$ in the étale cohomology of modular curves (see [Oht00]), and set

$$\mathbb{V}_{\boldsymbol{fgh}} := V_{\boldsymbol{f}} \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}} \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}.$$

This has rank 8 over $\mathcal{R}$, and by hypothesis its determinant can be written as $\det \mathbb{V}_{\boldsymbol{fgh}} = \mathcal{X}^2 \varepsilon_{\mathrm{cyc}}$ for a p-ramified Galois character $\mathcal{X}$ taking the value $(-1)^a$ at complex conjugation. Similarly as in [How07, Def. 2.1.3], we define the *critical twist*

$$\mathbb{V}_{\boldsymbol{fgh}}^\dagger := \mathbb{V}_{\boldsymbol{fgh}} \otimes \mathcal{X}^{-1}.$$

More generally, for any multiple N of $N_{\boldsymbol{\phi}}$ one can define Galois modules $V_{\boldsymbol{\phi}}(N)$ by working in tame level N; these split non-canonically into a finite direct sum of the $\mathbb{I}_{\boldsymbol{\phi}}$-adic representations $V_{\boldsymbol{\phi}}$ (see [DR17, §1.5.3]), and they define $\mathbb{V}_{\boldsymbol{fgh}}^\dagger(N)$ for any N divisible by $\mathrm{lcm}(N_{\boldsymbol{f}}, N_{\boldsymbol{g}}, N_{\boldsymbol{h}})$.

If f is a classical specialization of $\boldsymbol{f}$ with associated p-adic Galois representation V_f, we let $\mathbb{V}_{f,\boldsymbol{gh}}$ be the quotient of $\mathbb{V}_{\boldsymbol{fgh}}$ given by

$$\mathbb{V}_{f,\boldsymbol{gh}} := V_f \otimes_{\mathcal{O}} V_{\boldsymbol{g}} \hat{\otimes}_{\mathbb{I}} V_{\boldsymbol{h}},$$

and denote by $\mathbb{V}_{f,\boldsymbol{gh}}^\dagger$ the corresponding quotient of $\mathbb{V}_{\boldsymbol{fgh}}^\dagger$, and by $\mathbb{V}_{f,\boldsymbol{gh}}^\dagger(N)$ its level N counterpart.

2.4. **Theta elements and factorization.** We recall the factorization proven in [Hsi21, §8]. Let $f \in S_2(pN_f)$ be a p-stabilized newform of tame level N_f defined over $\mathcal{O}$, let $f^\circ \in S_2(N_f)$ be the associated newform, and let $\alpha_p = \alpha_p(f) \in \mathcal{O}^\times$ be the U_p-eigenvalue of f. Let K be an imaginary quadratic field of discriminant D_K prime to N_f. Write

$$N_f = N^+N^-$$

with N^+ (resp. N^-) divisible only by primes which are split (resp. inert) in K, and choose an ideal $\mathfrak{N}^+ \subset \mathcal{O}_K$ with $\mathcal{O}_K/\mathfrak{N}^+ \simeq \mathbf{Z}/N^+\mathbf{Z}$.

Assume that $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K, with our fixed embedding $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ inducing the prime $\mathfrak{p}$. Let Γ_∞ be the Galois group of the anticyclotomic $\mathbf{Z}_p$-extension K_∞/K, fix a topological generator $\boldsymbol{\gamma} \in \Gamma_\infty$, and identity $\mathcal{O}[\![\Gamma_\infty]\!]$ with the power series ring $\mathcal{O}[\![T]\!]$ via $\boldsymbol{\gamma} \mapsto 1+T$. For any prime-to-$p$ ideal $\mathfrak{a}$ of K, let $\sigma_{\mathfrak{a}}$ be the image of $\mathfrak{a}$ in the Galois group of the ray class field $K(p^\infty)/K$ of conductor p^∞ under the geometrically normalized reciprocity law map.

Theorem 2.3. *Let χ be a ring class character of K of conductor $c\mathcal{O}_K$ with values in $\mathcal{O}$, and assume that:*

(i) $(pN_f, cD_K) = 1$,
(ii) *N^- is the squarefree product of an odd number of primes,*
(iii) *if $q|N^-$ is a prime with $q \equiv 1 \pmod p$, then $\bar\rho_f$ is ramified at q.*

Then there exists a unique $\Theta_{f/K,\chi}(T) \in \mathcal{O}[\![T]\!]$ such that for every p-power root of unity ζ:

$$\Theta_{f/K,\chi}(\zeta-1)^2 = \frac{p^n}{\alpha_p^{2n}} \cdot \mathcal{E}_p(f,\chi,\zeta)^2 \cdot \frac{L(f^\circ/K \otimes \chi\epsilon_\zeta, 1)}{(2\pi)^2 \cdot \Omega_{f^\circ,N^-}} \cdot u_K^2 \sqrt{D_K}\chi\epsilon_\zeta(\sigma_{\mathfrak{N}^+}) \cdot \varepsilon_p,$$

where:

- *$n \geqslant 0$ is such that ζ has exact order p^n,*
- *$\epsilon_\zeta : \Gamma_\infty \to \mu_{p^\infty}$ be the character defined by $\epsilon_\zeta(\boldsymbol{\gamma}) = \zeta$,*
- $\mathcal{E}_p(f,\chi,\zeta) = \begin{cases} (1-\alpha_p^{-1}\chi(\mathfrak{p}))(1-\alpha_p\chi(\overline{\mathfrak{p}})) & \text{if } n = 0, \\ 1 & \text{if } n > 0, \end{cases}$
- *$\Omega_{f^\circ,N^-} = 4\|f^\circ\|^2_{\Gamma_0(N_f)} \cdot \eta^{-1}_{f,N^-}$ is the* Gross period *of f° (see* [Hsi21, p. 524]*),*
- *$\sigma_{\mathfrak{N}^+} \in \Gamma_\infty$ is the image of $\mathfrak{N}^+$ under the geometrically normalized Artin's reciprocity map,*
- *$u_K = |\mathcal{O}_K^\times|/2$, and $\varepsilon_p \in \{\pm1\}$ is the local root number of f° at p.*

Proof. See [BD96] for the first construction, and [CH18b, Thm. A] for the stated interpolation property. ☐

Remark 2.4. From the interpolation property of Theorem 2.3, one can show that the square of $\Theta_{f/K,\chi}(T)$ is essentially the anticyclomic restriction of the two-variable p-adic L-function constructed by Perrin-Riou [PR88].

When χ is the trivial character, we write $\Theta_{f/K,\chi}(T)$ simply as $\Theta_{f/K}(T)$. Suppose now that the p-stabilized newform f as in Theorem 2.3 is the specialization of a primitive Hida family $\boldsymbol{f} \in S^o(N_f, \mathbb{I})$ with branch character $\chi_{\boldsymbol{f}} = \mathbb{1}$ at an arithmetic point $x_1 \in \mathfrak{X}_{\mathbb{I}}^+$ of weight 2. Let $\ell \nmid pN_f$ be a prime split in K, and let χ be a ring class character of K of conductor $\ell^m\mathcal{O}_K$ for some $m > 0$. Denoting by the superscript τ the action of the non-trivial automorphism of $K/\mathbf{Q}$, write $\chi = \psi^{1-\tau}$ with ψ a ray class character modulo $\ell^m\mathcal{O}_K$. Set $C = D_K\ell^{2m}$ and let

$$\boldsymbol{g} = \boldsymbol{\theta}_\psi(S_2) \in \mathcal{O}[\![S_2]\!][\![q]\!], \quad \boldsymbol{g}^* = \boldsymbol{\theta}_{\psi^{-1}}(S_3) \in \mathcal{O}[\![S_3]\!][\![q]\!]$$

be the primitive CM Hida families of level C constructed in [Hsi21, §8.3].

The p-adic L-function $\mathscr{L}_p^f(\underline{\breve{\boldsymbol{f}}}, \underline{\breve{\boldsymbol{g}}}, \underline{\breve{\boldsymbol{g}}}^*)$ of Theorem 2.1 attached to the triple $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{g}^*)$ (taking $a = -1$ in (ev)) is an element in $\mathcal{R} = \mathbb{I}[\![S_2, S_3]\!]$; in the following we let

$$\mathscr{L}_p^f(\underline{\breve{f}}, \underline{\breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*}) \in \mathcal{O}[\![S]\!]$$

denote the restriction to the "line" $S = S_2 = S_3$ of the image of $\mathscr{L}_p^f(\breve{\underline{\boldsymbol{f}}}, \breve{\underline{\boldsymbol{g}}}, \breve{\underline{\boldsymbol{g}}}^*)$ under the specialization map at x_1.

Let $\mathbb{K}_\infty$ be the $\mathbf{Z}_p^2$-extension of K, and let $K_{\mathfrak{p}^\infty}$ denote the $\mathfrak{p}$-ramified $\mathbf{Z}_p$-extension in $\mathbb{K}_\infty$, with Galois group $\Gamma_{\mathfrak{p}^\infty} = \mathrm{Gal}(K_{\mathfrak{p}^\infty}/K)$. Let $\gamma_{\mathfrak{p}} \in \Gamma_{\mathfrak{p}^\infty}$ be a topological generator, and for the formal variable T let $\Psi_T : \mathrm{Gal}(\mathbb{K}_\infty/K) \to \mathcal{O}[\![T]\!]^\times$ be the universal character defined by

$$\Psi_T(\sigma) = (1+T)^{l(\sigma)}, \quad \text{where } \sigma|_{K_{\mathfrak{p}^\infty}} = \gamma_{\mathfrak{p}}^{l(\sigma)}. \tag{2.3}$$

The character $\Psi_T^{1-\tau}$ factors through Γ_∞ and yields an identification $\mathcal{O}[\![\Gamma_\infty]\!] \simeq \mathcal{O}[\![T]\!]$ corresponding to the topological generator $\gamma_{\mathfrak{p}}^{1-\tau} \in \Gamma_\infty$. Let p^b be the order of the p-part of the class number of K. Hereafter, we shall fix $\mathbf{v} \in \overline{\mathbf{Z}}_p^\times$ such that $\mathbf{v}^{p^b} = \varepsilon_{\mathrm{cyc}}(\gamma_{\mathfrak{p}}^{p^b}) \in 1 + p\mathbf{Z}_p$. Let $K(\chi, \alpha_p)/K$ (resp. $K(\chi)/K$) be the finite extension obtained by adjoining to K the values of χ and α_p (resp. the values of χ).

Proposition 2.5. *Assume that*

(i) N^- *is the squarefree product of an odd number of primes,*

(ii) $\bar{\rho}_f$ *is ramified at every prime* $q|N^-$ *with* $q \equiv 1 \pmod p$.

Set $T = \mathbf{v}^{-1}(1+S) - 1$. *Then*

$$\mathscr{L}_p^f(\breve{\underline{f}}, \breve{\underline{\boldsymbol{g}}}\breve{\underline{\boldsymbol{g}}}^*) = \pm \mathbf{w}^{-1} \cdot \Theta_{f/K}(T) \cdot C_{f,\chi} \cdot \sqrt{L^{\mathrm{alg}}(f/K \otimes \chi, 1)} \cdot \frac{\eta_{f^\circ}}{\eta_{f^\circ, N^-}},$$

where $\mathbf{w}$ *is an unit in* $\mathcal{O}[\![T]\!]$, $C_{f,\chi} \in K(\chi, \alpha_p)^\times$, *and*

$$L^{\mathrm{alg}}(f/K \otimes \chi, 1) := \frac{L(f/K \otimes \chi, 1)}{4\pi^2 \|f^\circ\|^2_{\Gamma_0(N_f)}} \in K(\chi).$$

Proof. This is [Hsi21, Prop. 8.1] specialized to $S = S_2 = S_3$, using the interpolation property of $\Theta_{f/K,\chi}(T)$ at $\zeta = 1$. (Note that the unit $\mathbf{w}$ is explicitly described in *loc. cit.*, but we omit it here.) □

Remark 2.6. The factorization of Proposition 2.5 reflects the decomposition of Galois representations

$$\mathbb{V}^\dagger_{f,\boldsymbol{g}\boldsymbol{g}^*} = \big(V_f(1) \otimes \mathrm{Ind}_K^{\mathbf{Q}} \Psi_T^{1-\tau}\big) \oplus \big(V_f(1) \otimes \mathrm{Ind}_K^{\mathbf{Q}} \chi\big). \tag{2.4}$$

Note that the first summand in (2.4) is the anticyclotomic deformation of $V_f(1)$, while the second is a fixed character twist of $V_f(1)$.

3. Coleman map for relative Lubin–Tate groups

In this section we review Perrin-Riou's theory [PR94] of big exponential maps, as extended by Kobayashi [Kob18] to $\mathbf{Z}_p$-extensions arising from torsion points on relative Lubin–Tate formal groups of height one. Applied to the localization of the anticyclotomic $\mathbf{Z}_p$-extension of an imaginary quadratic field K in which p splits, we then deduce, by the results of §2 and [DR17], a Coleman power series construction of the p-adic L-function $\Theta_{f/K}$ of Theorem 2.3. This new construction of $\Theta_{f/K}$ will play an important role in the proof of our main results.

3.1. **Preliminaries.** Fix a complete algebraic closure $\mathbf{C}_p$ of $\mathbf{Q}_p$. Let $\mathbf{Q}_p^{\mathrm{ur}} \subset \mathbf{C}_p$ be the maximal unramified extension of $\mathbf{Q}_p$, and let $\mathrm{Fr} \in \mathrm{Gal}(\mathbf{Q}_p^{\mathrm{ur}}/\mathbf{Q}_p)$ be the absolute Frobenius. Let $F \subset \mathbf{Q}_p^{\mathrm{ur}}$ be a finite unramified extension of $\mathbf{Q}_p$ with valuation ring $\mathscr{O}$ and set

$$R = \mathscr{O}[\![X]\!].$$

Let $\mathcal{F} = \mathrm{Spf}\, R$ be a relative Lubin–Tate formal group of height one defined over $\mathscr{O}$, and for each $n \in \mathbf{Z}$ set

$$\mathcal{F}^{(n)} := \mathcal{F} \times_{\mathrm{Spec}\,\mathscr{O}, \mathrm{Fr}^{-n}} \mathrm{Spec}\,\mathscr{O}.$$

The Frobenius morphism $\varphi_{\mathcal{F}} \in \mathrm{Hom}(\mathcal{F}, \mathcal{F}^{(-1)})$ induces a homomorphism $\varphi_{\mathcal{F}} \colon R \to R$ defined by

$$\varphi_{\mathcal{F}}(f) := f^{\mathrm{Fr}} \circ \varphi_{\mathcal{F}},$$

where f^{Fr} is the conjugate of f by Fr. Let $\psi_{\mathcal{F}}$ be the left inverse of $\varphi_{\mathcal{F}}$ satisfying

$$\varphi_{\mathcal{F}} \circ \psi_{\mathcal{F}}(f) = p^{-1} \sum_{x \in \mathcal{F}[p]} f(X \oplus_{\mathcal{F}} x). \tag{3.1}$$

Let F_∞/F be the Lubin–Tate $\mathbf{Z}_p^\times$-extension of F associated with $\mathcal{F}$, i.e., $F_\infty = \bigcup_{n=1}^{\infty} F(\mathcal{F}[p^n])$, and for every $n \geqslant -1$ let F_n be the subfield of F_∞ with $\mathrm{Gal}(F_n/F) \simeq (\mathbf{Z}/p^{n+1}\mathbf{Z})^\times$. (Hence, $F_{-1} = F$.) Letting $G_\infty = \mathrm{Gal}(F_\infty/F)$, there is a canonical decomposition

$$G_\infty \simeq \Delta \times \Gamma_\infty^{\mathcal{F}},$$

with Δ the torsion subgroup of G_∞ and $\Gamma_\infty^{\mathcal{F}} \simeq \mathbf{Z}_p$ the maximal torsion-free quotient of G_∞.

For every $a \in \mathbf{Z}_p^\times$, there is a unique formal power series $[a] \in R$ such that

$$[a]^{\mathrm{Fr}} \circ \varphi_{\mathcal{F}} = \varphi_{\mathcal{F}} \circ [a] \quad \text{and} \quad [a](X) \equiv aX \pmod{X^2}.$$

Letting $\varepsilon_{\mathcal{F}} \colon G_\infty \xrightarrow{\sim} \mathbf{Z}_p^\times$ be the Lubin–Tate character, we let $\sigma \in G_\infty$ act on $f \in R$ by

$$\sigma.f(X) := f([\varepsilon_{\mathcal{F}}(\sigma)](X)),$$

thus making R into an $\mathscr{O}[\![G_\infty]\!]$-module.

Lemma 3.1. *$R^{\psi_{\mathcal{F}}=0}$ is free of rank one over $\mathscr{O}[\![G_\infty]\!]$.*

Proof. This is [Kob18, Prop. 5.4]. □

Let V be a crystalline $G_{\mathbf{Q}_p}$-representation defined over a finite extension L of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}_L$. Let $\mathbf{D}(V) = \mathbf{D}_{\mathrm{cris},\mathbf{Q}_p}(V)$ be the filtered φ-module associated with V and set

$$\mathscr{D}_\infty(V) := \mathbf{D}(V) \otimes_{\mathbf{Z}_p} R^{\psi_{\mathcal{F}}=0}.$$

Fix an invariant differential $\omega_{\mathcal{F}} \in \Omega_R$, and let $\log_{\mathcal{F}} \in R \widehat{\otimes} \mathbf{Q}_p$ be the logarithm map satisfying

$$\log_{\mathcal{F}}(0) = 0 \quad \text{and} \quad d\log_{\mathcal{F}} = \omega_{\mathcal{F}},$$

where $d : R \to \Omega_R$ be the standard derivation.

Let $\epsilon = (\epsilon_n) \in T_p\mathcal{F} = \varprojlim \mathcal{F}^{(n)}[p^n]$ be a basis of the Tate module of $\mathcal{F}$, where the limit is with respect to the transition maps

$$\varphi^{\mathrm{Fr}^{-(n+1)}} \colon \mathcal{F}^{(n+1)}[p^{n+1}] \to \mathcal{F}^{(n)}[p^n].$$

One can associate to ϵ and $\omega_{\mathcal{F}}$ a p-adic period $t_\epsilon \in B_{\mathrm{cris}}^+$ such that

$$\mathbf{D}_{\mathrm{cris},F}(\varepsilon_{\mathcal{F}}) = F t_\epsilon^{-1} \quad \text{and} \quad \varphi t_\epsilon = \varpi t_\epsilon, \tag{3.2}$$

where ϖ is the uniformizer in F such that $\varphi_{\mathcal{F}}^*(\omega_{\mathcal{F}}^{\mathrm{Fr}}) = \varpi \cdot \omega_{\mathcal{F}}$ (see [Kob18, §9.2]). For $j \in \mathbf{Z}$, the Lubin–Tate twist $V\langle j \rangle := V \otimes_L \varepsilon_{\mathcal{F}}^j$ then satisfies

$$\mathbf{D}_{\mathrm{cris},F}(V\langle j \rangle) = \mathbf{D}(V) \otimes_{\mathbf{Q}_p} F t_\epsilon^{-j}.$$

There is a derivation $\mathrm{d}_\epsilon : \mathscr{D}_\infty(V\langle j \rangle) = \mathbf{D}_{\mathrm{cris},F}(V\langle j \rangle) \otimes_{\mathscr{O}} R^{\psi_{\mathcal{F}}=0} \to \mathscr{D}_\infty(V\langle j-1 \rangle)$ given by

$$d_\epsilon : f = \eta \otimes g \mapsto \eta t_\epsilon \otimes \partial g,$$

where $\partial \colon R \to R$ is defined by $df = \partial f \cdot \omega_{\mathcal{F}}$. These give rise to the map

$$\widetilde{\Delta} \colon \mathscr{D}_\infty(V) \to \bigoplus_{j \in \mathbf{Z}} \frac{\mathbf{D}_{\mathrm{cris},F}(V\langle -j \rangle)}{1 - \varphi} \tag{3.3}$$

sending $f \mapsto (\partial^j f(0) t_\epsilon^j \pmod{1-\varphi})_j$.

3.2. **Perrin-Riou's big exponential map.** For a finite extension K over $\mathbf{Q}_p$, let

$$\exp_{K,V}\colon \mathbf{D}(V)\otimes_{\mathbf{Q}_p} K \to \mathrm{H}^1(K,V)$$

be Bloch–Kato's exponential map [BK90, §3]. In this subsection, we recall the main properties of Perrin-Riou's map $\Omega_{V,h}$ interpolating $\exp_{K,V\langle j\rangle}$ over non-negative $j\in\mathbf{Z}$.

Let $V^* := \mathrm{Hom}_L(V, L(1))$ be the Kummer dual of V and denote by

$$[-,-]_V : \mathbf{D}(V^*)\otimes K \times \mathbf{D}(V)\otimes K \to L\otimes K$$

the K-linear extension of the de Rham pairing

$$\langle\, ,\, \rangle_{\mathrm{dR}}\colon \mathbf{D}(V^*)\times\mathbf{D}(V)\to L.$$

Let $\exp^*_{K,V} : \mathrm{H}^1(K,V)\to\mathbf{D}(V)\otimes K$ be the Bloch–Kato dual exponential map, which is characterized uniquely by

$$\mathrm{Tr}_{K/\mathbf{Q}_p}([x,\exp^*_{K,V}(y)]_V) = \langle \exp_{K,V^*}(x), y\rangle_{\mathrm{dR}},$$

for all $x\in\mathbf{D}(V^*)\otimes K$ and $y\in\mathrm{H}^1(K,V)$.

Choose a $\mathcal{O}_L$-lattice $T\subset V$ stable under the Galois action, and set $\widehat{\mathrm{H}}^1(F_\infty,T)=\varprojlim_n \mathrm{H}^1(F_n,T)$ and

$$\widehat{\mathrm{H}}^1(F_\infty,V)=\widehat{\mathrm{H}}^1(F_\infty,T)\otimes_{\mathbf{Z}_p}\mathbf{Q}_p,$$

which does not depend on the choice of T. Denote by

$$\mathrm{Tw}^j : \widehat{\mathrm{H}}^1(F_\infty,V)\simeq\widehat{\mathrm{H}}^1(F_\infty,V\langle j\rangle)$$

the twisting map by $\varepsilon^j_{\mathcal{F}}$. For a non-negative real number r, put

$$\mathscr{H}_{r,K}(X)=\left\{\sum_{n\geqslant 0,\tau\in\Delta} c_{n,\tau}\cdot\tau\cdot X^n\in K[\Delta][\![X]\!] \mid \sup_n |c_{n,\tau}|_p\, n^{-r}<\infty \text{ for all } \tau\in\Delta\right\},$$

where $|\cdot|_p$ is the normalized valuation of K with $|p|_p=p^{-1}$. Let $\boldsymbol{\gamma}$ be a topological generator of $\Gamma^{\mathcal{F}}_\infty$, and denote by $\mathscr{H}_{r,K}(G_\infty)$ the ring of elements $\{f(\boldsymbol{\gamma}-1)\colon f\in\mathscr{H}_{r,K}(X)\}$, so in particular $\mathscr{H}_{0,K}(G_\infty)=\mathcal{O}_K[\![G_\infty]\!]\otimes_{\mathcal{O}_K}K$. Put

$$\mathscr{H}_{\infty,K}(G_\infty)=\bigcup_{r\geqslant 0}\mathscr{H}_{r,K}(G_\infty).$$

Define the map

$$\Xi_{n,V}\colon \mathbf{D}(V)\otimes_{\mathbf{Q}_p}\mathscr{H}_{\infty,F}(X)\to\mathbf{D}(V)\otimes_{\mathbf{Q}_p}F_n$$

by

$$\Xi_{n,V}(G) := \begin{cases} p^{-(n+1)}\varphi^{-(n+1)}(G^{\mathrm{Fr}^{-(n+1)}}(\epsilon_n)) & \text{if } n\geqslant 0,\\ (1-p^{-1}\varphi^{-1})(G(0)) & \text{if } n=-1,\end{cases} \tag{3.4}$$

and let $\widetilde{\Lambda}=\mathbf{Z}_p[\![G_\infty]\!]$.

Theorem 3.2. *Let $\epsilon=(\epsilon_n)$ be a basis of $T_p\mathcal{F}$, let $h>0$ be such that $\mathbf{D}(V)=\mathrm{Fil}^{-h}\mathbf{D}(V)$, and assume that $\mathrm{H}^0(F_\infty,V)=0$. There exists $\widetilde{\Lambda}$-linear "big exponential map"*

$$\Omega^\epsilon_{V,h} : \mathscr{D}_\infty(V)^{\widetilde{\Delta}=0}\to\widehat{\mathrm{H}}^1(F_\infty,T)\otimes_{\widetilde{\Lambda}}\mathscr{H}_{\infty,F}(G_\infty)$$

such that for every $g\in\mathscr{D}_\infty(V)^{\widetilde{\Delta}=0}$ and $j\geqslant 1-h$ satisfies the interpolation property

$$\mathrm{pr}_{F_n}(\mathrm{Tw}^j\circ\Omega^\epsilon_{V,h}(g)) = (-1)^{h+j-1}(h+j-1)!\cdot\exp_{F_n,V\langle j\rangle}(\Xi_{n,V\langle j\rangle}(\mathrm{d}_\epsilon^{-j}G))\in\mathrm{H}^1(F_n,V\langle j\rangle),$$

where $G\in\mathbf{D}(V)\otimes_{\mathbf{Q}_p}\mathscr{H}_{h,F}(X)$ is a solution of the equation

$$(1-\varphi\otimes\varphi_{\mathcal{F}})G=g.$$

Moreover, these maps satisfy

$$\mathrm{Tw}^j \circ \Omega^\epsilon_{V,h} \circ \mathrm{d}^j_\epsilon = \Omega^\epsilon_{V\langle j\rangle,h+j},$$

and if $j \leqslant -h$ then

$$\exp^*_{F_n,V\langle j\rangle}(\mathrm{pr}_{F_n}(\mathrm{Tw}_j \circ \Omega^\epsilon_{V,h}(g))) = \frac{1}{(-h-j)!} \cdot \Xi_{n,V\langle j\rangle}(\mathrm{d}_\epsilon^{-j} G)) \in \mathbf{D}(V\langle j\rangle) \otimes_{\mathbf{Q}_p} F_n;$$

and if $D_{[s]} \subset \mathbf{D}(V)$ is a φ-invariant subspace in which all φ-eigenvalues have p-adic valuation at most s, then $\Omega^\epsilon_{V,h}$ maps $(D_{[s]} \otimes_{\mathbf{Z}_p} R^{\psi_{\mathcal{F}}=0})^{\widetilde{\Delta}=0}$ into $\widehat{\mathrm{H}}^1(F_\infty, T) \otimes_{\widetilde{\Lambda}} \mathscr{H}_{s+h,F}(G_\infty)$.

Proof. For $\mathcal{F} = \widehat{\mathbf{G}}_m$, the construction of $\Omega^\epsilon_{V,h}$ and its interpolation property for $j \geqslant 1-h$ is due to Perrin-Riou [PR94]; the interpolation formula for $j \leqslant -h$ is due to Colmez [Col98]. The extension of these results to $\mathbf{Z}_p$-extensions arising from relative Lubin–Tate formal groups of height one is given in [Kob18, Appendix]. □

3.3. **The Coleman map.** From now on, we assume that

$$\mathscr{D}_\infty(V)^{\widetilde{\Delta}=0} = \mathscr{D}_\infty(V), \tag{3.5}$$

i.e., $\widetilde{\Delta} = 0$ (note that by (3.3) this is a condition on the φ-eigenvalues on $\mathbf{D}_{\mathrm{cris},F}(V)$), and for simplicity for any field extension $M/\mathbf{Q}_p$ we write $\mathscr{H}_M$ for $\mathscr{H}_{0,M}(G_\infty)$. Let

$$[-,-]_V : \mathbf{D}(V^*) \otimes_{\mathbf{Q}_p} \mathscr{H}_F \times \mathbf{D}(V) \otimes_{\mathbf{Q}_p} \mathscr{H}_F \to L \otimes_{\mathbf{Q}_p} \mathscr{H}_F$$

be the pairing defined by

$$[\eta_1 \otimes \lambda_1, \eta_2 \otimes \lambda_2]_V = \langle \eta_1, \eta_2 \rangle_{\mathrm{dR}} \otimes \lambda_1 \lambda_2^\iota$$

for all $\lambda_1, \lambda_2 \in \mathscr{H}_F$.

Recall that $F_\infty = \bigcup_n F_n$, and let $\langle -,-\rangle_{F_n}$ be the local Tate pairing $\mathrm{H}^1(F_n, T^*) \times \mathrm{H}^1(F_n, T) \to \mathcal{O}_L$. Letting $x = (x_n)_n$ and $y = (y_n)_n$ be sequences in $\widehat{\mathrm{H}}^1(F_\infty, T^*)$ and $\widehat{\mathrm{H}}^1(F_\infty, T)$, define the $\mathcal{O}_L[\![G_\infty]\!]$-linear pairing

$$\langle -,-\rangle_{F_\infty} : \widehat{\mathrm{H}}^1(F_\infty, T^*) \times \widehat{\mathrm{H}}^1(F_\infty, T) \to \mathcal{O}_L[\![G_\infty]\!]$$

by letting $\langle x, y\rangle_{F_\infty}$ be the limit of the elements

$$\sum_{\sigma \in \mathrm{Gal}(F_n/F)} \langle x_n^{\sigma^{-1}}, y_n \rangle_{F_n} [\sigma] \in \mathcal{O}_L[\mathrm{Gal}(F_n/F)],$$

which are compatible under the natural projection maps $\mathcal{O}_L[\mathrm{Gal}(F_{n+1}/F)] \to \mathcal{O}_L[\mathrm{Gal}(F_n/F)]$. After inverting p, this extends to a pairing

$$\langle -,-\rangle_{F_\infty} : \widehat{\mathrm{H}}^1(F_\infty, V^*) \times \widehat{\mathrm{H}}^1(F_\infty, V) \to L \otimes_{\mathbf{Q}_p} \mathscr{H}_{\mathbf{Q}_p}. \tag{3.6}$$

Definition 3.3. Let $\boldsymbol{e} \in R^{\psi_{\mathcal{F}}=0}$ be a $\mathscr{O}[\![G_\infty]\!]$-module generator, and let ϵ a generator of $T_p\mathcal{F}$. The *Coleman map*

$$\mathrm{Col}^\epsilon_{\boldsymbol{e}} \colon \widehat{\mathrm{H}}^1(F_\infty, V^*) \to \mathbf{D}(V^*) \otimes_{\mathbf{Q}_p} \mathscr{H}_F$$

is the $L \otimes_{\mathbf{Q}_p} \mathscr{H}_F$-linear map uniquely characterized by

$$\mathrm{Tr}_{F/\mathbf{Q}_p}([\mathrm{Col}^\epsilon_{\boldsymbol{e}}(\mathbf{z}), \eta]_V) = \langle \mathbf{z}, \Omega^\epsilon_{V,h}(\eta \otimes \boldsymbol{e})\rangle_{F_\infty} \tag{3.7}$$

for all $\eta \in \mathbf{D}(V)$.

Let $\mathcal{Q}$ be the completion of $\mathbf{Q}_p^{\mathrm{ur}}$ in $\mathbf{C}_p$, with ring of integers $\mathcal{W}$, and set $F_n^{\mathrm{ur}} = F_n \mathbf{Q}_p^{\mathrm{ur}}$ for $-1 \leqslant n \leqslant \infty$ (so $F_{-1}^{\mathrm{ur}} = F^{\mathrm{ur}}$). Let $\sigma_0 \in \mathrm{Gal}(F_\infty^{\mathrm{ur}}/\mathbf{Q}_p)$ be such that $\sigma_0|_{\mathbf{Q}_p^{\mathrm{ur}}} = \mathrm{Fr}$ is the absolute Frobenius.

Fix an isomorphism

$$\rho : \widehat{\mathbf{G}}_m \simeq \mathcal{F} \tag{3.8}$$

defined over $\mathcal{W}$ and let $\rho : \mathcal{W}[\![T]\!] \simeq R \otimes_{\mathscr{O}} \mathcal{W}$ be the map defined by $\rho(f) = f \circ \rho^{-1}$, so

$$\varphi_{\mathcal{F}} \circ \rho = \rho^{\mathrm{Fr}} \circ \varphi_{\widehat{\mathbf{G}}_m}.$$

Fix also a $\mathscr{O}[\![G_\infty]\!]$-generator $\boldsymbol{e} \in R^{\psi_{\mathcal{F}}=0}$, and let $h_{\boldsymbol{e}} \in \mathcal{W}[\![G_\infty]\!]$ be such that $\rho(1+X) = h_{\boldsymbol{e}} \cdot \boldsymbol{e}$. Note that $\boldsymbol{e}(0) \in \mathscr{O}^\times$. Fix a sequence (ζ_{p^n}) of primitive p^n-th root of unity giving a generator of $T_p\widehat{\mathbf{G}}_m$, and let $\epsilon = (\epsilon_n)$ be the generator of $T_p\mathcal{F}$ given by

$$\epsilon_n = \rho^{\mathrm{Fr}^{-(n+1)}}(\zeta_{p^{n+1}} - 1) \in \mathcal{F}^{(n+1)}[p^{n+1}].$$

Let $t \in B_{\mathrm{cris}}^+$ be the p-adic period as in §3.1 associated to the generator $(\zeta_{p^{n+1}} - 1) \in T_p\widehat{\mathbf{G}}_m$ and the invariant differential $\omega_{\widehat{\mathbf{G}}_m} = \frac{dX}{1+X}$.

From now on, we suppose that $\mathrm{Fil}^{-1}\,\mathbf{D}(V) = \mathbf{D}(V)$ and $\mathrm{H}^0(F_\infty, V) = 0$, so the big exponential map $\Omega^\epsilon_{V,1}$ of Theorem 3.2 is defined. Let $\eta \in \mathbf{D}(V)$ be such that $\varphi\eta = \alpha\eta$, and suppose that η has slope s (i.e. $|\alpha|_p = p^{-s}$). For every $\mathbf{z} \in \widehat{\mathrm{H}}^1(F_\infty, V^*)$, we define

$$\mathrm{Col}^\eta(\mathbf{z}) := \sum_{j=1}^{[F:\mathbf{Q}_p]} \left[\mathrm{Col}^\epsilon_{\boldsymbol{e}}(\mathbf{z}^{\sigma_0^{-j}}), \eta\right] \cdot h_{\boldsymbol{e}} \cdot \sigma_0^j \in \mathscr{H}_{s+h,L\mathcal{Q}}(\widetilde{G}_\infty), \tag{3.9}$$

where $\widetilde{G}_\infty = \mathrm{Gal}(F_\infty/\mathbf{Q}_p)$, and $[-,-] : \mathbf{D}(V^*) \otimes \mathscr{H}_{\mathcal{Q}} \times \mathbf{D}(V) \otimes \mathscr{H}_{\mathcal{Q}} \to \mathscr{H}_{L\mathcal{Q}}$ is the image of $[-,-]_V$ under the natural map $L \otimes_{\mathbf{Q}_p} \mathscr{H}_{\mathcal{Q}} \to \mathscr{H}_{L\mathcal{Q}}$. We put

$$\mathbf{z}_{-j,n} := \mathrm{pr}_{F_n}(\mathrm{Tw}^{-j}(\mathbf{z})) \in \mathrm{H}^1(F_n, V^*\langle -j\rangle),$$

and say that a finite order character χ of $\widetilde{G}_\infty$ has conductor p^{n+1} if n is the smallest integer such that χ factors through $\mathrm{Gal}(F_n/\mathbf{Q}_p)$.

Theorem 3.4. *Let $\mathbf{z} \in \widehat{\mathrm{H}}^1(F_\infty, V^*)$ and let ψ be a p-adic character of $\widetilde{G}_\infty$ such that $\psi = \chi\varepsilon_{\mathcal{F}}^j$ with χ a finite order character of conductor p^{n+1}. If $j < 0$, then*

$$\mathrm{Col}^\eta(\mathbf{z})(\psi) = \frac{(-1)^{j-1}}{(-j-1)!}$$
$$\times \begin{cases} \left[\log_{F,V^*\langle -j\rangle}(\mathbf{z}_{-j,n}) \otimes t^{-j}, (1-p^{j-1}\varphi^{-1})(1-p^{-j}\varphi)^{-1}\eta\right] & \text{if } n = -1, \\ p^{(n+1)(j-1)}\boldsymbol{\tau}(\psi) \sum\limits_{\tau\in\mathrm{Gal}(F_n/\mathbf{Q}_p)} \chi^{-1}(\tau)\left[\log_{F_n,V^*\langle -j\rangle}(\mathbf{z}^\tau_{-j,n}) \otimes t^{-j}, \varphi^{-(n+1)}\eta\right] & \text{if } n \geqslant 0. \end{cases}$$

If $j \geqslant 0$, then

$$\mathrm{Col}^\eta(\mathbf{z})(\psi) = j!(-1)^j$$
$$\times \begin{cases} \left[\exp^*_{F,V^*\langle -j\rangle}(\mathbf{z}_{-j,n}) \otimes t^{-j}, (1-p^{j-1}\varphi^{-1})(1-p^{-j}\varphi)^{-1}\eta\right] & \text{if } n = -1, \\ p^{(n+1)(j-1)}\boldsymbol{\tau}(\psi) \sum\limits_{\tau\in\mathrm{Gal}(F_n/\mathbf{Q}_p)} \chi^{-1}(\tau)\left[\exp^*_{F_n,V^*\langle -j\rangle}(\mathbf{z}^\tau_{-j,n}) \otimes t^{-j}, \varphi^{-(n+1)}\eta\right] & \text{if } n \geqslant 0. \end{cases}$$

Here $\boldsymbol{\tau}(\psi)$ is the Gauss sum defined by

$$\boldsymbol{\tau}(\psi) := \sum_{\tau\in\mathrm{Gal}(F_n^{\mathrm{ur}}/F^{\mathrm{ur}})} \psi\varepsilon_{\mathrm{cyc}}^{-j}(\tau\sigma_0^{n+1})\zeta_{p^{n+1}}^{\tau\sigma_0^{n+1}}.$$

Proof. This follows from Theorem 3.2 by a direct computation (see [Kob18, Thm. 5.10], and [LZ14, Thm. 4.15] for a related computation). ∎

3.4. **Diagonal cycles and theta elements.** We now apply the local results of the preceding section to the global setting of §2. Assume that f, $\boldsymbol{g} = \boldsymbol{\theta}_\psi(S)$ and $\boldsymbol{g}^* = \boldsymbol{\theta}_{\psi^{-1}}(S)$ are as in §2.4. Keeping the notations from §2.3, by [DR17, §1] (see also [DR21] and [BSV21]) there exists a class

$$\kappa(f, \boldsymbol{g}\boldsymbol{g}^*) \in \mathrm{H}^1(\mathbf{Q}, \mathbb{V}^\dagger_{f,\boldsymbol{g}\boldsymbol{g}^*}(N)) \tag{3.10}$$

constructed from twisted diagonal cycles on the triple product of modular curves of tame level N.

Every triple of test vectors $\breve{\boldsymbol{F}} = (\breve{f}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{g}}^*)$ defines a $G_{\mathbf{Q}}$-equivariant projection $\mathbb{V}^\dagger_{f,\boldsymbol{g}\boldsymbol{g}^*}(N) \to \mathbb{V}^\dagger_{f,\boldsymbol{g}\boldsymbol{g}^*}$, and we put

$$\kappa(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*) := \mathrm{pr}_{\breve{\boldsymbol{F}}}(\kappa(f, \boldsymbol{g}\boldsymbol{g}^*)) \in \mathrm{H}^1(\mathbf{Q}, \mathbb{V}^\dagger_{f,\boldsymbol{g}\boldsymbol{g}^*}), \tag{3.11}$$

where $\mathrm{pr}_{\breve{\boldsymbol{F}}} : \mathrm{H}^1(\mathbf{Q}, \mathbb{V}^\dagger_{f,\boldsymbol{g}\boldsymbol{g}^*}(N)) \to \mathrm{H}^1(\mathbf{Q}, \mathbb{V}^\dagger_{f,\boldsymbol{g}\boldsymbol{g}^*})$ is the induced map on cohomology.

Since $\Psi_T^{1-\tau}$ gives the universal character of $\mathrm{Gal}(K_\infty/K)$, by the $G_{\mathbf{Q}}$-isomorphism (2.4) and Shapiro's lemma we have the identifications

$$\begin{aligned} \mathrm{H}^1(\mathbf{Q}, \mathbb{V}^\dagger_{f,\boldsymbol{g}\boldsymbol{g}^*}) &\simeq \mathrm{H}^1(\mathbf{Q}, V_f(1) \otimes \mathrm{Ind}_K^{\mathbf{Q}} \Psi_T^{1-\tau}) \oplus \mathrm{H}^1(\mathbf{Q}, V_f(1) \otimes \mathrm{Ind}_K^{\mathbf{Q}} \chi) \\ &\simeq \widehat{\mathrm{H}}^1(K_\infty, V_f(1)) \oplus \mathrm{H}^1(K, V_f(1) \otimes \chi). \end{aligned} \tag{3.12}$$

In the following, we write

$$\kappa(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*) = (\kappa_\infty(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*), \kappa_0(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*)) \tag{3.13}$$

according to this decomposition.

Let g and g^* be the weight 1 eigenform θ_ψ and $\theta_{\psi^{-1}}$, respectively, so that the specialization of $(\boldsymbol{g}, \boldsymbol{g}^*)$ at $T = 0$ (or equivalently, $S = \mathbf{v} - 1$) is a p-stabilization of the pair (g, g^*).

Lemma 3.5. *Assume that $L(f \otimes g \otimes g^*, 1) = 0$ and that $L(f/K \otimes \chi, 1) \neq 0$. Then for every choice of test vectors $\breve{\boldsymbol{F}} = (\breve{f}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{g}}^*)$ we have $\kappa_0(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*) = 0$.*

Proof. Let $\boldsymbol{\kappa} = \kappa(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*)$ and for every $? \in \{f, \boldsymbol{g}, \boldsymbol{g}^*\}$, let $\mathscr{F}^+ V_?$ be the rank one subspace of $V_?$ fixed by the inertia group at p. By (3.12), in order to prove the result it suffices to show that some specialization of κ has trivial image in $\mathrm{H}^1(K, V_f(1) \otimes \chi)$. Let

$$\kappa_{\breve{f},\breve{g}\breve{g}^*} := \boldsymbol{\kappa}|_{S=\mathbf{v}-1} \in \mathrm{H}^1(\mathbf{Q}, V_{fgg^*}) = \mathrm{H}^1(K, V_f(1)) \oplus \mathrm{H}^1(K, V_f(1) \otimes \chi),$$

where $V_{fgg^*} := V_f(1) \otimes V_g \otimes V_{g^*}$. By looking at the Hodge–Tate weights, we see that the Bloch–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, V_{fgg^*}) \subset \mathrm{H}^1(\mathbf{Q}, V_{fgg^*})$ is given by

$$\mathrm{Sel}(\mathbf{Q}, V_{fgg^*}) = \ker\left(\mathrm{H}^1(\mathbf{Q}, V_{fgg^*}) \overset{\partial_p \circ \mathrm{loc}_p}{\to} \mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}^- V_f(1) \otimes V_g \otimes V_{g^*})\right),$$

where ∂_p is the natural map induced by the projection $V_f \twoheadrightarrow \mathscr{F}^- V_f := V_f/\mathscr{F}^+ V_f$ (see e.g. [DR17, p. 634]). Thus it follows that

$$\mathrm{Sel}(\mathbf{Q}, V_{fgg^*}) = \mathrm{Sel}(K, V_f(1)) \oplus \mathrm{Sel}(K, V_f(1) \otimes \chi).$$

The implications $L(f \otimes g \otimes g^*, 1) = 0 \Longrightarrow \kappa_{\breve{f},\breve{g}\breve{g}^*} \in \mathrm{Sel}(\mathbf{Q}, V_{fgg^*})$ and $L(f/K \otimes \chi, 1) \neq 0 \Longrightarrow \mathrm{Sel}(K, V_f(1) \otimes \chi) = 0$, which follow from [DR17, Thm. C] and [CH15, Thm. 1], respectively, therefore yield the result. ☐

Suppose from now on that $f^\circ \in S_2(\Gamma_0(N_f))$ is the newform associated to an elliptic curve $E/\mathbf{Q}$ with good ordinary reduction at p. Thus $V_f(1) \simeq V_pE$ and from (3.13) we obtain an Iwasawa cohomology class

$$\kappa_\infty(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*) \in \widehat{\mathrm{H}}^1(K_\infty, V_pE).$$

Set $V = V_pE$ for the ease of notation. Note that $\mathrm{Fil}^{-1}\,\mathbf{D}(V) = \mathbf{D}(V)$ and, by the Weil pairing, $V^* \simeq V$. Let $\mathfrak{P}$ be the prime of $\overline{\mathbf{Q}}$ above p induced by our fixed embedding ι_p (inducing $\mathfrak{p}$ on

K), and for any subfield $H \subseteq \overline{\mathbf{Q}}$ denote by $\hat{H} = H_{\mathfrak{P}}$ the completion of H with respect to $\mathfrak{P}$. Then $\mathrm{Gal}(\hat{K}_\infty/\mathbf{Q}_p)$ is identified with the decomposition group of $\mathfrak{P}$ in $\Gamma_\infty = \mathrm{Gal}(K_\infty/K)$.

For any integer m let H_m be the ring class field of K of conductor m,, and put $F = \hat{H}_c$ for a fixed c prime to p. Let $\varpi \in K$ be a generator of $\mathfrak{p}^{[F:\mathbf{Q}_p]}$ and let F_∞/F be the Lubin–Tate $\mathbf{Z}_p$-extension associated with the uniformizer $\varpi/\overline{\varpi} \in \mathcal{O}_F$ (see [Kob18, §3.1]). As is well-known, we have

$$F_\infty = \bigcup_{n=0}^{\infty} \hat{H}_{cp^n}$$

(see e.g. [Shn16, Prop. 8.3]). In particular, F_∞ contains $\hat{K}_\infty$.

Let ω_E be the Néron differential of E, regarded as an element in $\mathbf{D}(\mathrm{H}^1_{\mathrm{et}}(E_{/\overline{\mathbf{Q}}}, \mathbf{Q}_p)) \simeq \mathbf{D}(V^*)$. Let $\alpha_p \in \mathbf{Z}_p^\times$ be the p-adic unit eigenvalue of the Frobenius map φ acting on $\mathbf{D}(V)$, and let $\eta \in \mathbf{D}(V) \simeq \mathbf{D}(\mathrm{H}^1_{\mathrm{et}}(E_{/\overline{\mathbf{Q}}}, \mathbf{Q}_p)) \otimes \mathbf{D}(\mathbf{Q}_p(1))$ be a φ-eigenvector of slope -1 such that

$$\varphi\eta = p^{-1}\alpha_p \cdot \eta \quad \text{and} \quad \langle \eta, \omega_E \otimes t^{-1} \rangle_{\mathrm{dR}} = 1. \tag{3.14}$$

Finally, note that hypothesis (3.5) holds since $\mathbf{D}(V)^{\varphi^{[F:\mathbf{Q}_p]}=(\varpi/\overline{\varpi})^j} = 0$ for any $j \in \mathbf{Z}$, given that the φ-eigenvalues of $\mathbf{D}(V)$ are p-Weil numbers while $\varpi/\overline{\varpi}$ is a 1-Weil number.

The second part of the next result recasts the "explicit reciprocity law" of [DR17, Thm. 5.3] (see also [DR21, Thm. 5.1] and [BSV21, Thm. A]) in terms of the Coleman map of §3.3.

Theorem 3.6. *Assume that $L(f \otimes g \otimes g^*, 1) = 0$ and that $L(f/K \otimes \chi, 1) \neq 0$. Then, for any test vectors $(\breve{f}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{g}}^*)$ we have*

$$\mathrm{Loc}_{\overline{\mathfrak{p}}}(\kappa_\infty(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*)) = 0,$$

and

$$\mathrm{Col}^\eta(\mathrm{Loc}_{\mathfrak{p}}(\kappa_\infty(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*))) = \mathscr{L}_p^f(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*) \cdot 2\alpha_p^{-1}(1 - \alpha_p^{-1}\chi(\overline{\mathfrak{p}}))^{-1}.$$

Proof. Let $\mathscr{F}^{++}\mathbb{V}^\dagger_{\boldsymbol{fgg}^*}$ be the rank four $G_{\mathbf{Q}_p}$-stable submodule of $\mathbb{V}^\dagger_{\boldsymbol{fgg}^*}$ defined by

$$\big[\mathscr{F}^+V \otimes \mathscr{F}^+V_{\boldsymbol{g}} \otimes V_{\boldsymbol{g}^*} + \mathscr{F}^+V \otimes V_{\boldsymbol{g}} \otimes \mathscr{F}^+V_{\boldsymbol{g}^*} + V \otimes \mathscr{F}^+V_{\boldsymbol{g}} \otimes \mathscr{F}^+V_{\boldsymbol{g}^*}\big] \otimes \mathcal{X}^{-1}.$$

The class $\kappa(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*) = (\kappa_\infty(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*), \kappa_0(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*)) \in \mathrm{H}^1(\mathbf{Q}, \mathbb{V}^\dagger_{\boldsymbol{fgg}^*})$ is known to land in the kernel of the composite map

$$\mathrm{H}^1(\mathbf{Q}, \mathbb{V}^\dagger_{\boldsymbol{fgg}^*}) \xrightarrow{\mathrm{Loc}_p} \mathrm{H}^1(\mathbf{Q}_p, \mathbb{V}^\dagger_{\boldsymbol{fgg}^*}) \to \mathrm{H}^1(\mathbf{Q}_p, \mathbb{V}^\dagger_{\boldsymbol{fgg}^*}/\mathscr{F}^{++}\mathbb{V}^\dagger_{\boldsymbol{fgg}^*})$$

(see e.g. [DR21, Prop. 5.8]). Using (2.4), we immediately find that

$$\mathscr{F}^{++}\mathbb{V}^\dagger_{\boldsymbol{fgg}^*} = V \otimes \Psi_T^{1-\tau} + \mathscr{F}^+V \otimes (\chi + \chi^{-1}),$$

and therefore, identifying $G_{\mathbf{Q}_p}$ with $G_{K_\mathfrak{p}}$ via our fixed embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, we obtain

$$\mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}^{++}\mathbb{V}^\dagger_{\boldsymbol{fgg}^*}) \simeq \mathrm{H}^1(K_\mathfrak{p}, V \otimes \Psi_T^{1-\tau}) \oplus \mathrm{H}^1(K_\mathfrak{p}, \mathscr{F}^+V \otimes \chi) \oplus \mathrm{H}^1(K_{\overline{\mathfrak{p}}}, \mathscr{F}^+V \otimes \chi).$$

This shows the vanishing of $\mathrm{Loc}_{\overline{\mathfrak{p}}}(\kappa_\infty(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*))$, and the second equality in the theorem follows from Lemma 3.5 and [DR17, Thm. 5.3]. □

Corollary 3.7. *Assume that $L(f \otimes g \otimes g^*, 1) = 0$ and that $L(f/K, \chi, 1) \neq 0$. Let $(\underline{\breve{f}}, \underline{\breve{\boldsymbol{g}}}, \underline{\breve{\boldsymbol{g}}}^*)$ be the triple of test vectors from Theorem 2.1. Then $\mathrm{Loc}_{\overline{\mathfrak{p}}}(\kappa_\infty(\underline{\breve{f}}, \underline{\breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*})) = 0$, and*

$$\mathrm{Col}^\eta(\mathrm{Loc}_\mathfrak{p}(\kappa_\infty(\underline{\breve{f}}, \underline{\breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*}))) = \pm\mathbf{w}^{-1} \cdot \Theta_{f/K}(T) \cdot \sqrt{L^{\mathrm{alg}}(f/K \otimes \chi, 1)} \cdot \frac{2C_{f,\chi}}{\alpha_p(1 - \alpha_p^{-1}\chi(\overline{\mathfrak{p}}))} \cdot \frac{\eta_f}{\eta_{f,N^-}},$$

where $\mathbf{w} \in \mathcal{O}[\![T]\!]^\times$ and $C_{f,\chi} \in K(\chi, \alpha_p)^\times$ are as in Proposition 2.5.

Proof. This is the combination of Theorem 3.6 and the factorization in Proposition 2.5. □

Remark 3.8. Corollary 3.7 places for the first time $\Theta_{f/K}(T)$ within the landscape of Perrin-Riou's vision [PR00], whereby p-adic L-functions ought to arise as the image of p-adic families of special cohomology classes under generalized Coleman power series maps. For a different class of anticyclotomic p-adic L-functions introduced by Bertolini–Darmon–Prasanna [BDP13], a similar result was obtained by the authors in [CH18a, Cas20].

4. Anticyclotomic derived p-adic heights

The main result of this section is Theorem 4.5, giving a formula for the anticyclotomic derived p-adic heights in terms of the Coleman map introduced before. This generalizes a formula of [Rub94] to arbitrary rank.

4.1. **The general theory.** Initiated in [BD95] and further developed in [How04], the theory of derived p-adic heights relates the degeneracies of the p-adic height to the failure of the p^∞-Selmer group of elliptic curves over a $\mathbf{Z}_p$-extension to be semi-simple as an Iwasawa module. Derived p-adic heights seem to have been rarely used for arithmetic applications in the previous literature[1], but they will play a key role in the proof of our results. In this section we briefly recall the results from [How04] (with a slight generalization) that we will need.

Let E be an elliptic curve over $\mathbf{Q}$ of conductor N with good ordinary reduction at $p > 2$. For any number field F, let $\mathrm{Sel}_{p^r}(E/F) \subseteq \mathrm{H}^1(F, E[p^r])$ be the p^r-Selmer group of E over F, and put

$$\mathrm{Sel}(F, T_pE) = \varprojlim_r \mathrm{Sel}_{p^r}(E/F)$$

and $\mathrm{Sel}(F, V_pE) = \mathrm{Sel}(F, T_pE) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. Let K be an imaginary quadratic field of discriminant prime to Np, and let K_∞/K be the anticyclotomic $\mathbf{Z}_p$-extension of K. Denote by K_n the subsection of K_∞ with $[K_n\colon K] = p^n$, and put

$$\mathrm{Sel}_{p^\infty}(E/K_\infty) = \varinjlim_n \mathrm{Sel}_{p^\infty}(E/K_n).$$

Finally, let $\Lambda = \mathbf{Z}_p[\![\mathrm{Gal}(K_\infty/K)]\!]$ be the anticyclotomic Iwasawa algebra, and denote by $J \subseteq \Lambda$ the augmentation ideal.

Theorem 4.1. *Let N^- be the largest factor of N divisible only by primes that are inert in K, and suppose that*

- *N^- is squarefree,*
- *$E[p]$ is ramified at every prime $q | N^-$.*

Then there is a filtration

$$\mathrm{Sel}(K, V_pE) = S_p^{(1)}(E/K) \supseteq S_p^{(2)}(E/K) \supseteq \cdots \supseteq S_p^{(i)}(E/K) \supseteq \cdots$$

and a sequence of height pairings

$$h_p^{(i)} : S_p^{(i)}(E/K) \times S_p^{(i)}(E/K) \to (J^i/J^{i+1}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

with the following properties:

(a) *$S_p^{(i+1)}(E/K)$ is the null-space of $h_p^{(i)}$.*
(b) *$S_p^{(\infty)}(E/K) := \bigcap_{i\geqslant 1} S_p^{(i)}(E/K)$ is the subspace of* $\mathrm{Sel}(K, V_pE)$ *consisting of universal norms for K_∞/K:*

$$S_p^{(\infty)}(E/K) = \bigcap_{n=1}^{\infty} \mathrm{cor}_{K_n/K}(\mathrm{Sel}(K_n, V_pE)),$$

[1] Perhaps by influence of *cyclotomic* Iwasawa theory, a context in which the p-adic height is conjectured to be non-degenerate, see [Sch85]. In contrast, in the anticyclotomic setting, as noted in [BD94, p. 76], degeneracies of the p-adic height pairing “seem to be the rule rather than the exception”.

where $\mathrm{cor}_{K_n/K} : \mathrm{Sel}(K_n, V_pE) \to \mathrm{Sel}(K, V_pE)$ *is the corestriction map.*

(c) $h_p^{(i)}$ *is symmetric (resp. alternating) for* i *odd (resp.* i *even).*

(d) $h_p^{(i)}(x^\tau, y^\tau) = (-1)^i h_p^{(i)}(x, y)$, *where* $\tau \in \mathrm{Gal}(K/\mathbf{Q})$ *is complex conjugation.*

(e) *Let*

$$e_i := \begin{cases} \dim_{\mathbf{Q}_p}(S_p^{(i)}(E/K)/S_p^{(i+1)}(E/K)) & \text{if } i < \infty, \\ \dim_{\mathbf{Q}_p} S_p^{(\infty)}(E/K) & \text{if } i = \infty. \end{cases}$$

Then there is a Λ*-module pseudo-isomorphism*

$$\mathrm{Sel}_{p^\infty}(E/K_\infty)^\vee \sim \big((\Lambda/J)^{\oplus e_1} \oplus \cdots \oplus (\Lambda/J^i)^{\oplus e_i} \oplus \cdots\big) \oplus \Lambda^{\oplus e_\infty} \oplus M'$$

with M' *a torsion* Λ*-module with characteristic ideal prime-to-*J.

Proof. This follows from Theorem 4.2 and Corollary 4.3 of [How04] when $N^- = 1$. We explain how to extend the result to squarefree N^- under the above hypothesis on $E[p]$.

Following the discussion in [*op. cit.*, §3] and adopting the notations there, we see that it suffices to show the vanishing of

$$\mathrm{H}^1_{\mathrm{ur}}(K_v, \mathbf{S}[p^k]) := \ker\big(\mathrm{H}^1(K_v, \mathbf{S}[p^k]) \to \mathrm{H}^1(K_v^{\mathrm{ur}}, \mathbf{S}[p^k])\big). \tag{4.1}$$

for every prime $v \nmid p$ inert in K, where $\mathbf{S}[p^k] = \varinjlim_n \mathrm{Ind}_{K_n/K} E[p^k]$. Since such primes v split completely in K_∞/K, by Shapiro's lemma and inflation-restriction we find

$$\begin{aligned} \mathrm{H}^1_{\mathrm{ur}}(K_v, \mathbf{S}[p^k]) &\simeq \ker\big(\mathrm{H}^1(K_v, E[p^k]) \otimes \Lambda^\vee \to \mathrm{H}^1(K_v^{\mathrm{ur}}, E[p^k]) \otimes \Lambda^\vee\big) \\ &\simeq \mathrm{H}^1(\mathbf{F}_v, E[p^k]^{I_v}) \otimes \Lambda^\vee \\ &= (E[p^k]^{I_v}/(\mathrm{Fr}_v - 1)E[p^k]^{I_v}) \otimes \Lambda^\vee, \end{aligned} \tag{4.2}$$

where $\mathbf{F}_v$ is the residue field of K_v, Fr_v is a Frobenius element at v, and $\Lambda^\vee = \mathrm{Hom}_{\mathbf{Z}_p}(\Lambda, \mathbf{Q}_p/\mathbf{Z}_p)$.

Since N^- is squarefree, any prime v as above is a prime of multiplicative reduction for E, so by Tate's uniformization we have

$$E[p^\infty] \sim \begin{pmatrix} \varepsilon_{\mathrm{cyc}} & * \\ 0 & 1 \end{pmatrix}$$

as G_{K_v}-modules, where $\varepsilon_{\mathrm{cyc}}$ is the p-adic cyclotomic character. Since $\bar\rho_{E,p}$ is ramified at v, the image of '$*$' in the above matrix generates $\mathbf{Q}_p/\mathbf{Z}_p$. Thus we see that

$$E[p^\infty]^{I_v}/(\mathrm{Fr}_v - 1)E[p^\infty]^{I_v} = 0,$$

which by (4.2) implies the vanishing of $\mathrm{H}^1_{\mathrm{ur}}(K_v, \mathbf{S}[p^k])$. □

We next recall Howard's abstract generalization of Rubin's height formula for derived p-adic heights. For every prime v of K above p, let $\mathscr{F}_v^+ T_pE$ be the kernel of the reduction map $T_pE \to T_p\tilde{E}$, where $\tilde{E}$ is the reduction of E modulo v. Letting $V = V_pE$, this induces the filtration $\mathscr{F}_v^+ V \subseteq V$. For every prime $v|p$ of K write

$$\widehat{\mathrm{H}}^1_{\mathrm{fin}}(K_{\infty,v}, V) = \bigoplus_{w|v} \widehat{\mathrm{H}}^1(K_{\infty,w}, \mathscr{F}_v^+ V),$$

where w runs over the places of K_∞ above v. The local pairings in (3.6) induce a semi-local pairing

$$\langle -, - \rangle_{K_{\infty,v}} : \widehat{\mathrm{H}}^1(K_{\infty,v}, V) \times \widehat{\mathrm{H}}^1_{\mathrm{fin}}(K_{\infty,v}, V) \to \Lambda \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

which induces a perfect duality between the $\widehat{\mathrm{H}}^1(K_{\infty,v}, V)/\widehat{\mathrm{H}}^1_{\mathrm{fin}}(K_{\infty,v}, V)$ and $\widehat{\mathrm{H}}^1_{\mathrm{fin}}(K_{\infty,v}, V)$. Every class $\mathbf{z} \in \widehat{\mathrm{H}}^1(K_\infty, V)$ defines a linear map

$$\mathcal{L}_{p,\mathbf{z}} = \sum_{v|p} \langle \mathrm{Loc}_v(\mathbf{z}), - \rangle_{K_{\infty,v}} : \widehat{\mathrm{H}}^1_{\mathrm{fin}}(K_{\infty,p}, V) = \bigoplus_{v|p} \widehat{\mathrm{H}}^1_{\mathrm{fin}}(K_{\infty,v}, V) \to \Lambda \otimes_{\mathbf{Z}_p} \mathbf{Q}_p,$$

Let $\mathrm{ord}(\mathcal{L}_{p,\mathbf{z}})$ be the largest integer r such that the image of $\mathcal{L}_{p,\mathbf{z}}$ is contained in J^r.

Theorem 4.2. *Let r be any positive integer with $r \leqslant \mathrm{ord}(\mathcal{L}_{p,\mathbf{z}})$. Then $z = \mathrm{pr}_K(\mathbf{z})$ belongs to $S_p^{(r)}(E/K)$ and for any $w \in S_p^{(r)}(E/K)$, we have*

$$h_p^{(r)}(z,w) = -\mathcal{L}_{p,\mathbf{z}}(\mathbf{w}_p) \pmod{J^{r+1}}$$

where $\mathbf{w}_p = (\mathbf{w}_v)_{v|p} \in \widehat{\mathrm{H}}^1_{\mathrm{fin}}(K_{\infty,p},V)$ is any semi-local class with $\mathrm{pr}_{K_v}(\mathbf{w}_v) = \mathrm{Loc}_v(w)$ for all $v|p$.

Proof. This is a reformulation of part (c) of Theorem 2.5 in [How04]. Note that the existence of $\mathbf{w}_p$ follows from the definition of $S_p^{(r)}(E/K)$ in *op. cit.*, and the fact that the image $\mathcal{L}_{p,\mathbf{z}}(\mathbf{w}_p) \in J^r/J^{r+1}$ is independent of the choice of $\mathbf{w}_p$ is shown in the proof. □

4.2. **Derived p-adic heights and the Coleman map.** Now we compute the local expression in Theorem 4.2 for the derived p-adic height pairing in terms of the Coleman map from §3, yielding our higher rank generalization of Rubin's formula.

We use the setting and notations introduced after Lemma 3.5. In particular, $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K, with $\mathfrak{p}$ the prime of K above p induced by our fixed embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Let $\hat{K}_\infty$ be the closure of the image of K_∞ in $\overline{\mathbf{Q}}_p$ under this embedding, and put

$$\Gamma_\infty = \mathrm{Gal}(K_\infty/K), \quad \hat{\Gamma}_\infty = \mathrm{Gal}(\hat{K}_\infty/\mathbf{Q}_p),$$

so naturally $\hat{\Gamma}_\infty$ is a subgroup of Γ_∞. Also, we put $F = \hat{H}_c$ for some fixed c prime to p, and $F_\infty = \hat{H}_{cp^\infty}$, which is a finite extension of $\hat{K}_\infty$.

Let $\boldsymbol{e} \in R^{\psi_{\mathcal{F}}=0}$ be a generator over $\mathscr{O}[\![G_\infty]\!]$ such that $\boldsymbol{e}(0) = 1$. Define

$$\mathbf{w}^\eta = \Omega^\epsilon_{V,1}(\eta \otimes \boldsymbol{e}) \in \widehat{\mathrm{H}}^1(F_\infty, V), \tag{4.3}$$

where $\Omega^\epsilon_{V,1}$ in is the big exponential map in Theorem 3.2.

As in §3.3, we let $\sigma_0 \in \mathrm{Gal}(F_\infty^{\mathrm{ur}}/\mathbf{Q}_p)$ be such that $\sigma_0|_{\mathbf{Q}_p^{\mathrm{ur}}} = \mathrm{Fr}$ is the absolute Frobenius.

Proposition 4.3. *Let $\mathbf{Q}_p^{\mathrm{cyc}}$ be the cyclotomic $\mathbf{Z}_p^\times$-extension of $\mathbf{Q}_p$. Let $\sigma_{\mathrm{cyc}} \in \mathrm{Gal}(F_\infty^{\mathrm{ur}}/\mathbf{Q}_p)$ be the Frobenius such that $\sigma_{\mathrm{cyc}}|_{\mathbf{Q}_p^{\mathrm{cyc}}} = 1$ and $\sigma_{\mathrm{cyc}}|_{\mathbf{Q}_p^{\mathrm{ur}}} = \mathrm{Fr}$. For each $\hat{\mathbf{z}} \in \widehat{\mathrm{H}}^1(\hat{K}_\infty, V)$, we have*

$$\langle \hat{\mathbf{z}}, \mathrm{cor}_{F_\infty/\hat{K}_\infty}(\mathbf{w}^\eta)\rangle_{\hat{K}_\infty} = \mathrm{pr}_{\hat{K}_\infty}(\mathrm{Col}^\eta(\hat{\mathbf{z}})) \sum_{i=1}^{[F:\mathbf{Q}_p]} \frac{\sigma^i_{\mathrm{cyc}}|_{\hat{K}_\infty}}{[F_\infty : \hat{K}_\infty]\cdot h_{\boldsymbol{e}}^{\mathrm{Fr}^i}} \in \mathcal{W}[\![\hat{\Gamma}_\infty]\!] \otimes \mathbf{Q}_p.$$

Proof. We first recall that for every $e \in (R \otimes_{\mathscr{O}} \mathcal{W})^{\psi_{\mathcal{F}}=0}$, the big exponential map $\Omega^\epsilon_{V,1}(\eta\otimes e)$ in Theorem 3.2 is given by

$$\Omega^\epsilon_{V,1}(\eta \otimes e) = (\exp_{F_n,V}(\Xi_{n,V}(G_e)))_{n=0,1,2,\ldots}, \tag{4.4}$$

where $G_e \in \mathbf{D}(V) \otimes \mathscr{H}_{1,\mathcal{Q}}(X)$ is a solution of $(1 - \varphi \otimes \varphi_{\mathcal{F}})G_e = \eta \otimes e$ and $\Xi_{n,V}$ is as in (3.4). Taking

$$G_e = G_{\boldsymbol{e}} = \sum_{m=0}^\infty (\varphi\otimes\varphi_{\mathcal{F}})^m(\eta\otimes\boldsymbol{e}) = \sum_{m=0}^\infty \varphi^m\eta\otimes\boldsymbol{e}^{\mathrm{Fr}^m},$$

we obtain

$$\begin{aligned}\Xi_{n,V}(G_{\boldsymbol{e}}) &= p^{-(n+1)}(\varphi^{-(n+1)}\otimes 1)G_{\boldsymbol{e}}^{\mathrm{Fr}^{-(n+1)}}(\epsilon_n)\\ &= \sum_{m=0}^\infty (p\varphi)^{-(n+1)}\varphi^m\eta\otimes\boldsymbol{e}^{\mathrm{Fr}^{m-(n+1)}}(\epsilon_{n-m}).\end{aligned} \tag{4.5}$$

Put $z_n = \mathrm{pr}_{\hat{K}_n}(\hat{\mathbf{z}})$ and $\hat{G}_n = \mathrm{Gal}(\hat{K}_n/\mathbf{Q}_p)$. From the definition of the Coleman map Col^ϵ_e, and using in (4.4) and (4.5), we thus find that

$$\begin{aligned}&\Big[\mathrm{pr}_{\hat{K}_n}(\mathrm{Col}^\epsilon_{\boldsymbol{e}}(\hat{\mathbf{z}})),\eta\Big]_V = \\ &\sum_{m=0}^{\infty}\left[\sum_{\gamma\in\hat{G}_n}\exp^*_{\hat{K}_n,V}(z_n^{\gamma^{-1}\sigma_0^{n+1-m}})\gamma,\ \sum_{\tau\in\hat{G}_n}(p\varphi)^{-(n+1)}\varphi^m\eta\otimes\boldsymbol{e}^{\mathrm{Fr}^{m-(n+1)}}(\epsilon_{n-m})^{\tau\sigma_0^{n+1-m}}\tau|_{\hat{K}_n}\right]_V,\end{aligned} \tag{4.6}$$

where $\exp^*_{\hat{K}_n,V}$ is the Bloch–Kato dual exponential map.

On the other hand, it is immediately seen that

$$\mathrm{pr}_{\hat{K}_n}(\langle\hat{\mathbf{z}},\mathrm{cor}_{F_\infty/\hat{K}_\infty}(\mathbf{w}^\eta)\rangle_{\hat{K}_\infty}) = \frac{1}{[F_\infty:\hat{K}_\infty]}\sum_{j=1}^{[F:\mathbf{Q}_p]}\mathrm{pr}_{\hat{K}_n}(\langle\hat{\mathbf{z}}^{\sigma_0^{-j}},\mathbf{w}^\eta\rangle_{F_\infty})\sigma_0^j|_{\hat{K}_n},$$

and from (4.6) we find that

$$\begin{aligned}\mathrm{pr}_{\hat{K}_n}(\langle\hat{\mathbf{z}}^{\sigma_0^{-j}},\mathbf{w}^\eta\rangle_{F_\infty}) &= \sum_{\gamma\in\hat{G}_n}\langle z_n^{\sigma_0^{-j}\gamma^{-1}},\exp_{F_n,V}(\Xi_{n,V}(G_{\boldsymbol{e}}))\rangle_{F_n}\gamma|_{\hat{K}_n}\\ &= \mathrm{Tr}_{F_n/\mathbf{Q}_p}\left(\left[\sum_{\gamma\in\hat{G}_n}\exp^*_{\hat{K}_n,V}(z_n^{\sigma_0^{-j}\gamma^{-1}})\gamma|_{\hat{K}_\infty},\Xi_{n,V}(G_{\boldsymbol{e}})\right]_V\right)\\ &= \sum_{m=0}^{\infty}\sum_{i=1}^{[F:\mathbf{Q}_p]}\left[\sum_{\gamma\in\hat{G}_n}\exp^*_{\hat{K}_n,V}(z_n^{\gamma^{-1}\sigma_0^{i-j+n+1-m}})\gamma,\ \sum_{\tau\in\hat{G}_n}(p\varphi)^{-(n+1)}\varphi^m\eta\otimes\boldsymbol{e}^{\mathrm{Fr}^{m-(n+1)}}(\epsilon_{n-m})^{\tau\sigma_0^{i+n+1-m}}\tau|_{\hat{K}_n}\right]\\ &= \sum_{i=1}^{[F:\mathbf{Q}_p]}\Big[\mathrm{pr}_{\hat{K}_n}(\mathrm{Col}^\epsilon_{\boldsymbol{e}}(\mathbf{z}^{\sigma_0^{-j}})^{\sigma_0^i}),\eta\Big].\end{aligned}$$

Taking the limit over n, we thus arrive at

$$\begin{aligned}\langle\hat{\mathbf{z}},\mathrm{cor}_{F_\infty/\hat{K}_\infty}(\mathbf{w}^\eta)\rangle_{\hat{K}_\infty} &= \frac{1}{[F_\infty:\hat{K}_\infty]}\sum_{j=1}^{[F:\mathbf{Q}_p]}\sum_{i=1}^{[F:\mathbf{Q}_p]}\Big[\mathrm{pr}_{\hat{K}_\infty}(\mathrm{Col}^\epsilon_{\boldsymbol{e}}(\hat{\mathbf{z}}^{\sigma_0^{-j}})^{\sigma_0^i}),\eta\Big]\sigma_0^j\\ &= \frac{1}{[F_\infty:\hat{K}_\infty]}\sum_{i=1}^{[F:\mathbf{Q}_p]}\mathrm{pr}_{\hat{K}_\infty}(\mathrm{Col}^\eta(\hat{\mathbf{z}})^{\sigma_0^i})\cdot\frac{1}{h_{\boldsymbol{e}}^{\sigma_0^i}},\end{aligned} \tag{4.7}$$

using (3.9) for the second equality. Finally, writing $g_\rho = \rho(1+X)$ for the isomorphism ρ in (3.8), one has $g_\rho^{\sigma_0^{-i}}(\epsilon_{i-1}) = \zeta_{p^i}\in\mathbf{Q}_p^{\mathrm{cyc}}$, which immediately implies the relation

$$\mathrm{pr}_{\hat{K}_\infty}(\mathrm{Col}^\eta(\hat{\mathbf{z}}))\cdot\sigma_{\mathrm{cyc}}^i = \mathrm{pr}_{\hat{K}_\infty}(\mathrm{Col}^\eta(\hat{\mathbf{z}})^{\sigma_0^i}).$$

Together with (4.7), this concludes the proof. □

We shall also need the following result.

Lemma 4.4. *The projection of* $\mathbf{w}^\eta$ *to* $\mathrm{H}^1(F,V)$ *is given by*

$$\mathrm{pr}_F(\mathbf{w}^\eta) = \exp_{F,V}\left(\frac{1-p^{-1}\varphi^{-1}}{1-\varphi}\eta\right).$$

Proof. Let $g = \eta \otimes \boldsymbol{e}$ and let $G(X) \in \mathbf{D}(V) \otimes \mathscr{H}_{1,\mathcal{Q}}(X)$ such that $(1 - \varphi \otimes \varphi_{\mathcal{F}})G = g$. Then

$$G(\epsilon_0) = \eta \otimes \boldsymbol{e}(\epsilon_0) - \eta + (1-\varphi)^{-1}\eta,$$

and by definition,

$$\mathrm{pr}_F(\mathbf{w}^\eta) = \mathrm{cor}_{F_0/F}(\Xi_{0,V}(G)), \tag{4.8}$$

where $\Xi_{0,V}(G)$ is as in (3.4). Equation (3.1) and the fact that $\psi_{\mathcal{F}}\boldsymbol{e}(X) = 0$ imply that

$$\sum_{\zeta \in \mathcal{F}^{\mathrm{Fr}^{-1}}[p]} \boldsymbol{e}^{\mathrm{Fr}^{-1}}(X \oplus_{\mathcal{F}} \zeta) = 0,$$

from where we obtain

$$\mathrm{Tr}_{F_0/F}(G^{\mathrm{Fr}^{-1}}(\epsilon_0)) = \sum_{\tau \in \mathrm{Gal}(F_0/F)} \eta \otimes \boldsymbol{e}(\epsilon_0^\tau) - \eta + (1-\varphi)^{-1}\eta = \frac{p\varphi - 1}{1-\varphi}\eta.$$

Together with (4.8), we thus see that

$$\mathrm{pr}_F(\mathbf{w}^\eta) = \exp_{F,V} \mathrm{Tr}_{F_0/F}\left(p^{-1}\varphi^{-1}(G^{\mathrm{Fr}^{-1}}(\epsilon_0))\right) = \exp_{F,V}\left((1-p^{-1}\varphi^{-1})(1-\varphi)^{-1}\eta\right),$$

concluding the proof. □

Recall the identification $K_{\mathfrak{p}} = \mathbf{Q}_p$, and let $\mathrm{H}^1_{\mathrm{fin}}(\mathbf{Q}_p, V) \subset \mathrm{H}^1(\mathbf{Q}_p, V)$ be the subspace given by $\mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}_{\mathfrak{p}}^+ V)$. As is well-known, $\mathrm{H}^1_{\mathrm{fin}}(\mathbf{Q}_p, V)$ agrees with the Bloch–Kato finite subspace. Let $\log_{\mathbf{Q},V} : \mathrm{H}^1_{\mathrm{fin}}(\mathbf{Q}_p, V) \to \mathbf{D}(V)$ be the Bloch–Kato logarithm map, and denote by $\log_{\omega_E,\mathfrak{p}}$ the composition

$$\log_{\omega,\mathfrak{p}} : \mathrm{H}^1(\mathbf{Q}_p, V) \xrightarrow{\log_{\mathbf{Q},V}} \mathbf{D}(V) \xrightarrow{\langle -,\omega_E \otimes t^{-1}\rangle_{\mathrm{dR}}} \mathbf{Q}_p. \tag{4.9}$$

For a global class $\mathbf{z} \in \widehat{\mathrm{H}}^1(K_\infty, V)$, put

$$\mathrm{Col}^\eta(\mathrm{Loc}_{\mathfrak{p}}(\mathbf{z})) := \sum_{\sigma \in \Gamma_\infty/\hat{\Gamma}_\infty} \mathrm{Col}^\eta(\mathrm{Loc}_{\mathfrak{P}}(\mathbf{z}^{\sigma^{-1}}))\sigma \in \mathcal{W}[\![\Gamma_\infty]\!], \tag{4.10}$$

where $\mathrm{Loc}_{\mathfrak{P}} : \widehat{\mathrm{H}}^1(K_\infty, V) \to \widehat{\mathrm{H}}^1(\hat{K}_\infty, V)$ is the restriction map, and let J be the augmentation ideal of $\mathcal{W}[\![\Gamma_\infty]\!]$.

Theorem 4.5. *Let $\mathbf{z} \in \widehat{\mathrm{H}}^1(K_\infty, V)$, and denote by $\mathfrak{r}$ be the largest integer r such that*

$$\mathrm{Col}^\eta(\mathrm{Loc}_{\mathfrak{p}}(\mathbf{z})) \in J^r \quad \textit{and} \quad \mathrm{Col}^\eta(\mathrm{Loc}_{\mathfrak{p}}(\overline{\mathbf{z}})) \in J^r,$$

where $\overline{\mathbf{z}} = \mathbf{z}^\tau$ for the complex conjugation $\tau \in \mathrm{Gal}(K/\mathbf{Q})$. Then for every $0 < r \leqslant \mathfrak{r}$, the class $z = \mathrm{pr}_K(\mathbf{z})$ belongs to $S_p^{(r)}(E/K)$ and for every $x \in S_p^{(r)}(E/K)$, we have

$$h_p^{(r)}(z,x) = -\frac{1-p^{-1}\alpha_p}{1-\alpha_p^{-1}} \cdot \left(\mathrm{Col}^\eta(\mathrm{Loc}_{\mathfrak{p}}(\mathbf{z})) \cdot \log_{\omega,\mathfrak{p}}(x) + \mathrm{Col}^\eta(\mathrm{Loc}_{\mathfrak{p}}(\overline{\mathbf{z}})) \cdot \log_{\omega,\mathfrak{p}}(\overline{x})\right) \pmod{J^{r+1}},$$

where $\overline{x} = x^\tau$.

Proof. The inclusion $z \in S_p^{(r)}(E/K)$ follows immediately from Theorem 4.2. Let $x \in S_p^{(r)}(E/K)$, and put

$$\mathbf{w}_{\mathfrak{P}} := \mathrm{cor}_{F_\infty/\hat{K}_\infty}(\mathbf{w}^\eta) \in \widehat{\mathrm{H}}^1_{\mathrm{fin}}(\hat{K}_\infty, V).$$

Then, since $\dim_{\mathbf{Q}_p} \mathrm{H}^1_{\mathrm{fin}}(\mathbf{Q}_p, V) = 1$, we can write

$$\mathrm{Loc}_{\mathfrak{p}}(x) = c \cdot \mathrm{pr}_{\mathbf{Q}_p}(\mathbf{w}_{\mathfrak{P}})$$

for some $c \in \mathbf{Q}_p$. Since $\mathrm{pr}_{\mathbf{Q}_p}(\mathbf{w}_{\mathfrak{P}}) = \mathrm{cor}_{F/\mathbf{Q}_p}(\mathbf{w}^\eta)$, from Lemma 4.4 and (3.14) we see that

$$\langle \log_{\mathbf{Q}_p,V}(\mathrm{pr}_{\mathbf{Q}_p}(\mathbf{w}_{\mathfrak{P}})), \omega_E \otimes t^{-1}\rangle_{\mathrm{dR}} = [F : \mathbf{Q}_p] \cdot \frac{1-\alpha_p^{-1}}{1-p^{-1}\alpha_p},$$

from where we deduce that

$$c = \frac{1-p^{-1}\alpha_p}{1-\alpha_p^{-1}} \cdot [F:\mathbf{Q}_p]^{-1} \cdot \log_{\omega_E,\mathfrak{p}}(x).$$

Together with the formula in Theorem 4.2, this gives the equality

$$h_p^{(r)}(z,x) = -\frac{1-p^{-1}\alpha_p}{1-\alpha_p^{-1}} \cdot [F:\mathbf{Q}_p]^{-1}$$
$$\times \left(\sum_{\sigma\in\Gamma_\infty/\hat{\Gamma}_\infty} \log_{\omega_E,\mathfrak{p}}(x)\cdot \langle \mathrm{Loc}_{\mathfrak{P}}(\mathbf{z}^{\sigma^{-1}}), \mathbf{w}_{\mathfrak{P}}\rangle_{\hat{K}_\infty}\sigma + \log_{\omega_E,\mathfrak{p}}(\overline{x})\cdot \langle \mathrm{Loc}_{\mathfrak{P}}(\overline{\mathbf{z}}^{\sigma^{-1}}), \mathbf{w}_{\mathfrak{P}}\rangle_{\hat{K}_\infty}\sigma\right)$$

in J^r/J^{r+1}. Since $h_{\boldsymbol{e}} \equiv 1 \pmod J$, as is immediate from the defining relation $\rho(1+X) = h_{\boldsymbol{e}}\cdot\boldsymbol{e}$ and the fact that $\boldsymbol{e}(0) = 1$, the result now follows from Proposition 4.3. □

5. Proof of the main results

We begin by recalling the setting of Theorem A in the Introduction. Let $E/\mathbf{Q}$ be an elliptic curve of conductor N with good ordinary reduction at the prime $p > 3$, and assume that E has root number $+1$ and $L(E,1) = 0$ (so, of course, $\mathrm{ord}_{s=1}L(E,s) \geqslant 2$). Let K be an imaginary quadratic field of discriminant prime to N in which $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ splits, with $\mathfrak{p}$ the prime of K above p induced by our fixed embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Let ψ be a ray class character of K of conductor prime to Np, and as in Conjecture 1.2 assume that

(a) $L(E^K,1)\cdot L(E/K,\chi,1) \neq 0$,
(b) $\chi(\overline{\mathfrak{p}}) \neq 1$,

where $\chi = \psi/\psi^\tau$. In addition, we assume that

(c) $E[p]$ is irreducible as a $G_{\mathbf{Q}}$-module,
(d) N^- is the squarefree product of an odd number of primes,
(e) $E[p]$ is ramified at every prime $q|N^-$,

where N^- is the maximal factor of N divisible only by primes inert in K. Let (f,g,g^*) be the triple consisting of the newform $f \in S_2(\Gamma_0(N))$ associated to E and the weight one theta series associated to ψ and ψ^{-1}, respectively. Finally, put $\alpha = \psi(\overline{\mathfrak{p}})$ and $\beta = \psi(\mathfrak{p})$.

5.1. **Generalized Kato classes.** By construction, the Hida families

$$\boldsymbol{g} = \boldsymbol{g}_\alpha = \boldsymbol{\theta}_\psi(S), \quad \boldsymbol{g}^* = \boldsymbol{g}^*_{\alpha^{-1}} = \boldsymbol{\theta}_{\psi^{-1}}(S) \in \mathcal{O}[\![S]\!][\![q]\!]$$

considered in §2.4 specialize at $S = \mathbf{v}-1$ to g_α and $g^*_{\alpha^{-1}}$, the p-stabilizations of g and g^* with U_p-eigenvalue α and α^{-1}, respectively. Thus for every choice of test vectors $(\breve{f}, \breve{\boldsymbol{g}}_\alpha, \breve{\boldsymbol{g}}^*_{\alpha^{-1}})$ the $\mathcal{O}[\![S]\!]$-adic class $\kappa(\breve{f}, \breve{\boldsymbol{g}}_\alpha \breve{\boldsymbol{g}}^*_{\alpha^{-1}})$ in (3.11) specializes to the *generalized Kato class*

$$\kappa_{\alpha,\alpha^{-1}}(f,g,g^*) := \kappa(\breve{f}, \breve{\boldsymbol{g}}_\alpha \breve{\boldsymbol{g}}^*_{\alpha^{-1}})|_{S=\mathbf{v}-1} \in \mathrm{H}^1(\mathbf{Q}, V_{fgg^*}),$$

where $V_{fgg^*} := V_f \otimes V_g \otimes V_{g^*}$.

Varying over the possible combinations of roots of the Hecke polynomial at p for g and g^*, we thus obtain the four generalized Kato classes

$$\kappa_{\alpha,\alpha^{-1}}(f,g,g^*),\ \kappa_{\alpha,\beta^{-1}}(f,g,g^*),\ \kappa_{\beta,\alpha^{-1}}(f,g,g^*),\ \kappa_{\beta,\beta^{-1}}(f,g,g^*) \in \mathrm{H}^1(\mathbf{Q}, V_{fgg^*}). \tag{5.1}$$

Note the $G_{\mathbf{Q}}$-module decomposition (1.7) yields

$$\begin{aligned}\mathrm{H}^1(\mathbf{Q}, V_{fgg^*}) &\simeq \mathrm{H}^1(\mathbf{Q}, V_pE) \oplus \mathrm{H}^1(\mathbf{Q}, V_pE\otimes \mathrm{ad}^0V_p(g))\\ &\simeq \mathrm{H}^1(\mathbf{Q}, V_pE) \oplus \mathrm{H}^1(\mathbf{Q}, V_pE^K) \oplus \mathrm{H}^1(K, V_pE\otimes\chi),\end{aligned}$$

where E^K is the twist of E by the quadratic character corresponding to K.

Lemma 5.1. *The projections to* $\mathrm{H}^1(\mathbf{Q}, V_pE)$ *of each of the classes in* (5.1) *lands in* $\mathrm{Sel}(\mathbf{Q}, V_pE)$.

Proof. Since we are assuming $L(E,1) = 0$ and (a) above, the result follows from the vanishing of $\mathrm{Sel}(\mathbf{Q}, V_pE^K)$ and $\mathrm{Sel}(K, V_pE \otimes \chi)$ by the same argument as in Lemma 3.5. □

5.2. **Vanishing of** $\kappa_{\alpha,\beta^{-1}}(f,g,g^*)$ **and** $\kappa_{\beta,\alpha^{-1}}(f,g,g^*)$**.** This part follows easily from the work of Darmon–Rotger [DR21] and Bertolini–Seveso–Venerucci [BSV21].

Proposition 5.2. $\kappa_{\alpha,\beta^{-1}}(f,g,g^*) = \kappa_{\beta,\alpha^{-1}}(f,g,g^*) = 0$.

Proof. Let

$$\boldsymbol{g}_\alpha = \boldsymbol{\theta}_{\psi,\alpha}(S_2) \in \mathcal{O}[\![S_2]\!][\![q]\!], \quad \boldsymbol{g}^*_{\beta^{-1}} = \boldsymbol{\theta}_{\psi^{-1},\beta^{-1}}(S_3) \in \mathcal{O}[\![S_3]\!][\![q]\!]$$

be CM Hida families as in §2.4, but passing through the specialization $(g_\alpha, g_{\beta^{-1}})$ rather than $(g_\alpha, g_{\alpha^{-1}})$. Let

$$\kappa(f, \boldsymbol{g}_\alpha \boldsymbol{g}^*_{\beta^{-1}})(S_2, S_3) \in \mathrm{H}^1(\mathbf{Q}, \mathbb{V}^\dagger_{f\boldsymbol{g}_\alpha\boldsymbol{g}^*_{\beta^{-1}}}) \tag{5.2}$$

be the two-variable restriction of the three-variable cohomology class constructed in [DR21] and [BSV21] (after a choice of test vectors $\breve{\boldsymbol{g}}_\alpha$, $\breve{\boldsymbol{g}}^*_{\beta^{-1}}$ that we omit from the notation), and consider the further restriction

$$\boldsymbol{\kappa}^\iota := \kappa(f, \boldsymbol{g}_\alpha \boldsymbol{g}^*_{\beta^{-1}})(\mathbf{v}(1+T) - 1, \mathbf{v}(1+T)^{-1} - 1) \in \mathrm{H}^1(\mathbf{Q}, \mathbb{V}^\dagger_{f\boldsymbol{g}_\alpha(\boldsymbol{g}^*_{\beta^{-1}})^\iota}),$$

where $\mathbb{V}^\dagger_{f\boldsymbol{g}_\alpha(\boldsymbol{g}^*_{\beta^{-1}})^\iota} \simeq (V_pE \otimes \mathrm{Ind}_K^{\mathbf{Q}}\chi) \oplus (V_pE \otimes \mathrm{Ind}_K^{\mathbf{Q}}\Psi_T^{1-\tau})$. Thus $\boldsymbol{\kappa}^\iota$ is the restriction of (5.2) to the line of weights $(\ell, 2-\ell)$ (*cf.* $\kappa(\breve{f}, \breve{\boldsymbol{g}}\breve{\boldsymbol{g}}^*)$ in (3.11), where the line (ℓ,ℓ) is considered). Then by definition,

$$\kappa_{\alpha,\beta^{-1}}(f,g,g^*) = \boldsymbol{\kappa}^\iota(\mathbf{v}-1, \mathbf{v}-1).$$

As in Theorem 3.6, by [DR21, Prop. 5.8] the restriction $\mathrm{Loc}_p(\boldsymbol{\kappa}^\iota)$ belongs to the natural image of $\mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}^{++}\mathbb{V}^\dagger_{f\boldsymbol{g}_\alpha(\boldsymbol{g}^*_{\beta^{-1}})^\iota})$ in $\mathrm{H}^1(\mathbf{Q}_p, \mathbb{V}^\dagger_{f\boldsymbol{g}_\alpha(\boldsymbol{g}^*_{\beta^{-1}})^\iota})$, where

$$\mathscr{F}^{++}\mathbb{V}^\dagger_{f\boldsymbol{g}_\alpha(\boldsymbol{g}^*_{\beta^{-1}})^\iota} = V_pE \otimes \chi^{-1} + \mathscr{F}^+V_pE \otimes (\Psi_T^{1-\tau} + \Psi_T^{1-\tau}).$$

Thus the projection $\boldsymbol{\kappa}^\iota_\infty$ of $\boldsymbol{\kappa}^\iota$ to $\mathrm{H}^1(\mathbf{Q}, V_pE \otimes \mathrm{Ind}_K^{\mathbf{Q}}\Psi_T^{1-\tau}) \simeq \widehat{\mathrm{H}}^1(K_\infty, V_pE)$ is crystalline at p, and therefore defines a Selmer class for V_pE over the K_∞/K. Since under our hypotheses the space of such anticyclotomic universal norms is trivial by Cornut–Vatsal [CV05], we conclude that $\boldsymbol{\kappa}^\iota_\infty = 0$. As in the proof of Theorem 3.6, it follows that $\kappa_{\alpha,\beta^{-1}}(f,g,g^*) = 0$. The vanishing of $\kappa_{\beta,\alpha^{-1}}(f,g,g^*)$ is shown in the same manner. □

5.3. **The leading term formula.** Let $J \subseteq \Lambda$ be the augmentation ideal, and let

$$\mathfrak{r} = \mathrm{ord}_J(\Theta_{f/K}) := \sup\{s \geqslant 0 \mid \Theta_{f/K} \in J^s\}.$$

Since $\Theta_{f/K}$ is nonzero by [Vat03], $\mathfrak{r}$ is a well-defined non-negative integer, and since $L(E/K,1) = 0$ under our hypotheses, $\mathfrak{r} > 0$ by the interpolation property. Let

$$\mathrm{Sel}(K, V_pE) = S_p^{(1)} \supseteq S_p^{(2)} \supseteq \cdots \supseteq S_p^{(i)} \supseteq \cdots \supseteq S_p^{(\infty)} \tag{5.3}$$

be the filtration in Theorem 4.1, where we have put $S_p^{(i)} = S_p^{(i)}(E/K)$ for the ease of notation, and let

$$h_p^{(i)} : S_p^{(i)} \times S_p^{(i)} \to (J^i/J^{i+1}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

be the associated derived p-adic height pairings. Since we assume that N^- is the squarefree product of an odd number of primes, we have $S_p^{(\infty)} = 0$ by part (b) of Theorem 4.1 and the work of Cornut–Vatsal [CV05].

Theorem 5.3. *Let $\mathfrak{r} = \mathrm{ord}_J(\Theta_{f/K})$. Then*

$$\kappa_{\alpha,\alpha^{-1}}(f,g,g^*) \in S_p^{(\mathfrak{r})}, \tag{5.4}$$

and that for every for every $x \in S_p^{(\mathfrak{r})}$ we have

$$h_p^{(\mathfrak{r})}(\kappa_{\alpha,\alpha^{-1}}(f,g,g^*), x) = \frac{1-p^{-1}\alpha_p}{1-\alpha_p^{-1}} \cdot \Theta_{f/K} \cdot \log_{\omega_E,\mathfrak{p}}(x) \cdot C \pmod{J^{\mathfrak{r}+1}}, \tag{5.5}$$

where α_p is the p-adic unit root of $X^2 - a_p(E)X + p = 0$ and C is a non-zero algebraic number with $C^2 \in K(\chi, \alpha_p)^\times$.

Proof. This is the combination of Corollary 3.7 and Theorem 4.5. □

5.4. **Non-vanishing of $\kappa_{\alpha,\alpha^{-1}}(f,g,g^*)$.** Here we prove the implication (1.10) in Theorem A. Thus suppose that $\dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) = 2$. Since $L(E^K, 1) \neq 0$, we have $\mathrm{Sel}(\mathbf{Q}, V_pE^K) = 0$ by [Kol88] (or, alternatively, [Kat04]), and therefore

$$\mathrm{Sel}(K, V_pE) = \mathrm{Sel}(\mathbf{Q}, V_pE), \quad (r^+, r^-) = (2, 0), \tag{5.6}$$

where $r^\pm$ denotes the dimension of the $\pm$-eigenspace of $\mathrm{Sel}(K, V_pE)$ under the action of the complex conjugation τ. Since τ acts as -1 on J/J^2, part (4) of Theorem 4.1 gives

$$h_p^{(i)}(x^\tau, y^\tau) = (-1)^r h_p^{(i)}(x, y), \tag{5.7}$$

and hence from (5.6) we see that for i odd, the null-space of $h_p^{(i)}$ (i.e., $S_p^{(i+1)}$) is either zero or two-dimensional, with the latter case occurring as long as $S_p^{(i)} \neq 0$. Since on the other hand $h_p^{(i)}$ is a non-degenerate alternating pairing on $S_p^{(i)}/S_p^{(i+1)}$ for even values of i, unless $S_p^{(i)} = 0$, it follows that (5.3) reduces to

$$\mathrm{Sel}(\mathbf{Q}, V_pE) = S_p^{(1)} = S_p^{(2)} = \cdots = S_p^{(r)} \supsetneq S_p^{(r+1)} = \cdots = S_p^{(\infty)} = 0 \tag{5.8}$$

for some even $r \geqslant 2$. By Theorem 4.1, we deduce that there is a Λ-module pseudo-isomorphism

$$\mathrm{Sel}_{p^\infty}(E/K_\infty)^\vee \sim (\Lambda/J^r)^{\oplus 2} \oplus M',$$

where M' is a torsion Λ-module with characteristic ideal prime-to-J. Therefore letting $\mathcal{L}_p \in \Lambda$ be any generator of the characteristic ideal of $\mathrm{Sel}_{p^\infty}(E/K_\infty)^\vee$, we have

$$\mathrm{ord}_J(\mathcal{L}_p) = 2r.$$

Finally, the divisibility $(\Theta_{f/K}^2) \supseteq (\mathcal{L}_p)$ arising from [SU14, §3.6.3] implies that $r \geqslant \mathfrak{r}$, and hence $S_p^{(\mathfrak{r})} = \mathrm{Sel}(\mathbf{Q}, V_pE)$ by (5.8). Since by our hypothesis that $\mathrm{Sel}(\mathbf{Q}, V_pE) \neq \ker(\mathrm{Loc}_p)$ we can find $x \in \mathrm{Sel}(\mathbf{Q}, V_pE)$ with $\log_{\omega_E,\mathfrak{p}}(x) \neq 0$, the non-vanishing of $\kappa_{\alpha,\alpha^{-1}}(f,g,g^*)$ now follows from the leading term formula (5.5).

Remark 5.4. The same argument as above with β in place of α establishes the non-vanishing of $\kappa_{\beta,\beta^{-1}}(f,g,g^*)$ under the given hypotheses.

5.5. **Analogue of Kolyvagin's theorem for $\kappa_{\alpha,\alpha^{-1}}(f,g,g^*)$.** Here we prove the implication (1.9) in Theorem A. As in §5.4, we see that $\mathrm{Sel}(K, V_pE) = \mathrm{Sel}(\mathbf{Q}, V_pE)$ and the non-trivial jumps in (5.3) can only occur at even values of i. Thus (5.3) reduces to

$$\mathrm{Sel}(\mathbf{Q}, V_pE) = S_p^{(1)} = \cdots = S_p^{(2r_1)} \supsetneq S_p^{(2r_1+1)} = \cdots = S_p^{(2r_t)} \supsetneq S_p^{(2r_t+1)} = \cdots = S_p^{(\infty)} = 0 \tag{5.9}$$

for some $1 \leqslant r_1 \leqslant \cdots \leqslant r_t$, and by Theorem 4.1 we have

$$\mathrm{Sel}_{p^\infty}(E/K_\infty)^\vee \sim (\Lambda/J^{2r_1})^{d_1} \oplus \cdots \oplus (\Lambda/J^{2r_t})^{\oplus d_t} \oplus M'$$

where $d_i = \dim_{\mathbf{Q}_p}(S_p^{(2r_i)}/S_p^{(2r_i+1)}) \geqslant 2$ and M' is as in §5.4. Letting $\mathcal{L}_p \in \Lambda$ be a generator of the characteristic ideal of $\mathrm{Sel}_{p^\infty}(E/K_\infty)^\vee$, we therefore have

$$\mathrm{ord}_J(\mathcal{L}_p) = 2(r_1 d_1 + \cdots + r_t d_t), \quad \dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) = d_1 + \cdots + d_t. \tag{5.10}$$

Suppose now that $\kappa_{\alpha,\alpha^{-1}}(f,g,g^*) \neq 0$. By (5.4), it follows that $S_p^{(\mathfrak{r})} \neq 0$ and therefore

$$\mathfrak{r} \leqslant 2r_t. \tag{5.11}$$

On the other hand, the divisibility $(\mathcal{L}_p) \supseteq (\Theta_{f/K}^2)$ established in [BD05] (as refined in [PW11]) implies that $r_1 d_1 + \cdots + r_t d_t \leqslant \mathfrak{r}$; together with (5.11) this yields

$$2r_t \geqslant r_1 d_1 + \cdots + r_t d_t \geqslant 2(r_1 + \cdots + r_t),$$

from which we conclude that $t = 1$, $d_1 = 2$, and $\dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) = 2$.

5.6. **Proof of Theorem B.** This will follow from essentially the same argument as in §5.4, but without the need to appeal to [SU14].

Let the hypotheses be as in the statement of Theorem B, and assume that $\mathrm{ord}_T(\Theta_{f/K}) = 2$. Then Theorem 5.3 gives the inclusion

$$\kappa_{\alpha,\alpha^{-1}}(f,g,g^*) \in S_p^{(2)}.$$

As in §5.4, the assumption that $L(E^K, 1) \neq 0$ implies that $\mathrm{Sel}(K, V_pE) = \mathrm{Sel}(\mathbf{Q}, V_pE)$. Since by (5.7) the τ-eigenspaces of $\mathrm{Sel}(K, V_pE)$ are isotropic under $h_p^{(1)}$, we see that

$$S_p^{(2)} = \mathrm{Sel}(K, V_pE) = \mathrm{Sel}(\mathbf{Q}, V_pE).$$

Finally, since our assumption that $E(\mathbf{Q})$ has positive rank implies that $\mathrm{Sel}(\mathbf{Q}, V_pE) \neq \ker(\mathrm{Loc}_p)$ (see Remark 1.5) the non-vanishing of $\kappa_{\alpha,\alpha^{-1}}(f,g,g^*)$ follows from the leading term formula of Theorem 5.3. The same argument yields the non-vanishing of $\kappa_{\beta,\beta^{-1}}(f,g,g^*)$.

5.7. **Application to the strong elliptic Stark conjecture.** We keep the setting from the beginning of this section, but assume in addition that $\#\text{Ш}(E/\mathbf{Q})[p^\infty] < \infty$.

As explained in [DR16, §4.5.3], the p-adic regulators appearing in the *elliptic Stark conjectures* of [DLR15] all vanish in the setting we have placed ourselves in. As a remedy, in [DR16] they formulated a strengthening of those conjectures in terms of certain *enhanced regulators*; in our setting they are given (modulo $\mathbf{Q}^\times$) by

$$\mathrm{Log}_p(P \wedge Q) = P \otimes \log_p(Q) - Q \otimes \log_p(P)$$

where (P, Q) is any basis of $E(\mathbf{Q}) \otimes_{\mathbf{Z}} \mathbf{Q}$. The strong elliptic Stark conjecture then predicts that the generalized Kato classes $\kappa_{\alpha,\alpha^{-1}}(f,g,g^*)$ and $\kappa_{\beta,\beta^{-1}}(f,g,g^*)$ both agree with $\mathrm{Log}_p(P \wedge Q)$ up to a nonzero algebraic constant.

In the direction of this conjecture, our methods show that $\kappa_{\alpha,\alpha^{-1}}(f,g,g^*)$ and $\kappa_{\beta,\beta^{-1}}(f,g,g^*)$ span the same p-adic line as $\mathrm{Log}_p(P \wedge Q)$ inside the 2-dimensional $\mathrm{Sel}(\mathbf{Q}, V_pE)$.

To state the application, we identify $J^{\mathfrak{r}}/J^{\mathfrak{r}+1}$ with $\mathbf{Z}_p$ in the usual manner by choosing a topological generator of Γ_∞, and let $\Theta_{f/K}^{(\mathfrak{r})} \in \mathbf{Z}_p \smallsetminus \{0\}$ denote the image of $\Theta_{f/K} \pmod{J^{\mathfrak{r}+1}}$ under this identification.

Theorem 5.5. *Let the setting be as in the beginning of Section 5, and let* $\mathfrak{r} = \mathrm{ord}_J(\Theta_{f/K})$. *Then, as elements of* $\mathrm{Sel}(\mathbf{Q}, V_pE) \simeq E(\mathbf{Q}) \otimes_{\mathbf{Z}} \mathbf{Q}_p$, *we have*

$$\kappa_{\alpha,\alpha^{-1}}(f,g,g^*) = C \cdot \frac{1 - p^{-1}\alpha_p}{1 - \alpha_p^{-1}} \cdot \frac{\Theta_{f/K}^{(\mathfrak{r})}}{h_p^{(\mathfrak{r})}(P,Q)} \cdot \mathrm{Log}_p(P \wedge Q),$$

where C is nonzero and such that $C^2 \in K(\chi, \alpha_p)^\times$. The same result holds of $\kappa_{\beta,\beta^{-1}}(f,g,g^*)$.

Proof. Immediate from the leading term formula of §5.3 applied to $x = P$ and Q. $\square$

Remark 5.6. The term $h_p^{(\mathfrak{r})}(P,Q)$ recovers the derived p-adic regulator R_{der} introduced in [BD95]. Thus Theorem 5.5 links the conjectural algebraicity of the ratio between $\kappa_{\alpha,\alpha^{-1}}(f,g,g^*)$ and $\mathrm{Log}_p(P\wedge Q)$, as predicted in [DR16, §4.5.3], to a refinement of the p-adic Birch and Swinnerton-Dyer conjecture in [BD96, Conjecture 4.3] formulated in terms of R_{der}.

6. Appendix. Non-vanishing of $\kappa_{\alpha,\alpha^{-1}}(f,g,g^*)$: Numerical examples

In this appendix, we exhibit the first examples of elliptic curves E over $\mathbf{Q}$ of rank two with non-vanishing generalized Kato classes by numerically verifying the conditions in Theorem B.

Setting. In the examples tabulated below, we take elliptic curves $E/\mathbf{Q}$ with

$$\mathrm{ord}_{s=1}L(E,s)=2=\mathrm{rank}_{\mathbf{Z}}\,E(\mathbf{Q})$$

of conductor $N\in\{q,2q\}$, with q an odd prime, and pairs $(p,-d)$ consisting of a prime $p>3$ and a squarefree integer $-d<0$ such that:

- $K=\mathbf{Q}(\sqrt{-d})$ has class number one, q is inert in K, and $L(E^K,1)\neq 0$,
- p splits in K and $E[p]$ is irreducible as a $G_{\mathbf{Q}}$-module.

Note that such pairs $(p,-d)$ can be easily produced. Indeed, [Rib90, Thm. 1.1] implies that $E[p]$ must ramify at $N^-=q$, and the irreducibility of $E[p]$ can be verified either by [Maz78] when $p\geqslant 11$ or by checking (from e.g. Cremona's tables) that E does not admit any rational m-isogenies for $m>3$.

For every such triple $(E,p,-d)$, there is a ring class character χ of K of ℓ-power conductor for some prime $\ell\nmid Np$ such that $L(E/K,\chi,1)\neq 0$. (In fact, there are infinitely many such χ, as follows from [Vat03, Thm. 1.3] and its extension in [CH18b, Thm. D].) Writing $\chi=\psi/\psi^\tau$ and letting $g=\theta_\psi$ and $g^*=\theta_{\psi^{-1}}$ we then have the class

$$\kappa_{\alpha,\alpha^{-1}}(f,g,g^*)\in\mathrm{Sel}(\mathbf{Q},V_pE)$$

as in §5.2 (see Lemma 5.1). By Theorem B, to verify the nonvanishing of $\kappa_{\alpha,\alpha^{-1}}(f,g,g^*)$ it suffices to check that

$$\mathrm{ord}_T(\Theta_{f/K})=2. \tag{6.1}$$

Verifying order of vanishing 2. Let B be the definite quaternion algebra over $\mathbf{Q}$ of discriminant q, let $R\subset B$ be an Eichler order of level N/q, and let $\mathrm{Cl}(R)$ be the class group of R. Let

$$\phi_f:\mathrm{Cl}(R)\to\mathbf{Z}$$

be the Hecke eigenfunction associated to f by Jacquet–Langlands, normalized so that $\phi_f\not\equiv 0\pmod p$. Fix an isomorphism $i_p:R\otimes\mathbf{Z}_p\simeq\mathrm{M}_2(\mathbf{Z}_p)$ and an optimal embedding $\mathcal{O}_K\hookrightarrow R$ such that K is sent to a subspace consisting of diagonal matrices, and for $a\in\mathbf{Z}_p^\times$ and $n\geqslant 0$ put

$$r_n(a)=i_p^{-1}(\begin{pmatrix}1 & ap^{-n}\\ 0 & 1\end{pmatrix})\in\widehat{B}^\times,$$

where $\widehat{B}=B\otimes_{\mathbf{Z}}\widehat{\mathbf{Z}}$ is the adelic completion of B.

Consider the sequence $\{P_n^a\}_{n\geqslant 0}$ of right R-ideals given by $P_n^a:=(r_n(a)\widehat{R})\cap B$, and define the n-th theta element $\Theta_{f/K,n}\in\mathbf{Z}_p[T]$ by

$$\Theta_{f/K,n}:=\frac{1}{\alpha_p^{n+1}}\sum_{i=0}^{p^n-1}\sum_{a\in\mu_{p-1}}\Big(\alpha_p\cdot\phi_f(P_n^{a\mathbf{u}^i})-\phi_f(P_{n+1}^{a\mathbf{u}^i})\Big)(1+T)^i,$$

where α_p is the p-adic unit root of $x^2-a_p(E)x+p$ and $\mathbf{u}=1+p$.

By the definition of $\Theta_{f/K}$ (see e.g. [BD96, §2.7]), we have

$$\Theta_{f/K}\equiv\Theta_{f/K,n}\pmod{(1+T)^{p^n}-1}.$$

Since $(p^n, (1+T)^{p^n}-1) \subset (p^n, T^p)$, in the examples listed in the following tables we could verify (6.1) by computing $\Theta_{f/K,n} \bmod (p^n, T^p)$ for $n = 2$ and 3, respectively. The computations were done using the Brandt module package in SAGE.

TABLE 1. Examples with $\mathrm{ord}_T(\Theta_{f/K}) = 2$ determined mod (p^2, T^p)

E	p	$-d$	$\Theta_{f/K} \bmod (p^2, T^p)$
389a1	11	-2	$10T^2 + 69T^3 + T^4 + 103T^5 + 106T^6 + 66T^7 + 11T^8 + 55T^9 + 110T^{10}$
433a1	11	-7	$88T^2 + 22T^3 + 86T^4 + 7T^5 + 10T^6 + 12T^7 + 29T^8 + 88T^9 + 48T^{10}$
446c1	7	-3	$22T^2 + 27T^3 + 3T^4 + 16T^5 + 11T^6$
563a1	5	-1	$18T^2 + 9T^3 + 5T^4$
643a1	5	-1	$T^2 + 21T^4$
709a1	11	-2	$27T^2 + 114T^3 + 3T^4 + 14T^5 + 36T^6 + 15T^7 + 42T^8 + 44T^9 + 91T^{10}$
718b1	5	-19	$3T^2 + 20T^3 + 12T^4$
794a1	7	-3	$47T^2 + 23T^3 + 8T^4 + 24T^5 + 7T^6$
997b1	11	-2	$71T^2 + 41T^3 + 83T^4 + 19T^5 + 114T^6 + 111T^7 + 101T^8 + 46T^9 + 102T^{10}$
997c1	11	-2	$54T^2 + 38T^3 + 36T^4 + 81T^5 + 82T^6 + 18T^7 + 72T^8 + 95T^9 + 4T^{10}$
1034a1	5	-19	$22T^2 + 4T^3 + 6T^4$
1171a1	5	-1	$6T^2 + 6T^3 + 20T^4$
1483a1	13	-1	$128T^2 + 148T^3 + 127T^4 + 162T^5 + 30T^6 + 149T^7 + 141T^8 + 97T^9 + 49T^{10} + 13T^{11} + 29T^{12}$
1531a1	5	-1	$16T^2 + 7T^3 + 21T^4$
1613a1	17	-2	$128T^2 + 165T^3 + 224T^4 + 287T^5 + 140T^6 + 211T^7 + 147T^8 + 160T^9 + 59T^{10} + 122T^{11} + 195T^{12} + 43T^{13} + 207T^{14} + 214T^{15} + 285T^{16}$
1627a1	13	-1	$101T^2 + 151T^3 + 58T^4 + 104T^5 + 3T^6 + 165T^7 + 128T^8 + 63T^9 + 17T^{10} + 55T^{11} + 166T^{12}$
1907a1	13	-1	$72T^2 + 131T^3 + 32T^4 + 142T^5 + 84T^6 + 104T^7 + 90T^8 + 105T^9 + 38T^{10} + 92T^{11} + 116T^{12}$
1913a1	7	-3	$41T^2 + 16T^3 + 28T^4 + 23T^5 + 14T^6$
2027a1	13	-1	$54T^2 + 128T^3 + 65T^4 + 93T^5 + 83T^6 + 161T^7 + 113T^8 + 133T^9 + 49T^{10} + 151T^{11} + 13T^{12}$

TABLE 2. Examples with $\mathrm{ord}_T(\Theta_{f/K}) = 2$ determined mod (p^3, T^p)

E	p	$-d$	$\Theta_{f/K} \bmod (p^3, T^p)$
571b1	5	-1	$100T^2 + 100T^3 + 15T^4$
1621a1	11	-2	$1089T^2 + 807T^4 + 986T^5 + 586T^6 + 1098T^7 + 772T^8 + 228T^9 + 1296T^{10}$

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