# NONVANISHING OF GENERALISED KATO CLASSES AND IWASAWA MAIN CONJECTURES

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Abstract. A construction due to Darmon–Rotger gives rise to generalised Kato classes $\kappa_p(E)$ in the $p$-adic Selmer group $\mathrm{Sel}(\mathbf{Q}, V_pE)$ of elliptic curves $E/\mathbf{Q}$ of positive even analytic rank, where $p > 3$ is any prime of good ordinary reduction for $E$. In [DR16], they conjectured that $\kappa_p(E) \neq 0$ precisely when $\mathrm{Sel}(\mathbf{Q}, V_pE)$ is two-dimensional. The first cases of this conjecture were obtained by the author with M.-L. Hsieh [CH22]. In this note we give a new proof of the implication

$$\kappa_p(E) \neq 0 \quad \Longrightarrow \quad \dim_{\mathbf{Q}_p} \mathrm{Sel}(\mathbf{Q}, V_pE) = 2$$

established in *op. cit.*, and show that the converse implication holds if *and only if* the restriction map $\mathrm{loc}_p : \mathrm{Sel}(\mathbf{Q}, V_pE) \to E(\mathbf{Q}_p)\hat{\otimes}\mathbf{Q}_p$ is nonzero. The present approach is an adaptation to the non-CM case of the method introduced by the author [Cas22] in the case of CM elliptic curves.

## 1. Introduction

After the grounbreaking works of Gross–Zagier and Kolyvagin in the 1980s, the construction of non-torsion rational points (or more generally, Selmer classes) on elliptic curves of $E/\mathbf{Q}$ with $\mathrm{ord}_{s=1}L(E, s) \geq 2$, akin to the construction of Heegner points in the cases of analytic rank 0 or 1, is widely regarded as one of the central open problems in number theory.

In a recent series of spectacular works [DR14, DR17, DR22, BSV20, BSV22b] (culminating in the collective volume [BDR+22] with applications to the theory of Stark–Heegner points), Darmon–Rotger and Bertolini–Seveso–Venerucci revisited the construction of diagonal cycle classes due to Gross–Kudla [GK92] and Gross–Schoen [GS95], obtaining in particular an interpolation of these classes in $p$-adic families. Directly from their geometric construction, one obtains classes attached to triples $(f, g, h)$ of cuspidal eigenforms of "balanced" weights $(k, l, m)$ (meaning that none of $k$, $l$, or $m$ is larger than the sum of the other two), but after $p$-adic interpolation one also gets classes for weights beyond this range, such as the "unbalanced" weight $(2, 1, 1)$.

Of special relevance to the Birch–Swinnerton-Dyer conjecture and its equivariant refinements is the case where $f$ is the newform of weight 2 associated to an elliptic curve $E/\mathbf{Q}$, and $g$ and $h$ are the weight 1 cuspidal eigenforms associated (as a consequence of the proof of Serre's conjecture [KW09]) to degree 2 odd irreducible Artin representations $\varrho_1$ and $\varrho_2$, respectively. In this case, one of the main results of [DR17] and [BSV22b] relates the resulting *generalised Kato classes*

$$\kappa_p(f, g, h) \in \mathrm{H}^1(\mathbf{Q}, V_pE \otimes \varrho), \quad \varrho := \varrho_1 \otimes \varrho_2,$$

obtained from a specialisation in weights $(2, 1, 1)$ of a $p$-adic family of diagonal cycles classes, to the value at $s = 1$ of the twisted Hasse–Weil $L$-funcion $L(E \otimes \varrho, s)$. Together with global duality, this relation leads to a proof of the implication

$$L(E \otimes \varrho, 1) \neq 0 \quad \Longrightarrow \quad \mathrm{Hom}_{G_{\mathbf{Q}}}(V_\varrho^\vee, E(H) \otimes L) = 0,$$

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where $L$ is any number field over which $\varrho$ is defined (assumed to admit an embedding into $\mathbf{Q}_p$ for simplicity), $V_\varrho^\vee$ is the linear dual of an $L$-vector space affording $\varrho$, and $H$ is the fixed field of $\ker(\varrho)$. More precisely, by virtue of the explicit reciprocity law obtained in *loc. cit.*, the nonvanishing of $L(E\otimes\varrho,1)$ implies that the class $\kappa_p(f,g,h)$ is *non-crystalline* at $p$, from where the bound on the $\varrho$-isotypical component of $E(H)$ follows by global duality.

Interestingly, the same explicit reciprocity law shows that the generalised Kato classes $\kappa_p(f,g,h)$ are Selmer whenever $L(E\otimes\varrho,1)=0$. Moreover, the representations $\varrho$ for which the construction applies necessarily have real traces, and in many cases the sign in the functional of the self-dual $L$-function $L(E\otimes\varrho,s)$ is $+1$. Thus the results of [DR17] and [BSV22b] provide a construction of (possibly zero *a priori*!) Selmer classes in situations where $\mathrm{ord}_{s=1}L(E\otimes\varrho,s)\geq 2$.

In [DR16], Darmon–Rotger carried out a systematic study of their construction in relation with the Birch–Swinnerton-Dyer conjecture and the elliptic Stark conjecture of [DLR15]. (More recently, a similar study was carried out by Bertolini–Seveso–Venerucci [BSV22a], which in particular allows one to interpret and refine some of the rationality conjectures in [DR16] in terms of a $p$-adic Birch–Swinnerton-Dyer conjecture for $p$-adic Garrett–Rankin $L$-functions.) In particular, when $L(E\otimes\varrho,1)=0$ with sign $+1$ they conjectured the equivalence

$$\kappa_p(f,g,h)\neq 0 \quad \overset{?}{\Longleftrightarrow} \quad \dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q},V_pE\otimes\varrho)=2. \tag{1.1}$$

This leads to the expectation, when combined with the equivariant Birch–Swinnerton-Dyer conjecture, that $\kappa_p(f,g,h)$ is a nonzero class in $\mathrm{Sel}(\mathbf{Q},V_pE\otimes V_\varrho)$ if and only if $\mathrm{ord}_{s=1}L(E\otimes\varrho,s)=2$.

Of special interest to the Birch–Swinnerton-Dyer conjecture is the case in which $\varrho$ contains the trivial representation. This occurs precisely when $\varrho_2\simeq\varrho_1^\vee$, so that

$$V_\varrho\simeq L\oplus\mathrm{ad}^0(V_{\varrho_1}),$$

where $\mathrm{ad}^0(V_{\varrho_1})$ is the 3-dimensional representation of $G_{\mathbf{Q}}$ acting on the trace zero endomorphisms of $V_{\varrho_1}$. By the Artin formalism, we then have

$$L(E\otimes\varrho,s)=L(E,s)\cdot L(E\otimes\mathrm{ad}^0(\varrho_1)).$$

In particular, if $L(E\otimes\mathrm{ad}^0(\varrho_1))\neq 0$, denoting by $\kappa_p(E)$ the image of $\kappa_p(f,g,h)$ under the resulting natural projection $\mathrm{H}^1(\mathbf{Q},V_pE\otimes\varrho)\to\mathrm{H}^1(\mathbf{Q},V_pE)$, by the Bloch–Kato conjecture [BK90] (which in this case predicts that the dimensions of the Selmer groups for $V_pE$ and $V_pE\otimes\varrho$ are the same) the equivalence (1.1) amounts to the conjectural equivalence

$$\kappa_p(E)\neq 0 \quad \overset{?}{\Longleftrightarrow} \quad \dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q},V_pE)=2. \tag{1.2}$$

In [CH22], the author and M.-L. Hsieh proved the first cases of conjecture (1.2), in situations where $\varrho_1$ is induced from a finite order character of an imaginary quadratic field in which $p$ splits. The key ingredient in the proof was a leading coefficient formula for the $p$-adic $L$-function introduced in [BD96] in terms of Howard's derived $p$-adic heights. [How04].

The main result of this note (see Theorem 5.2.3) includes a new proof of [CH22, Thm. A]. The approach is an adaptation of the method introduced by the author in [Cas22], where the case of CM elliptic curves $E/\mathbf{Q}$ is treated, and clarifies the role played by the restriction map at $p$

$$\mathrm{loc}_p:\mathrm{Sel}(\mathbf{Q},V_pE)\to E(\mathbf{Q}_p)\hat{\otimes}\mathbf{Q}_p.$$

Indeed, in *loc. cit.* it was shown that (under mild hypotheses) the nonvanishing of $\kappa_p(E)$ implies that $\mathrm{Sel}(\mathbf{Q},V_pE)$ is 2-dimensional, while for the proof of the converse implication it was necessary to assume that

$$\mathrm{Sel}(\mathbf{Q},V_pE)\neq\ker(\mathrm{loc}_p). \tag{$\mathrm{Loc}_p$}$$

A novel insight of the approach in this paper is that, assuming $\dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) = 2$, condition ($\mathrm{Loc}_p$) is in fact *necessary* for the nonvanishing of $\kappa_p(E)$.

Similarly as in [Cas22], our approach is based on a study of the relation between two different formulations of the Iwasawa–Greenberg main conjecture for triple products studied in [ACR23]:

(IMC-1) one in terms of Hsieh's triple product $p$-adic $L$-functions [Hsi21];

(IMC-2) another without reference to $p$-adic $L$-functions, phrased in terms of the $p$-adic family of diagonal cycles used in the construction of $\kappa_p(f, g, h)$ and $\kappa_p(E)$.

In the case where $\varrho_1 = \varrho_2^\vee$ is dual to $\varrho_2$, and is induced from a finite order Hecke character of an imaginary quadratic field $K$, conjecture (IMC-1) can be related to the Iwasawa main conjecture for $E/K$ in the anticyclotomic setting, and using global duality and the explicit reciprocity laws of [DR22] and [BSV22b], one can easily show that (in general) conjectures (IMC-1) and (IMC-2) are equivalent. Thus, from the results on the anticyclotomic main conjecture stemming from the works of Bertolini–Darmon [BD05] and Skinner–Urban [SU14], we deduce under mild hypotheses a proof of conjecture (IMC-2), from where the proof of our main result follow easily from a variant of Mazur's control theorem and global duality.

We conclude this introduction by noting that even though throughout the paper $p$ is assumed to be a prime of *good* ordinary reduction, it seems possible to extend our results to the multiplicative case; it would then be interesting to compare the resulting $\kappa_p(E)$ with the $p$-adic limits of Heegner points studied in [DF23, FG23] (see also [Cas18] for a related construction).

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## 2. $p$-adic $L$-functions

Fix a prime $p > 3$. Let $K$ be an imaginary quadratic field of discriminant $D_K$ and assume that

$$(p) = \mathfrak{p}\overline{\mathfrak{p}} \text{ splits in } K,$$

with $\mathfrak{p}$ the prime of $K$ above $p$ induced by a fixed embedding $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. In this section we recall from [Hsi21] the construction of the triple product $p$-adic $L$-function for Hida families, and its relation with the anticyclotomic $p$-adic $L$-functions of Bertolini–Darmon [BD96] in the case where two of the Hida families are CM Hida families attached to dual ray class characters of $K$.

### 2.1. Triple product $p$-adic $L$-function.

Let $\mathbb{I}$ be a normal domain finite flat over

$$\Lambda := \mathscr{O}[\![1 + p\mathbf{Z}_p]\!],$$

where $\mathscr{O}$ is the ring of integers of a finite extension of $\mathbf{Q}_p$. For a positive integer $N$ prime $p$ and a Dirichlet character $\chi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathscr{O}^\times$, we denote by $S^o(N, \chi, \mathbb{I}) \subset \mathbb{I}[\![q]\!]$ the space of ordinary $\mathbb{I}$-adic cusp forms of tame level $N$ and branch character $\chi$ as defined in [Hsi21, §3.1].

Denote by $\mathfrak{X}_{\mathbb{I}}^+ \subset \mathrm{Spec}\,\mathbb{I}(\overline{\mathbf{Q}}_p)$ the set of *arithmetic points* of $\mathbb{I}$, consisting of the ring homomorphisms $Q : \mathbb{I} \to \overline{\mathbf{Q}}_p$ such that $Q|_{1+p\mathbf{Z}_p}$ is given by $z \mapsto z^{k_Q}\epsilon_Q(z)$ for some $k_Q \in \mathbf{Z}_{\geq 2}$ called the *weight of* $Q$ and $\epsilon_Q(z) \in \mu_{p^\infty}$. As in [Hsi21, §3.1], we say that $\boldsymbol{f} = \sum_{n=1}^\infty a_n(\boldsymbol{f})q^n \in S^o(N, \chi, \mathbb{I})$

is a *primitive Hida family* if for every $Q \in \mathfrak{X}^+_{\mathbb{I}}$ the specialization $\boldsymbol{f}_Q$ gives the $q$-expansion of an ordinary $p$-stabilised newform of weight $k_Q$ and tame conductor $N$. Attached to such $\boldsymbol{f}$ we let $\mathfrak{X}^{\rm cls}_{\mathbb{I}} \subset \mathfrak{X}^+_{\mathbb{I}}$ be the set of ring homomorphisms $Q$ as above with $k_Q \in \mathbf{Z}$ such that $\boldsymbol{f}_Q$ is the $q$-expansion of a classical modular form.

For $\boldsymbol{f}$ a primitive Hida family of tame level $N$, we let

$$\rho_{\boldsymbol{f}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathrm{Frac}\,\mathbb{I})$$

denote the associated Galois representation, where $\mathrm{Frac}\,\mathbb{I}$ is the field of fractions of $\mathbb{I}$. It will be convenient for us to take $\rho_{\boldsymbol{f}}$ to be the *dual* of that in [Hsi21, §3.2]; in particular, $\det \rho_{\boldsymbol{f}} = \chi_{\mathbb{I}} \cdot \varepsilon_{\rm cyc}$ in the notations of *loc. cit.*, where $\varepsilon_{\rm cyc}$ is the $p$-adic cyclotomic character. We assume that the associated the residual representation $\bar{\rho}_{\boldsymbol{f}}$ is absolutely irreducible, and also denote by

$$\rho_{\boldsymbol{f}} : G_{\mathbf{Q}} \to \mathrm{Aut}_{\mathbb{I}}(V_{\boldsymbol{f}}) \simeq \mathrm{GL}_2(\mathbb{I})$$

the realisation of $\rho_{\boldsymbol{f}}$ on a module $V_{\boldsymbol{f}} \simeq \mathbb{I}^{\oplus 2}$. By [Wil88, Thm. 2.2.2], restricted to $G_{\mathbf{Q}_p}$ the Galois representation $V_{\boldsymbol{f}}$ fits into a short exact sequence

$$0 \to V_{\boldsymbol{f}}^+ \to V_{\boldsymbol{f}} \to V_{\boldsymbol{f}}^- \to 0,$$

where $V_{\boldsymbol{f}}^-$ is free of rank 1 over $\mathbb{I}$, with the $G_{\mathbf{Q}_p}$-action given by the unramified character sending an arithmetic Frobenius $\mathrm{Fr}_p$ to $a_p(\boldsymbol{f})$.

Associated with $\boldsymbol{f}$ there is a $\mathbb{I}$-algebra homomorphism

$$\lambda_{\boldsymbol{f}} : \mathbb{T}(N, \mathbb{I}) \to \mathbb{I}$$

where $\mathbb{T}(N, \mathbb{I})$ is the Hecke algebra acting on $\oplus_\chi S^o(N, \chi, \mathbb{I})$, where $\chi$ runs over the characters of $(\mathbf{Z}/pN\mathbf{Z})^\times$. Let $\mathbb{T}_{\mathfrak{m}}$ be the local component of $\mathbb{T}(N, \mathbb{I})$ through which $\lambda_{\boldsymbol{f}}$ factors, and following [Hid88] define the *congruence ideal* $C(\boldsymbol{f})$ of $\boldsymbol{f}$ by

$$C(\boldsymbol{f}) := \lambda_{\boldsymbol{f}}(\mathrm{Ann}_{\mathbb{T}_{\mathfrak{m}}}(\ker \lambda_{\boldsymbol{f}})) \subset \mathbb{I}.$$

When, in addition to absolutely irreducible, $\bar{\rho}_{\boldsymbol{f}}$ is also $p$-distinguished (i.e., the semi-simplification of $\bar{\rho}_{\boldsymbol{f}}|_{G_{\mathbf{Q}_p}}$ is non-scalar), it follows from the results of [Wil95] and [Hid88] that $C(\boldsymbol{f})$ is generated by a nonzero element $\eta_{\boldsymbol{f}} \in \mathbb{I}$.

2.1.1. *Triple products of Hida families.* Let

$$(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in S^o(N_f, \chi_f, \mathbb{I}_f) \times S^o(N_g, \chi_g, \mathbb{I}_g) \times S^o(N_h, \chi_h, \mathbb{I}_h)$$

be a triple of primitive Hida families with

$$\chi_f \chi_g \chi_h = \omega^{2a} \text{ for some } a \in \mathbf{Z}, \tag{2.1}$$

where $\omega$ is the Teichmüller character. Put

$$\mathcal{R} = \mathbb{I}_f \hat{\otimes}_{\mathscr{O}} \mathbb{I}_g \hat{\otimes}_{\mathscr{O}} \mathbb{I}_h,$$

which is a finite extension of the three-variable Iwasawa algebra $\Lambda \hat{\otimes}_{\mathscr{O}} \Lambda \hat{\otimes}_{\mathscr{O}} \Lambda$, and let

$$\mathfrak{X}^{\boldsymbol{f}}_{\mathcal{R}} := \left\{(Q_1, Q_2, Q_3) \in \mathfrak{X}^+_{\mathbb{I}_\varphi} \times \mathfrak{X}^{\rm cls}_{\mathbb{I}_g} \times \mathfrak{X}^{\rm cls}_{\mathbb{I}_h} \;:\; k_{Q_1} \geq k_{Q_2} + k_{Q_3} \text{ and } k_{Q_1} \equiv k_{Q_2} + k_{Q_3} \pmod 2\right\}$$

be the weight space for $\mathcal{R}$ in the so-called $\boldsymbol{f}$*-unbalanced range*.

Let $\mathbf{V} = V_{\boldsymbol{f}} \hat{\otimes}_{\mathscr{O}} V_{\boldsymbol{g}} \hat{\otimes}_{\mathscr{O}} V_{\boldsymbol{h}}$ be the triple tensor product Galois representation attached to $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$, and writing $\det \mathbf{V} = \mathcal{X}^2 \varepsilon_{\rm cyc}$ (as is possible by (2.1)) define

$$\mathbf{V}^\dagger := \mathbf{V} \otimes \mathcal{X}^{-1}, \tag{2.2}$$

which is a self-dual twist of $\mathbf{V}$. Define the rank four $G_{\mathbf{Q}_p}$-invariant subspace $\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}^\dagger) \subset \mathbf{V}^\dagger$ by

$$\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}^\dagger) := V_{\boldsymbol{f}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}} \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}} \otimes \mathcal{X}^{-1}. \tag{2.3}$$

For every $\underline{Q} = (Q_1, Q_2, Q_3) \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$ we denote by $\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}_{\underline{Q}}^\dagger) \subset \mathbf{V}_{\underline{Q}}^\dagger$ the corresponding specialisations. Finally, for every rational prime $\ell$ denote by $\varepsilon_\ell(\mathbf{V}_{\underline{Q}}^\dagger)$ the epsilon factor attached to the restriction of $\mathbf{V}_{\underline{Q}}^\dagger$ to $G_{\mathbf{Q}_\ell}$ as in [Tat79, p. 21], and assume that

(2.4) for some $\underline{Q} \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$, we have $\varepsilon_\ell(\mathbf{V}_{\underline{Q}}^\dagger) = +1$ for all prime factors $\ell$ of $N_f N_g N_h$.

As explained in [Hsi21, §1.2], condition (2.4) is independent of $\underline{Q}$, and it implies that the sign in the functional equation for the triple product $L$-function

$$L(\mathbf{V}_{\underline{Q}}^\dagger, s)$$

(with center at $s = 0$) is $+1$ for all $\underline{Q} \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$.

**Theorem 2.1.1.** *Let $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ be a triple of primitive Hida families as above satisfying conditions (2.1) and (2.4). Assume in addition that:*

- $\gcd(N_f, N_g, N_h)$ *is square-free,*
- *the residual representation $\bar{\rho}_{\boldsymbol{f}}$ is absolutely irreducible and $p$-distinguished,*

*and fix a generator $\eta_{\boldsymbol{f}}$ of the congruence ideal of $\boldsymbol{f}$. Then there exists a unique element*

$$\mathscr{L}_p^{\boldsymbol{f}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in \mathcal{R}$$

*such that for all $\underline{Q} = (Q_0, Q_1, Q_2) \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$ of weight $(k_0, k_1, k_2)$ with $\epsilon_{Q_0} = 1$ we have*

$$(\mathscr{L}_p^{\boldsymbol{f}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})(\underline{Q}))^2 = \Gamma_{\mathbf{V}_{\underline{Q}}^\dagger}(0) \cdot \frac{L(\mathbf{V}_{\underline{Q}}^\dagger, 0)}{(\sqrt{-1})^{2k_0} \cdot \Omega_{\boldsymbol{f}_{Q_0}}^2} \cdot \mathcal{E}_p(\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}_{\underline{Q}}^\dagger)) \cdot \prod_{\ell \in \Sigma_{\mathrm{exc}}} (1 + \ell^{-1})^2,$$

*where:*

- $\Gamma_{\mathbf{V}_{\underline{Q}}^\dagger}(0) = \Gamma_{\mathbf{C}}(c_{\underline{Q}})\Gamma_{\mathbf{C}}(c_{\underline{Q}} + 2 - k_1 - k_2)\Gamma_{\mathbf{C}}(c_{\underline{Q}} + 1 - k_1)\Gamma_{\mathbf{C}}(c_{\underline{Q}} + 1 - k_2)$*, with*

$$c_{\underline{Q}} = (k_0 + k_1 + k_2 - 2)/2$$

  *and $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$;*
- $\Omega_{\boldsymbol{f}_{Q_0}}$ *is the canonical period*

$$\Omega_{\boldsymbol{f}_{Q_0}} := (-2\sqrt{-1})^{k_0+1} \cdot \frac{\|\boldsymbol{f}_{Q_0}^\circ\|_{\Gamma_0(N_f)}^2}{\iota_p(\eta_{\boldsymbol{f}_{Q_0}})} \cdot \Big(1 - \frac{\chi_f'(p)p^{k_0-1}}{\alpha_{Q_0}^2}\Big)\Big(1 - \frac{\chi_f'(p)p^{k_0-2}}{\alpha_{Q_0}^2}\Big),$$

  *with $\boldsymbol{f}_{Q_0}^\circ \in S_{k_0}(N_f)$ the newform of conductor $N_f$ associated with $\boldsymbol{f}_{Q_0}$, $\chi_f'$ the prime-to-$p$ part of $\chi_f$, and $\alpha_{Q_0}$ the specialization of $a_p(\boldsymbol{f}) \in \mathbb{I}_f^\times$ at $Q_0$;*
- $\mathcal{E}_p(\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}_{\underline{Q}}^\dagger))$ *is the modified $p$-Euler factor*

$$\mathcal{E}_p(\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}_{\underline{Q}}^\dagger)) := \frac{L_p(\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}_{\underline{Q}}^\dagger), 0)}{\varepsilon_p(\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}_{\underline{Q}}^\dagger)) \cdot L_p(\mathbf{V}_{\underline{Q}}^\dagger / \mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}_{\underline{Q}}^\dagger), 0)} \cdot \frac{1}{L_p(\mathbf{V}_{\underline{Q}}^\dagger, 0)},$$

*and $\Sigma_{\mathrm{exc}}$ is an explicitly defined subset of the prime factors of $N_f N_g N_h$,* [Hsi21, p. 416].

*Proof.* This is [Hsi21, Thm. A], which also includes an interpolations formula in the cases where $\epsilon_{Q_0}$ is not necessarily trivial. □

2.2. **CM Hida families.** Let $K_\infty$ be the unique $\mathbf{Z}_p^2$-extension of $K$, and let $K_{\mathfrak{p}^\infty}$ be the maximal subfield of $K_\infty$ unramified outside $\mathfrak{p}$. Put

$$\Gamma_\infty := \mathrm{Gal}(K_\infty/K) \simeq \mathbf{Z}_p^2, \qquad \Gamma_{\mathfrak{p}^\infty} := \mathrm{Gal}(K_{\mathfrak{p}^\infty}/K) \simeq \mathbf{Z}_p.$$

For every ideal $\mathfrak{C} \subset \mathcal{O}_K$ we let $K(\mathfrak{C})$ be the ray class field of $K$ of conductor $\mathfrak{C}$ (so in particular $K_{\mathfrak{p}^\infty}$ is the maximal $\mathbf{Z}_p$-extension of $K$ inside $K(\mathfrak{p}^\infty)$). Let $\mathrm{Art}_\mathfrak{p}$ be the restriction of the global Artin map to $K_\mathfrak{p}^\times$, with geometric normalisation. Identifying $\mathbf{Z}_p^\times$ and $\mathcal{O}_{K_\mathfrak{p}}^\times$ via $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, the map $\mathrm{Art}_\mathfrak{p}$ induces an embedding $1 + p\mathbf{Z}_p \to \Gamma_{\mathfrak{p}^\infty}$. Write $I_\mathfrak{p}^{\mathrm{w}} = \mathrm{Art}_\mathfrak{p}(1 + p\mathbf{Z}_p)|_{K_{\mathfrak{p}^\infty}}$ and let $b \geq 0$ be such that $[\Gamma_{\mathfrak{p}^\infty} : I_\mathfrak{p}^{\mathrm{w}}] = p^b$. (Note that $b = 0$ if the class number of $K$ is coprime to $p$.)

Fix a topological generator $\gamma_\mathfrak{p} \in \Gamma_{\mathfrak{p}^\infty}$ with $\gamma_\mathfrak{p}^{p^b} = \mathrm{Art}_\mathfrak{p}(1+p)|_{K_{\mathfrak{p}^\infty}}$, and for each variable $S$ let $\Psi_S : \Gamma_\infty \to \mathscr{O}[\![S]\!]^\times$ be the universal character given by

$$\Psi_S(\sigma) = (1 + S)^{l(\sigma)},$$

where $l(\sigma) \in \mathbf{Z}_p$ is such that $\sigma|_{K_{\mathfrak{p}^\infty}} = \gamma_\mathfrak{p}^{l(\sigma)}$. Let $\mathbf{v} \in \mathscr{O}$ be such that $\mathbf{v}^{p^b} = 1 + p$ (after enlarging $\mathscr{O}$ if necessary). For any finite order character $\psi : G_K \to \mathscr{O}^\times$ of conductor $\mathfrak{C}$ put

$$\boldsymbol{\theta}_\psi(S)(q) = \sum_{(\mathfrak{a},\mathfrak{p}\mathfrak{C})=1} \psi(\sigma_\mathfrak{a}) \Psi^{-1}_{\mathbf{v}^{-1}(1+S)-1}(\sigma_\mathfrak{a}) q^{\mathbf{N}(\mathfrak{a})} \in \mathscr{O}[\![S]\!][\![q]\!],$$

where $\sigma_\mathfrak{a} \in \mathrm{Gal}(K(\mathfrak{C}\mathfrak{p}^\infty)/K)$ is the Artin symbol of $\mathfrak{a}$. Then $\boldsymbol{\theta}_\psi(S)$ is a primitive Hida family defined over $\mathscr{O}[\![S]\!]$ of tame level $D_K \mathbf{N}(\mathfrak{C})$ and tame character $(\psi \circ \mathscr{V})\eta_{K/\mathbf{Q}}\omega^{-1}$, where

$$\mathscr{V} : G_\mathbf{Q}^{\mathrm{ab}} \to G_K^{\mathrm{ab}}$$

is the transfer map and $\eta_{K/\mathbf{Q}}$ is the quadratic character associated to $K/\mathbf{Q}$.

2.3. **Anticyclotomic $p$-adic $L$-functions.** Let $f \in S_2(\Gamma_0(pN_f))$, with $p \nmid N_f$, be a $p$-ordinary $p$-stabilised newform of tame level $N_f$ defined over $\mathscr{O}$. Assume that $f$ is the ordinary $p$-stabilisation of the newform $f^\circ \in S_2(\Gamma_0(N_f))$, and let $\alpha_p \in \mathscr{O}^\times$ be the $U_p$-eigenvalue of $f$. Write

$$N_f = N^+ N^- \tag{2.5}$$

with $N^+$ (resp. $N^-$) divisible only by primes which are split (resp. inert) in $K$, and fix an ideal $\mathfrak{N}^+ \subset \mathcal{O}_K$ with $\mathcal{O}_K/\mathfrak{N}^+ \simeq \mathbf{Z}/N^+\mathbf{Z}$.

Let $\Gamma^-$ be the Galois group of the anticyclotomic $\mathbf{Z}_p$-extension $K_\infty^-/K$. By definition, the map $\sigma \mapsto l(\sigma^{1-\mathbf{c}}|_{K_{\mathfrak{p}^\infty}})$ factor through $\Gamma^-$, and we let $\gamma^-$ be the topological generator of $\Gamma^-$ mapping to 1 under the resulting isomorphism $\Gamma^- \simeq \mathbf{Z}_p$. We then identity $\mathscr{O}[\![\Gamma^-]\!]$ with the power series ring $\mathscr{O}[\![T]\!]$ via $\gamma^- \mapsto 1 + T$.

**Theorem 2.3.1.** *Let $\chi$ be a ring class character of $K$ of conductor $c\mathcal{O}_K$ with values in $\mathscr{O}$, and assume that:*

(i) $(pN_f, cD_K) = 1$,
(ii) *$N^-$ is the squarefree product of an odd number of primes,*
(iii) *if $q \mid N^-$ is a prime with $q \equiv 1 \pmod p$, then $\bar\rho_f$ is ramified at $q$.*

*Then there exists a unique* $\Theta_{f/K,\chi}(T) \in \mathscr{O}[\![T]\!]$ *such that for every* $p$*-power root of unity* $\zeta$*:*

$$\Theta_{f/K,\chi}(\zeta-1)^2 = \frac{p^n}{\alpha_p^{2n}} \cdot \mathcal{E}_p(f,\chi,\zeta)^2 \cdot \frac{L(f^\circ/K \otimes \chi\epsilon_\zeta, 1)}{(2\pi)^2 \cdot \Omega_{f^\circ,N^-}} \cdot u_K^2 \sqrt{D_K}\chi\epsilon_\zeta(\sigma_{\mathfrak{N}^+}) \cdot \varepsilon_p,$$

*where:*

- $n \geq 0$ *is such that* $\zeta$ *has exact order* $p^n$*,*
- $\epsilon_\zeta : \Gamma_\infty^- \to \mu_{p^\infty}$ *be the character defined by* $\epsilon_\zeta(\gamma^-) = \zeta$*,*
- $\mathcal{E}_p(f,\chi,\zeta) = \begin{cases} (1-\alpha_p^{-1}\chi(\mathfrak{p}))(1-\alpha_p\chi(\bar{\mathfrak{p}})) & \text{if } n=0, \\ 1 & \text{if } n>0, \end{cases}$
- $\Omega_{f^\circ,N^-} = 4\|f^\circ\|^2_{\Gamma_0(N_f)} \cdot \eta^{-1}_{f,N^-}$ *is the* Gross period *of* $f^\circ$ *(see* [Hsi21, p. 524]*),*
- $u_K = |\mathcal{O}_K^\times|/2$*,*
- $\varepsilon_p \in \{\pm 1\}$ *is the local root number of* $f^\circ$ *at* $p$*.*

*Proof.* See [BD96] for the original construction, and [CH18, Thm. A] for the stated interpolation property. ☐

When $\chi$ is the trivial character, we write $\Theta_{f/K,\chi}(T)$ simply as $\Theta_{f/K}(T)$.

2.4. **Factorisation of triple product $p$-adic $L$-functions.** Let $f \in S_2(pN_f)$ be a $p$-stabilised newform as in the preceding section. By Hida theory, $f$ is the specialisation of a unique primitive Hida family $\boldsymbol{f} \in S^o(N_f,\mathbb{I})$ at an arithmetic point $Q_0 \in \mathfrak{X}_{\mathbb{I}}^+$ of weight 2. Let $\ell \nmid pN_f$ be a prime split in $K$, and let $\chi$ be a ring class character of $K$ of conductor $\ell^m\mathcal{O}_K$ for some $m>0$. Denoting by $\mathbf{c}$ the non-trivial automorphism of $K/\mathbf{Q}$, write $\chi = \psi/\psi^{\mathbf{c}}$ with $\psi$ a ray class character modulo $\ell^m\mathcal{O}_K$ with $\psi^{\mathbf{c}}(\sigma) = \psi(\mathbf{c}\sigma\mathbf{c}^{-1})$. Let

$$\boldsymbol{g} = \boldsymbol{\theta}_\psi(S_1) \in \mathscr{O}[\![S_1]\!][\![q]\!], \quad \boldsymbol{g}^* = \boldsymbol{\theta}_{\psi^{-1}}(S_2) \in \mathscr{O}[\![S_2]\!][\![q]\!] \tag{2.6}$$

be the primitive CM Hida families (of tame level $C = D_K\ell^{2m}$) attached to $\psi$ and $\psi^{-1}$, respectively.

The triple $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{g}^*)$ satisfies conditions (2.1) and (2.4) and the associated triple product $p$-adic $L$-function $\mathscr{L}_p^{\boldsymbol{f}}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{g}^*)$ of Theorem 2.1.1 is an element in $\mathcal{R} = \mathbb{I}\hat{\otimes}_{\mathscr{O}}\mathscr{O}[\![S_1]\!]\hat{\otimes}_{\mathscr{O}}\mathscr{O}[\![S_2]\!] \simeq \mathbb{I}[\![S_1,S_2]\!]$. Put $S = S_1$. In the following, we let

$$\mathscr{L}_p^{\boldsymbol{f}}(f,\underline{\boldsymbol{g}\boldsymbol{g}}^*) \in \mathscr{O}[\![S]\!] \tag{2.7}$$

be the image of $\mathscr{L}_p^{\boldsymbol{f}}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{g}^*)$ under the map $\mathbb{I}[\![S_1,S_2]\!] \to \mathscr{O}[\![S_1,S_2]\!]$ given by the specialisation $Q_0 : \mathbb{I} \to \mathscr{O}$ composed with the quotient $\mathscr{O}[\![S_1,S_2]\!] \to \mathscr{O}[\![S_1,S_2]\!]/(S_1-S_2)$.

Let $K(\chi)$ be the field obtained by adjoining to $K$ the values of $\chi$, and put $K(\chi,\alpha_p) = K(\chi)(\alpha_p)$.

**Proposition 2.4.1.** *Assume that:*

(i) $N^-$ *is the squarefree product of an odd number of primes,*
(ii) *if* $q \mid N^-$ *is a prime with* $q \equiv 1 \pmod p$*, then* $\bar{\rho}_f$ *is ramified at* $q$*.*

*Set* $T = \mathbf{v}^{-1}(1+S)-1$*. Then*

$$\mathscr{L}_p^{\boldsymbol{f}}(f,\underline{\boldsymbol{g}\boldsymbol{g}}^*)(S) = \pm\mathbf{w}^{-1} \cdot \Theta_{f/K}(T) \cdot C_{f,\chi} \cdot \sqrt{L^{\rm alg}(f/K \otimes \chi, 1)} \cdot \frac{\eta_{f^\circ}}{\eta_{f^\circ,N^-}},$$

*where* $\mathbf{w}$ *is a unit in* $\mathscr{O}[\![T]\!]$*,* $C_{f,\chi} \in K(\chi,\alpha_p)^\times$*, and*

$$L^{\rm alg}(f/K \otimes \chi, 1) := \frac{L(f/K \otimes \chi, 1)}{4\pi^2\|f^\circ\|^2_{\Gamma_0(N_f)}} \in K(\chi).$$

*Proof.* This is immediate from [Hsi21, Prop. 8.1] and the interpolation property of $\Theta_{f/K,\chi}(0)$. ☐

## 3. Selmer group decompositions

In this section we introduce two different Selmer groups associated to triple products of modular forms, and relate them to anticyclotomic Selmer groups attached to a single modular form.

### 3.1. **Selmer groups for triple products of modular form.**

Let $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ be a triple of primitive Hida families as in §2.1.1 satisfying (2.1), and let $\mathbf{V}^\dagger = \mathbf{V} \otimes \mathcal{X}^{-1}$ be the self-dual twist of the associated big Galois representation.

**Definition 3.1.1.** Put

$$\mathscr{F}_p^{\rm bal}(\mathbf{V}^\dagger) = \mathscr{F}_p^2(\mathbf{V}^\dagger) := \big(V_{\boldsymbol{f}}^+ \otimes V_{\boldsymbol{g}}^+ \otimes V_{\boldsymbol{h}} + V_{\boldsymbol{f}}^+ \otimes V_{\boldsymbol{g}} \otimes V_{\boldsymbol{h}}^+ + V_{\boldsymbol{f}} \otimes V_{\boldsymbol{g}}^+ \otimes V_{\boldsymbol{h}}^+\big) \otimes \mathcal{X}^{-1},$$

and define the *balanced local condition* $\mathrm{H}^1_{\rm bal}(\mathbf{Q}_p, \mathbf{V}^\dagger)$ by

$$\mathrm{H}^1_{\rm bal}(\mathbf{Q}_p, \mathbf{V}^\dagger) := \mathrm{im}\big\{\mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}_p^{\rm bal}(\mathbf{V}^\dagger)) \to \mathrm{H}^1(\mathbf{Q}_p, \mathbf{V}^\dagger)\big\}.$$

Similarly, put

$$\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}^\dagger) := \big(V_{\boldsymbol{f}}^+ \otimes V_{\boldsymbol{g}} \otimes V_{\boldsymbol{h}}\big) \otimes \mathcal{X}^{-1},$$

and define the $\boldsymbol{f}$*-unbalanced local condition* $\mathrm{H}^1_{\boldsymbol{f}}(\mathbf{Q}_p, \mathbf{V}^\dagger)$ by

$$\mathrm{H}^1_{\boldsymbol{f}}(\mathbf{Q}_p, \mathbf{V}^\dagger) := \mathrm{im}\big\{\mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}^\dagger)) \to \mathrm{H}^1(\mathbf{Q}_p, \mathbf{V}^\dagger)\big\}.$$

It is easy to see that the maps appearing in these definitions are injective, and in the following we shall use this to identify $\mathrm{H}^1_?(\mathbf{Q}_p, \mathbf{V}^\dagger)$ with $\mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}_p^?(\mathbf{V}^\dagger))$ for $? \in \{\mathrm{bal}, \boldsymbol{f}\}$. Fix a finite set of primes $S$ containing $\infty$ and the primes dividing $N_fN_gN_g$, and let $G_{\mathbf{Q},S}$ be the Galois group of the maximal extension of $\mathbf{Q}$ unramified outside $S$.

**Definition 3.1.2.** Let $? \in \{\mathrm{bal}, \boldsymbol{f}\}$, and define the Selmer group $\mathrm{Sel}^?(\mathbf{Q}, \mathbf{V}^\dagger)$ by

$$\mathrm{Sel}^?(\mathbf{Q}, \mathbf{V}^\dagger) := \ker\bigg\{\mathrm{H}^1(G_{\mathbf{Q},S}, \mathbf{V}^\dagger) \to \frac{\mathrm{H}^1(\mathbf{Q}_p, \mathbf{V}^\dagger)}{\mathrm{H}^1_?(\mathbf{Q}_p, \mathbf{V}^\dagger)}\bigg\}$$

We call $\mathrm{Sel}^{\rm bal}(\mathbf{Q}, \mathbf{V}^\dagger)$ (resp. $\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q}, \mathbf{V}^\dagger)$) the *balanced* (resp. $\boldsymbol{f}$*-unbalanced*) Selmer group.

Let $\mathbf{A}^\dagger = \mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{V}^\dagger, \mu_{p^\infty})$ be the arithmetic dual of $\mathbf{V}^\dagger$, and for $? \in \{\mathrm{bal}, \boldsymbol{f}\}$ define $\mathrm{H}^1_?(\mathbf{Q}_p, \mathbf{A}^\dagger) \subset \mathrm{H}^1(\mathbf{Q}_p, \mathbf{A}^\dagger)$ to be the orthogonal complement of $\mathrm{H}^1_?(\mathbf{Q}_p, \mathbf{V}^\dagger)$ under the local Tate duality

$$\mathrm{H}^1(\mathbf{Q}_p, \mathbf{V}^\dagger) \times \mathrm{H}^1(\mathbf{Q}_p, \mathbf{A}^\dagger) \to \mathbf{Q}_p/\mathbf{Z}_p.$$

Similarly as above, we then define the balanced and $\boldsymbol{f}$-unbalanced Selmer groups with coefficients in $\mathbf{A}^\dagger$ by

$$\mathrm{Sel}^?(\mathbf{Q}, \mathbf{A}^\dagger) := \ker\bigg\{\mathrm{H}^1(G_{\mathbf{Q},S}, \mathbf{A}^\dagger) \to \frac{\mathrm{H}^1(\mathbf{Q}_p, \mathbf{A}^\dagger)}{\mathrm{H}^1_?(\mathbf{Q}_p, \mathbf{A}^\dagger)} \times \prod_{\ell \in S \smallsetminus \{p\}} \mathrm{H}^1(\mathbf{Q}_\ell, \mathbf{A}^\dagger)\bigg\},$$

and let $X^?(\mathbf{Q}, \mathbf{A}^\dagger) = \mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{Sel}^?(\mathbf{Q}, \mathbf{A}^\dagger), \mathbf{Q}_p/\mathbf{Z}_p)$ denote the Pontryagin dual of $\mathrm{Sel}^?(\mathbf{Q}, \mathbf{A}^\dagger)$.

3.2. **Anticyclotomic Selmer groups for modular forms.** Let $f \in S_2(\Gamma_0(pN_f))$ be an ordinary $p$-stabilised newform and $K/\mathbf{Q}$ an imaginary quadratic field as in §2.3.

Let $V_f$ be the $p$-adic Galois representation associated to $f$. We adopt the convention that if $f$ corresponds to the isogeny class of an elliptic curve $E/\mathbf{Q}$, then $V_f \simeq V_pE$ (rather than its dual). By $p$-ordinarity, restricted to $G_{\mathbf{Q}_p}$ the representation $V_f$ fits into a short exact sequence

$$0 \to V_f^+ \to V_f \to V_f^- \to 0$$

with $V_f^\pm$ both 1-dimensional, and with the $G_{\mathbf{Q}_p}$-action on $V_f^-$ given by the unramified character sending arithmetic Frobenius to $\alpha_p \in \mathscr{O}^\times$, the $U_p$-eigenvalue of $f$.

Below we fix $\Sigma$ to be any finite set of places of $K$ containing $\infty$ and the prime dividing $pN_f$, and for any field extension $L/K$ let $G_{L,\Sigma}$ the Galois group of the maximal extension of $L$ unramified outside $\Sigma$.

**Definition 3.2.1.** Let $L$ be a finite field extension of $K$, and let $\mathscr{F} = \{\mathscr{F}_v(V_f)\}_{v|p}$ be a collection a $G_{K_v}$-stable subpaces $\mathscr{F}_v(V_f) \subset V_f$ for $v \mid p$. We define the *Greenberg Selmer group* $\mathrm{Sel}_{\mathscr{F}}(L, V_f)$ by

$$\mathrm{Sel}_{\mathscr{F}}(L, V_f) := \ker\biggl\{\mathrm{H}^1(G_{L,\Sigma}, V_f) \to \prod_w \frac{\mathrm{H}^1(L_w, V_f)}{\mathrm{H}^1_{\mathscr{F}}(L_w, V_f)}\biggr\},$$

where $w$ runs over the finite primes of $L$ lying above a prime $v \in \Sigma$, and

$$\mathrm{H}^1_{\mathscr{F}}(L_w, V_f) := \begin{cases} \ker\{\mathrm{H}^1(L_w, V_f) \to \mathrm{H}^1(L_w^{\mathrm{ur}}, V_f)\} & \text{if } w \nmid p, \\ \mathrm{im}\{\mathrm{H}^1(L_w, \mathscr{F}_v(V_f)) \to \mathrm{H}^1(L_w, V_f)\} & \text{if } w \mid v \mid p. \end{cases}$$

We shall be particularly interested in the following choices of $\mathscr{F}$:

- The *relaxed-strict* Selmer group $\mathrm{Sel}_{\emptyset,0}(L, V_f)$ obtained by taking

$$\mathscr{F}_v(V_f) = \begin{cases} V_f & \text{if } v = \mathfrak{p}, \\ 0 & \text{if } v = \overline{\mathfrak{p}}. \end{cases}$$

- The *ordinary* Selmer group $\mathrm{Sel}(L, V_f)$ obtained by taking $\mathscr{F}_v(V_f) = V_f^+$ for all $v \mid p$.

For a $G_K$-stable lattice $T_f \subset V_f$, we let $\mathrm{H}^1_{\mathscr{F}}(L_w, T_f)$ be the inverse image of $\mathrm{H}^1_{\mathscr{F}}(L_w, V_f)$ under the natural map $\mathrm{H}^1(L_w, T_f) \to \mathrm{H}^1(L_w, V_f)$, and define $\mathrm{Sel}_{\mathscr{F}}(L, T_f)$ by the same recipe as above. Then, for $A_f := \mathrm{Hom}_{\mathbf{Z}_p}(T_f, \mu_{p^\infty})$, we define the *dual Selmer group* $\mathrm{Sel}_{\mathscr{F}^*}(L, A_f)$ by

$$\mathrm{Sel}_{\mathscr{F}^*}(L, A_f) := \ker\biggl\{\mathrm{H}^1(G_{L,\Sigma}, A_f) \to \prod_w \frac{\mathrm{H}^1(L_w, A_f)}{\mathrm{H}^1_{\mathscr{F}^*}(L_w, A_f)}\biggr\}$$

where $\mathrm{H}^1_{\mathscr{F}^*}(L_w, A_f)$ is the orthogonal complement of $\mathrm{H}^1_{\mathscr{F}}(L_w, T_f)$ under local Tate duality

$$\mathrm{H}^1(L_w, T_f) \times \mathrm{H}^1(L_w, A_f) \to \mathbf{Q}_p/\mathbf{Z}_p.$$

Write $K_m^-$ for the subextension of the the anticyclotomic $\mathbf{Z}_p$-extension $K_\infty^-$ with $[K_m^- : K] = p^m$, and put

$$\mathrm{Sel}_{\mathscr{F}}(K_\infty^-, T_f) := \varprojlim_m \mathrm{Sel}_{\mathscr{F}}(K_m^-, T_f), \quad \mathrm{Sel}_{\mathscr{F}}(K_\infty^-, A_f) := \varinjlim_m \mathrm{Sel}_{\mathscr{F}}(K_m^-, A_f),$$

where the limits are with respect to the corestriction and restriction maps, respectively. Let $\Psi : G_K \to \mathscr{O}[\![\Gamma^-]\!]^\times$ be the character arising from the projection $G_K \to \Gamma^-$. Writing $T_f \otimes \Psi^{-1}$

(resp. $A_f \otimes \Psi$) for the module $T_f \otimes_{\mathscr{O}} \mathscr{O}[\![\Gamma^-]\!]$ (resp. $A_f \otimes_{\mathscr{O}} \mathscr{O}[\![\Gamma^-]\!]$) with $G_K$-action on the second factor given by $\Psi^{-1}$ (resp. $\Psi$), we then have natural $\mathscr{O}[\![\Gamma^-]\!]$-module pseudo-isomorphisms

$$\mathrm{Sel}_{\mathscr{F}}(K_\infty^-, T_f) \sim \mathrm{Sel}_{\mathscr{F}}(K, T_f \otimes \Psi^{-1}), \quad \mathrm{Sel}_{\mathscr{F}}(K_\infty^-, A_f) \sim \mathrm{Sel}_{\mathscr{F}}(K, A_f \otimes \Psi), \tag{3.1}$$

where the Selmer groups on the right-hand side are defined is the same way as in Definition 3.2.1, with $\mathscr{F}_v(T_f \otimes \Psi^{-1}) := (\mathscr{F}_v(V_f) \cap T_f) \otimes \Psi^{-1}$ and $\mathscr{F}_v(A_f \otimes \Psi) := \mathrm{Hom}_{\mathbf{Z}_p}(T_f/(\mathscr{F}_v(V_f) \cap T_f), \mu_{p^\infty}) \otimes \Psi$.

3.3. **Decomposition of triple product Selmer groups.** Suppose now that $\boldsymbol{f}$ is the Hida family passing through the $p$-stabilised newform $f \in S_2(\Gamma_0(pN_f))$, and $(\boldsymbol{g}, \boldsymbol{g}^*) = (\boldsymbol{\theta}_\psi(S_1), \boldsymbol{\theta}_{\psi^{-1}}(S_2))$ are CM Hida families as in (2.6).

For any primitive Hida family $\boldsymbol{\phi}$, we now let $V_{\boldsymbol{\phi}}$ be the realization of $\rho_{\boldsymbol{\phi}}$ arising in the $p$-adic étale cohomology of the $p$-tower of modular curves as in Ohta's works [Oht99, Oht00], following the conventions in [KLZ17, §7.2] (except that a "Hida family" for us is a "branch" in their sense).

In the case of $\boldsymbol{g}$ (and similarly $\boldsymbol{g}^*$), when $\chi := \psi/\psi^{\mathbf{c}}$ satisfies

$$\bar{\chi}|_{G_{K_{\mathfrak{p}}}} \neq 1, \tag{3.2}$$

where $G_{K_{\mathfrak{p}}} \subset G_K$ is a decomposition group at $\mathfrak{p}$, it follows from a slight extension of the isomorphism in [LLZ15, Cor. 5.2.5] (see [BL18, §3.2.3]) that

$$V_{\boldsymbol{g}} \simeq \mathrm{Ind}_K^{\mathbf{Q}}(\psi^{-1}\Psi_{T_1}),$$

where $T_1 = \mathbf{v}^{-1}(1+S_1) - 1$. Suppose $Q_0 : \mathbb{I} \to \mathscr{O}$ is such that $f$ is the specialisation of $\boldsymbol{f}$ at $Q_0$, and write $\mathbb{V}^\dagger_{f\boldsymbol{g}\boldsymbol{g}^*}$ for the resulting specialisation of $\mathbf{V}^\dagger$. Similarly setting $T_2 = \mathbf{v}^{-1}(1+S_2) - 1$ and noting that $(\det V_{\boldsymbol{g}})(\det V_{\boldsymbol{g}^*}) = \Psi_{T_1}\Psi_{T_2} \circ \mathscr{V}$, we then have

$$\begin{aligned} \mathbb{V}^\dagger_{f\boldsymbol{g}\boldsymbol{g}^*} &\simeq T_f \otimes \mathrm{Ind}_K^{\mathbf{Q}}(\psi^{-1}\Psi_{T_1}) \otimes \mathrm{Ind}_K^{\mathbf{Q}}(\psi\Psi_{T_2}) \otimes (\Psi_{T_1}^{1/2}\Psi_{T_2}^{1/2} \circ \mathscr{V})^{-1} \\ &\simeq \big(T_f \otimes \mathrm{Ind}_K^{\mathbf{Q}}\Psi_{W_1}^{1-\mathbf{c}}\big) \oplus \big(T_f \otimes \mathrm{Ind}_K^{\mathbf{Q}}\chi^{-1}\Psi_{W_2}^{1-\mathbf{c}}\big), \end{aligned} \tag{3.3}$$

where $T_f$ is the specialisation of $V_{\boldsymbol{f}}$ at $Q_0$ and

$$W_1 = \mathbf{v}^{-1}(1+S_1)^{1/2}(1+S_2)^{1/2} - 1, \quad W_2 = (1+S_1)^{1/2}(1+S_2)^{-1/2} - 1,$$

In particular, together with Shapiro's lemma this gives

$$\mathrm{H}^1(\mathbf{Q}, \mathbb{V}^\dagger_{f\boldsymbol{g}\boldsymbol{g}^*}) \simeq \mathrm{H}^1(K, T_f \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus \mathrm{H}^1(K, T_f \otimes \chi^{-1}\Psi_{W_2}^{1-\mathbf{c}}) \tag{3.4}$$

**Proposition 3.3.1.** *Suppose $\psi$ is a ray class character of $K$ such that $\chi = \psi/\psi^{\mathbf{c}}$ satisfies (3.2). Then under the isomorphism (3.4), the balanced Selmer group decomposes as*

$$\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^\dagger_{f\boldsymbol{g}\boldsymbol{g}^*}) \simeq \mathrm{Sel}_{\emptyset,0}(K, T_f \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus \mathrm{Sel}(K, T_f \otimes \chi^{-1}\Psi_{W_2}^{1-\mathbf{c}}),$$

*and the $\boldsymbol{f}$-unbalanced Selmer group decomposes as*

$$\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q}, \mathbb{V}^\dagger_{f\boldsymbol{g}\boldsymbol{g}^*}) \simeq \mathrm{Sel}(K, T_f \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus \mathrm{Sel}(K, T_f \otimes \chi^{-1}\Psi_{W_2}^{1-\mathbf{c}}).$$

*Proof.* This is shown in [CD23, Prop. 5.3.1]; we recall the details for the convenience of the reader. From (3.3), we see that

$$\mathbb{V}^\dagger_{f\boldsymbol{g}\boldsymbol{g}^*}|_{G_{\mathbf{Q}_p}} \simeq \big(T_f \otimes \Psi_{W_1}^{1-\mathbf{c}}\big) \oplus \big(T_f \otimes \Psi_{W_1}^{\mathbf{c}-1}\big) \oplus \big(T_f \otimes \chi^{-1}\Psi_{W_2}^{1-\mathbf{c}}\big) \oplus \big(T_f \otimes \chi\Psi_{W_2}^{\mathbf{c}-1}\big)$$

noting that $\chi^{\mathbf{c}} = \chi^{-1}$, and we find that the balanced local condition is given by

$$\mathscr{F}_p^{\mathrm{bal}}(\mathbb{V}^\dagger_{f\boldsymbol{g}\boldsymbol{g}^*}) \simeq \big(T_f \otimes \Psi_{W_1}^{1-\mathbf{c}}\big) \oplus \big(T_f^+ \otimes \chi^{-1}\Psi_{W_1}^{1-\mathbf{c}}\big) \oplus \big(T_f^+ \otimes \chi\Psi_{W_2}^{\mathbf{c}-1}\big), \tag{3.5}$$

where $T_f^+ \subset T_f$ is the specialisation of $V_{\boldsymbol{f}}^+$ at $Q_0$. Put $\widetilde{\mathbb{V}}^\dagger_{\boldsymbol{fgg}^*} = (T_f \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus (T_f \otimes \chi^{-1}\Psi_{W_2}^{1-\mathbf{c}})$, and note that

$$\mathrm{H}^1(\mathbf{Q}, \mathbb{V}^\dagger_{\boldsymbol{fgg}^*}) \simeq \mathrm{H}^1(K, \widetilde{\mathbb{V}}^\dagger_{\boldsymbol{fgg}^*}) \tag{3.6}$$

by Shapiro's lemma. For $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$, letting $\mathscr{F}_{\mathfrak{q}}^{\mathrm{bal}}(\widetilde{\mathbb{V}}^\dagger_{\boldsymbol{fgg}^*})$ be the submodule of $\widetilde{\mathbb{V}}^\dagger_{\boldsymbol{fgg}^*}$ corresponding to $\mathscr{F}_p^{\mathrm{bal}}(\mathbb{V}^\dagger_{\boldsymbol{fgg}^*})$, from (3.5) we have

$$\begin{cases} \mathscr{F}_{\mathfrak{p}}^{\mathrm{bal}}(\widetilde{\mathbb{V}}^\dagger_{\boldsymbol{fgg}^*}) \simeq \big(T_f \otimes \Psi_{W_1}^{1-\mathbf{c}}\big) \oplus \big(T_f^+ \otimes \chi^{-1}\Psi_{W_2}^{1-\mathbf{c}}\big), \\ \mathscr{F}_{\overline{\mathfrak{p}}}^{\mathrm{bal}}(\widetilde{\mathbb{V}}^\dagger_{\boldsymbol{fgg}^*}) \simeq T_f^+ \otimes \chi^{-1}\Psi_{W_2}^{1-\mathbf{c}}, \end{cases} \tag{3.7}$$

using again that $\chi^{\mathbf{c}} = \chi^{-1}$ for the second isomorphism. As a result, under the isomorphism (3.6) we have

$$\begin{aligned} &\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^\dagger_{\boldsymbol{fgg}^*}) \simeq \\ &\quad \ker\Big\{\mathrm{H}^1(G_{K,\Sigma}, \widetilde{\mathbb{V}}^\dagger_{\boldsymbol{fgg}^*}) \to \frac{\mathrm{H}^1(K_{\mathfrak{p}}, \widetilde{\mathbb{V}}^\dagger_{\boldsymbol{fgg}^*})}{\mathrm{H}^1(K_{\mathfrak{p}}, (T_f \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus (T_f^+ \otimes \chi^{-1}\Psi_{W_2}^{1-\mathbf{c}}))} \times \frac{\mathrm{H}^1(K_{\overline{\mathfrak{p}}}, \widetilde{\mathbb{V}}^\dagger_{\boldsymbol{fgg}^*})}{\mathrm{H}^1(K_{\overline{\mathfrak{p}}}, T_f^+ \otimes \chi^{-1}\Psi_{W_2}^{1-\mathbf{c}})}\Big\}, \end{aligned}$$

yielding the result in this case. Similarly, we find that the $\boldsymbol{f}$-balanced local condition is given by

$$\begin{cases} \mathscr{F}_{\mathfrak{p}}^{\boldsymbol{f}}(\widetilde{\mathbb{V}}^\dagger_{\boldsymbol{fgg}^*}) \simeq \big(T_f^+ \otimes \Psi_{W_1}^{1-\mathbf{c}}\big) \oplus \big(T_f^+ \otimes \chi^{-1}\Psi_{W_2}^{1-\mathbf{c}}\big), \\ \mathscr{F}_{\overline{\mathfrak{p}}}^{\boldsymbol{f}}(\widetilde{\mathbb{V}}^\dagger_{\boldsymbol{fgg}^*}) \simeq \big(T_f^+ \otimes \Psi_{W_1}^{1-\mathbf{c}}\big) \oplus \big(T_f^+ \otimes \chi^{-1}\Psi_{W_2}^{1-\mathbf{c}}\big), \end{cases}$$

and as above we arrive at the claimed description of $\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q}, \mathbb{V}^\dagger_{\boldsymbol{fgg}^*})$. □

As a consequence, we also obtain decompositions for corresponding Selmer groups with coefficients in $\mathbb{A}^\dagger_{\boldsymbol{fgg}^*} = \mathrm{Hom}_{\mathbf{Z}_p}(\mathbb{V}^\dagger_{\boldsymbol{fgg}^*}, \mu_{p^\infty})$, mirroring in the case of $\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q}, \mathbb{A}^\dagger_{\boldsymbol{fgg}^*})$ the factorisation of $p$-adic $L$-functions in Proposition 2.4.1. Put $A_f = \mathrm{Hom}_{\mathbf{Z}_p}(T_f, \mu_{p^\infty})$.

**Corollary 3.3.2.** *Under the asumption in Proposition 3.3.1, we have the decompositions*

$$\begin{aligned} \mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{A}^\dagger_{\boldsymbol{fgg}^*}) &\simeq \mathrm{Sel}_{0,\emptyset}(K, A_f \otimes \Psi_{W_1}^{\mathbf{c}-1}) \oplus \mathrm{Sel}(K, A_f \otimes \chi\Psi_{W_2}^{\mathbf{c}-1}), \\ \mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q}, \mathbb{A}^\dagger_{\boldsymbol{fgg}^*}) &\simeq \mathrm{Sel}(K, A_f \otimes \Psi_{W_1}^{\mathbf{c}-1}) \oplus \mathrm{Sel}(K, A_f \otimes \chi\Psi_{W_2}^{\mathbf{c}-1}). \end{aligned}$$

*Proof.* This is immediate from Proposition 3.3.1 and local Tate duality. □

## 4. Iwasawa main conjectures

Keeping the setting from §3.3, and put $\Lambda_2 = \mathscr{O}[\![S_1, S_2]\!]$ and

$$\mathbb{V}^\dagger_{\boldsymbol{f}\underline{\boldsymbol{gg}}^*} = \mathbb{V}^\dagger_{\boldsymbol{fgg}^*} \hat{\otimes}_{\Lambda_2} \Lambda_2/(S_1 - S_2),$$

where $\Lambda_2 = \mathscr{O}[\![S_1, S_2]\!]$. In this section we recall the diagonal cycle main conjecture formulated in [ACR23] specialised to our setting, and explain its relation with the anticyclotomic Iwasawa main conjecture for $f$.

4.1. **Diagonal cycle main conjecture.** Let $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{g}^*)\in \mathrm{H}^1(\mathbf{Q},\mathbf{V}^\dagger)$ be the *big diagonal class* constructed in [BSV22b, §8.1], where $\boldsymbol{f}$ is the primitive Hida family passing through $f$, and denote by

$$\kappa(f,\underline{\boldsymbol{gg}}^*)\in \mathrm{H}^1(\mathbf{Q},\mathbb{V}^\dagger_{f\underline{\boldsymbol{gg}}^*}) \tag{4.1}$$

its specialisation. More precisely, from *loc. cit.* one directly obtains a class as above with coefficient in a representation $\mathbf{V}^\dagger(N)$ isomorphic (non-canonically) to finitely many copies of $\mathbf{V}^\dagger$, where $N=\mathrm{lcm}(N_f,N_g,N_{g^*})=\mathrm{lcm}(N_f,D_K\ell)$; to obtain (4.1) we apply the projection $\mathbf{V}^\dagger(N)\to\mathbf{V}^\dagger$ corresponding to the choice of level-$N$ test vectors for $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{g}^*)$ furnished by [Hsi21, Thm. A].

From the decompositions in Proposition 3.3.1 we obtain

$$\begin{aligned}\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbb{V}^\dagger_{f\underline{\boldsymbol{gg}}^*})&\simeq \mathrm{Sel}_{\emptyset,0}(K,T_f\otimes\Psi_T^{1-\mathbf{c}})\oplus\mathrm{Sel}(K,T_f\otimes\chi^{-1}\Psi_{W_2}^{1-\mathbf{c}})_{/W_2},\\ \mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q},\mathbb{V}^\dagger_{f\underline{\boldsymbol{gg}}^*})&\simeq \mathrm{Sel}(K,T_f\otimes\Psi_T^{1-\mathbf{c}})\oplus\mathrm{Sel}(K,T_f\otimes\chi^{-1}\Psi_{W_2}^{1-\mathbf{c}})_{/W_2},\end{aligned} \tag{4.2}$$

where $T=T_1=\mathbf{v}^{-1}(1+S_1)-1$ and the subscript $/W_2$ denotes the cokernel of multiplication by $W_2$. Put

$$\mathscr{F}_p^3(\mathbb{V}^\dagger_{f\boldsymbol{gg}^*}):=T_f^+\hat{\otimes}_{\mathscr{O}}V_{\boldsymbol{f}}^+\hat{\otimes}_{\mathscr{O}}V_{\boldsymbol{g}^*}^+\otimes\mathcal{X}^{-1},$$

and denote by $\mathscr{F}_p^3(\mathbb{V}^\dagger_{f\underline{\boldsymbol{gg}}^*})$ the resulting submodule of $\mathbb{V}^\dagger_{f\underline{\boldsymbol{gg}}^*}$. Similarly defining $\mathscr{F}_p^{\mathrm{bal}}(\mathbb{V}^\dagger_{f\underline{\boldsymbol{gg}}^*})$ from $\mathscr{F}_p^{\mathrm{bal}}(\mathbb{V}^\dagger_{f\boldsymbol{gg}^*})=\mathscr{F}_p^2(\mathbb{V}^\dagger_{f\boldsymbol{gg}^*})$, in terms of the description in (3.7) we find that

$$\begin{cases}\mathscr{F}_{\mathfrak{p}}^{\mathrm{bal}}(\widetilde{\mathbb{V}}^\dagger_{f\underline{\boldsymbol{gg}}^*})/\mathscr{F}_{\mathfrak{p}}^3(\widetilde{\mathbb{V}}^\dagger_{f\underline{\boldsymbol{gg}}^*})\simeq\big(T_f^-\otimes\Psi_T^{1-\mathbf{c}}\big)\oplus\big(T_f^+\otimes\chi^{-1}\Psi_{W_2}^{1-\mathbf{c}}\big)_{/W_2},\\ \mathscr{F}_{\overline{\mathfrak{p}}}^{\mathrm{bal}}(\widetilde{\mathbb{V}}^\dagger_{f\underline{\boldsymbol{gg}}^*})/\mathscr{F}_{\overline{\mathfrak{p}}}^3(\widetilde{\mathbb{V}}^\dagger_{f\underline{\boldsymbol{gg}}^*})\simeq(T_f^+\otimes\chi^{-1}\Psi_{W_2}^{1-\mathbf{c}})_{/W_2},\end{cases} \tag{4.3}$$

where $T_f^-:=T_f/T_f^+$.

It follows from [BSV22b, Cor. 8.2] that $\kappa(f,\underline{\boldsymbol{gg}}^*)$ lies in the balanced Selmer group $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbb{V}^\dagger_{f\underline{\boldsymbol{gg}}^*})$. Therefore, viewing this class in $\mathrm{H}^1(K,\widetilde{\mathbb{V}}^\dagger_{f\underline{\boldsymbol{gg}}^*})$ via the isomorphism (3.6), we can consider the image of $\mathrm{loc}_{\mathfrak{p}}(\kappa(f,\underline{\boldsymbol{gg}}^*))$ under the natural map

$$p_f^-:\mathrm{H}^1\big(K_{\mathfrak{p}},\mathscr{F}_{\mathfrak{p}}^{\mathrm{bal}}(\widetilde{\mathbb{V}}^\dagger_{f\underline{\boldsymbol{gg}}^*})\big)\to\mathrm{H}^1\big(K_{\mathfrak{p}},\mathscr{F}_{\mathfrak{p}}^{\mathrm{bal}}(\widetilde{\mathbb{V}}^\dagger_{f\underline{\boldsymbol{gg}}^*})/\mathscr{F}_{\mathfrak{p}}^3(\widetilde{\mathbb{V}}^\dagger_{f\underline{\boldsymbol{gg}}^*})\big)\to\mathrm{H}^1(K_{\mathfrak{p}},T_f^-\otimes\Psi_T^{1-\mathbf{c}})$$

arising from the projection onto the first factor in (4.3).

Put

$$\Lambda=\mathscr{O}[\![T]\!].$$

Let $u:=1+p$, and for any $\Lambda$-module $M$ and integer $k$, denote by $M_k$ the specialisation of $M$ at $T=u^{k-2}-1$. Letting $\Phi$ be the field of fractions of $\mathscr{O}$, it is then easy to see that the Bloch–Kato logarithm and dual exponential maps define isomorphisms

$$\begin{aligned}\log_{\mathfrak{p}}:\mathrm{H}^1(K_{\mathfrak{p}},T_f^-\otimes\Psi_T^{1-\mathbf{c}})_k\otimes\mathbf{Q}_p\to\Phi,\quad k\geq 3,\\ \exp^*_{\mathfrak{p}}:\mathrm{H}^1(K_{\mathfrak{p}},T_f^-\otimes\Psi_T^{1-\mathbf{c}})_k\otimes\mathbf{Q}_p\to\Phi,\quad k=2.\end{aligned} \tag{4.4}$$

The following fundamental result due to Bertolini–Seveso–Venerucci and Darmon–Rotger relates $p_f^-(\mathrm{loc}_{\mathfrak{p}}(\kappa(f,\underline{\boldsymbol{gg}}^*)))$ to the restricted triple product $p$-adic $L$-function $\mathscr{L}_p^{\boldsymbol{f}}(f,\underline{\boldsymbol{gg}}^*)$ in (2.7).

**Theorem 4.1.1** (Explicit reciprocity law)**.** *There is an injective $\Lambda$-module homomorphism*

$$\mathrm{Log}^{\boldsymbol{f}}:\mathrm{H}^1(K_{\mathfrak{p}},T_f^-\otimes\Psi_T^{1-\mathbf{c}})\to\Lambda$$

*with pseudo-null cokernel satisfying for any* $\mathfrak{Z} \in \mathrm{H}^1(K_{\mathfrak{p}}, T_f^- \otimes \Psi_T^{1-\mathbf{c}})$ *the interpolation property*

$$\mathrm{Log}^{\boldsymbol{f}}(\mathfrak{Z})_k = c_k \times \begin{cases} \log_{\mathfrak{p}}(\mathfrak{Z}_k) & \text{if } k \geq 3, \\ \exp_{\mathfrak{p}}(\mathfrak{Z}_k) & \text{if } k = 2, \end{cases}$$

*where* $c_k$ *is an explicit nonzero constant, and such that*

$$\mathrm{Log}^{\boldsymbol{f}}\left(p_f^-(\mathrm{loc}_{\mathfrak{p}}(\kappa(f, \underline{\boldsymbol{gg}}^*)))\right) = \mathscr{L}_p^{\boldsymbol{f}}(f, \underline{\boldsymbol{gg}}^*).$$

*Proof.* The map $\mathrm{Log}^{\boldsymbol{f}}$ is obtained by specialising the three-variable big regulator map of [BSV22b, §7.1], and the explicit reciprocity law is a consequence of the results of [BSV22b, Thm. A] and [DR22, Thm. 10]. The details are explained in [ACR23, Prop. 7.3] and [CD23, Thm. 5.1.1]. □

The next result shows the equivalence between a "diagonal cycle main conjecture" in the spirit of [PR87, Conj. B] and the Iwasawa–Greenberg main conjecture for $\mathscr{L}_p^{\boldsymbol{f}}(f, \underline{\boldsymbol{gg}}^*)$.

**Proposition 4.1.2.** *The following statements (1)-(2) are equivalent:*

(1) $\kappa(f, \underline{\boldsymbol{gg}}^*)$ *is not* $\Lambda$*-torsion,*

$$\mathrm{rank}_\Lambda\left(\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^\dagger_{f\underline{\boldsymbol{gg}}^*})\right) = \mathrm{rank}_\Lambda\left(X^{\mathrm{bal}}(\mathbf{Q}, \mathbb{A}^\dagger_{f\underline{\boldsymbol{gg}}^*})\right) = 1,$$

*and the following equality holds in* $\Lambda \otimes \mathbf{Q}_p$*:*

$$\mathrm{char}_\Lambda\left(X^{\mathrm{bal}}(\mathbf{Q}, \mathbb{A}^\dagger_{f\underline{\boldsymbol{gg}}^*})_{\mathrm{tors}}\right) = \mathrm{char}_\Lambda\left(\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^\dagger_{f\underline{\boldsymbol{gg}}^*})/(\kappa(f, \underline{\boldsymbol{gg}}^*))\right)^2.$$

*where the subscript* tors *denotes the* $\Lambda$*-torsion submodule.*

(2) $\mathscr{L}_p^{\boldsymbol{f}}(f, \underline{\boldsymbol{gg}}^*)$ *is nonzero, the modules* $\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q}, \mathbb{V}^\dagger_{f\underline{\boldsymbol{gg}}^*})$ *and* $X^{\boldsymbol{f}}(\mathbf{Q}, \mathbb{A}^\dagger_{f\underline{\boldsymbol{gg}}^*})$ *are both* $\Lambda$*-torsion, and*

$$\mathrm{char}_\Lambda\left(X^{\boldsymbol{f}}(\mathbf{Q}, \mathbb{A}^\dagger_{f\underline{\boldsymbol{gg}}^*})\right) = \left(\mathscr{L}_p^{\boldsymbol{f}}(f, \underline{\boldsymbol{gg}}^*)^2\right)$$

*in* $\Lambda \otimes \mathbf{Q}_p$*.*

*Proof.* This follows from an argument by now standard using Theorem 4.1.1 and global duality as in [ACR23, Thm. 7.15]. □

4.2. **Anticyclotomic main conjecture for modular forms.** For our later use, here we record some known results on the anticyclotomic Iwasawa main conjecture for modular forms.

Let $\mathbb{F}$ be the residue field of $\mathscr{O}$, and denote by

$$\bar{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbb{F})$$

the residual representation associated to $f$. Factor $N_f = N^+N^-$ as in (2.5). Following [KPW17], we say that *Assumption (A)* holds if:

- $\bar{\rho}_f$ is absolutely irreducible,
- $\bar{\rho}_f$ is ramified at every prime $q \mid N^-$ with $q \equiv \pm 1 \pmod{p}$,
- either $p > 5$ or the image of $\bar{\rho}_f$ contains a conjugate of $\mathrm{GL}_2(\mathbb{F}_p)$.

Note that this is the same as Assumption (A) from [KPW17], except that the latter also includes the non-anomalous condition $a_p(f) \not\equiv 1 \pmod{p}$; that the next result holds without this condition follows from recent advances on Ihara's lemma, [MS21].

**Theorem 4.2.1.** *Suppose that the ordinary $p$-stabilised newform $f \in S_2(\Gamma_0(pN_f))$ is $p$-old, and that $N^-$ is the squarefree product of an odd number of primes. Under Assumption (A), the module $\mathrm{Sel}(K, A_f \otimes \Psi_T^{\mathbf{c}-1})$ is $\Lambda$-cotorsion, and*

$$\mathrm{char}_\Lambda\big(\mathrm{Sel}(K, A_f \otimes \Psi_T^{\mathbf{c}-1})^\vee\big) = \big(\Theta_{f/K}(T)^2\big)$$

*in $\Lambda \otimes \mathbf{Q}_p$.*

*Proof.* This follows from the combination of the main result of Bertolini–Darmon [BD05], as later refined by several authors (see [PW11, CH15, KPW17]), showing the $\Lambda$-cotornionness $\mathrm{Sel}(K, A_f \otimes \Psi_T^{\mathbf{c}-1})$ and the integral divisibility

$$\mathrm{char}_\Lambda\big(\mathrm{Sel}(K, A_f \otimes \Psi_T^{\mathbf{c}-1})^\vee\big) \supset \big(\Theta_{f/K}(T)^2\big), \tag{4.5}$$

and the converse divisibility that follows from the two-variable main conjecture for $f/K$ proved in the work of Skinner–Urban and Wan [SU14, Wan15] (after inverting $p$ in the latter case) restricted to the anticyclotomic line. □

**Remark 4.2.2.** By using the Euler system divisibility from [CD23, Thm. 5.6.1] (using diagonal cycles rather than congruence arguments and Heegner points), the ramification hypothesis on $\bar{\rho}_f$ in Theorem 4.2.1 can be replaced by the "big image" hypothesis in [*op. cit.*, §3.3.2].

## 5. Main result

In this section we state and prove the main result of this note toward the nonvanishing conjectures of [DR16] in the setting of rank two elliptic curves.

5.1. **Generalised Kato classes.** Let $E/\mathbf{Q}$ be an elliptic curve of conductor $N_f$, and let $p > 3$ a prime of good ordinary reduction for $E$. Let $f \in S_2(\Gamma_0(pN_f))$ be the ordinary $p$-stabilisation of the newform associated to $E$, and let $(\boldsymbol{g}, \boldsymbol{g}^*) = (\boldsymbol{\theta}_\psi(S_1), \boldsymbol{\theta}_{\psi^{-1}}(S_2))$ be a dual pair of primitive CM Hida families as in (2.6). When $\chi = \psi/\psi^{\mathbf{c}}$ satisfies (3.2), we have

$$\mathrm{H}^1(\mathbf{Q}, \mathbb{V}^\dagger_{f\underline{\boldsymbol{g}\boldsymbol{g}^*}}) \simeq \mathrm{H}^1(K, T_f \otimes \Psi_T^{1-\mathbf{c}}) \oplus \mathrm{H}^1(K, T_f \otimes \chi^{-1}\Psi_{W_2}^{1-\mathbf{c}})_{/W_2} \tag{5.1}$$

by specialising (3.4). The following is a special case of the construction of generalised Kato classes by Darmon–Rotger [DR16] in the *adjoint case.*

**Definition 5.1.1.** Let $\kappa_p(E)$ be the image of $\kappa(f, \underline{\boldsymbol{g}\boldsymbol{g}^*})$ under the composition

$$\mathrm{H}^1(\mathbf{Q}, \mathbb{V}^\dagger_{f\underline{\boldsymbol{g}\boldsymbol{g}^*}}) \to \mathrm{H}^1(K, T_f \otimes \Psi_T^{1-\mathbf{c}}) \to \mathrm{H}^1(K, T_f) \simeq \mathrm{H}^1(K, T_pE),$$

where the first arrow is the projection onto the first direct summand in (5.1), and the second is induced by the multiplication by $T$ on $T_f \otimes \Psi_T^{1-\mathbf{c}}$.

By (4.2), the inclusion $\kappa(f, \underline{\boldsymbol{g}\boldsymbol{g}^*}) \in \mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^\dagger_{f\underline{\boldsymbol{g}\boldsymbol{g}^*}})$ from [BSV22b, Cor. 8.2] implies that

$$\kappa_p(E) \in \mathrm{Sel}_{\emptyset,0}(K, T_pE). \tag{5.2}$$

In the following we shall view $\kappa_p(E)$ as a class with coefficients in $V_pE$.

**Proposition 5.1.2.** *The following implication holds*

$$L(f/K, 1) = 0 \quad \Longrightarrow \quad \kappa_p(E) \in \mathrm{Sel}(K, V_pE).$$

*Proof.* By (5.2), the inclusion $\kappa_p(E) \in \mathrm{Sel}(K, V_pE)$ holds if and only if $\mathrm{res}_{\mathfrak{p}}(\kappa_p(E))$ vanishes under the natural map $\mathrm{H}^1(K_{\mathfrak{p}}, V_f) \to \mathrm{H}^1(K_{\mathfrak{p}}, V_f^-)$. Since from the combination of Theorem 4.1.1, Proposition 2.4.1, and Theorem 2.3.1 yield the equivalence

$$\exp_{\mathfrak{p}}(\mathrm{res}_{\mathfrak{p}}(\kappa_p(E))) = 0 \quad \Longleftrightarrow \quad L(f/K, 1) \cdot L(f/K \otimes \chi, 1) = 0,$$

and the Bloch–Kato dual in (4.4) is an isomorphism, the result follows. □

Denote by $E^K$ the twist of $E$ by the quadratic character corresponding to $K/\mathbf{Q}$.

**Corollary 5.1.3.** *Suppose $L(E, 1) = 0$ and $L(E^K, 1) \neq 0$. Then $\kappa_p(E) \in \mathrm{Sel}(\mathbf{Q}, V_pE)$.*

*Proof.* Since Kato's results [Kat04] show that $\mathrm{Sel}(\mathbf{Q}, V_pE^K) = 0$ when $L(E^K, 1) \neq 0$, this follows from Proposition 5.1.2 and the isomorphism $\mathrm{Sel}(K, V_pE) \simeq \mathrm{Sel}(\mathbf{Q}, V_pE) \oplus \mathrm{Sel}(\mathbf{Q}, V_pE^K)$. □

**Remark 5.1.4.** The proof of Proposition 5.1.2 show that when $L(f/K, 1) = 0$, the class $\kappa_p(E)$ lies in fact in the kernel of the restriction map on $\mathrm{Sel}(K, V_pE)$ at the primes above $p$. In particular, in the setting of Corollary 5.1.3, $\kappa_p(E)$ lands in the *strict* Selmer group

$$\mathrm{Sel}_0(\mathbf{Q}, V_pE) = \ker\left\{\mathrm{Sel}(\mathbf{Q}, V_pE) \xrightarrow{\mathrm{loc}_p} E(\mathbf{Q}_p)\hat{\otimes}\mathbf{Q}_p\right\}.$$

5.2. **Nonvanishing of $\kappa_p(E)$ in rank two.** Denote by

$$\rho_f : G_{\mathbf{Q}} \to \mathrm{Aut}_{\mathbf{F}_p}(E[p]) \simeq \mathrm{GL}_2(\mathbf{F}_p)$$

the mod $p$ Galois representation associated to $E$, and assume that:

(h1) $\bar{\rho}_f$ is absolutely irreducible,
(h2) there exists a prime $q \| N_f$, and if $q \equiv \pm 1 \pmod{p}$ then $\bar{\rho}_f$ is ramified at $q$,
(h3) either $p > 5$ of the image of $\bar{\rho}_f$ contains a conjugate of $\mathrm{GL}_2(\mathbb{F}_p)$.

We shall consider generalised Kato classes as in §5.1 attached to the following.

**Choice 5.2.1.** Let $q$ be a prime as in (h2) above, and let $\ell \nmid 6pN_f$ be a prime. Choose an imaginary quadratic field $K$ and a ring class character $\chi$ of $K$ of conductor dividing $\ell^\infty \mathcal{O}_K$ such that:

(1) $q$ is inert in $K$,
(2) every prime factor of $N_f/q$ splits in $K$,
(3) $\ell$ splits in $K$,
(4) $\ell$ is ordinary for $E$,
(5) $L(E^K, 1) \neq 0$ and $L(E/K, \chi, 1) \neq 0$,
(6) $\chi|_{G_{K_{\mathfrak{p}}}} \neq 1$.

**Remark 5.2.2.** The existence of (infinitely many) $K$ satisfying (1)–(3) and such that $L(E^K, 1) \neq 0$ follows from [BFH90]; for any such $K$, the results of [Vat03] (see also [CH18, Thm. D]) ensure that $L(E/K, \chi, 1) \neq 0$ for all but finitely many $\chi$ of $\ell$-power conductor.

Fix $(K, \chi)$ as in Choice 5.2.1. Write $\chi = \psi/\psi^{\mathbf{c}}$ with $\psi$ a ray class character modulo $\ell^m \mathcal{O}_K$ for some $m > 0$, let $(\boldsymbol{g}, \boldsymbol{g}^*) = (\boldsymbol{\theta}_\psi(S_1), \boldsymbol{\theta}_{\psi^{-1}}(S_2))$ be the associated primitive CM Hida families as in (2.6), and let $\kappa_p(E) \in \mathrm{H}^1(K, V_pE)$ the corresponding generalised Kato class.

By Corollary 5.1.3, when $L(E, 1) = 0$ we have $\kappa_p(E) \in \mathrm{Sel}(\mathbf{Q}, V_pE)$. Moreover, by our choice of $K$ and $\chi$, when further $E/\mathbf{Q}$ has sign $+1$ (so in particular $\mathrm{ord}_{s=1} L(E, s) \geq 2$), the nonvanishing conjecture of Darmon–Rotger [DR16, Conj. 3.2] (as specialised in [*op. cit.*, §4.5.3] to the setting of rank two elliptic curves) predicts the equivalence

$$\kappa_p(E) \neq 0 \quad \overset{?}{\Longleftrightarrow} \quad \dim_{\mathbf{Q}_p} \mathrm{Sel}(\mathbf{Q}, V_pE) = 2.$$

The following is the main result of this note in the direction of this conjecture.

**Theorem 5.2.3.** *Suppose that $L(E, s)$ vanishes to positive even order at $s = 1$. Then*

$$\kappa_p(E) \neq 0 \quad \Longrightarrow \quad \dim_{\mathbf{Q}_p} \mathrm{Sel}(\mathbf{Q}, V_pE) = 2.$$

*Conversely, if $\dim_{\mathbf{Q}_p} \mathrm{Sel}(\mathbf{Q}, V_pE) = 2$ then $\kappa_p(E) \neq 0$ if and only if the restriction map*

$$\mathrm{loc}_p : \mathrm{Sel}(\mathbf{Q}, V_pE) \to E(\mathbf{Q}_p)\hat{\otimes}\mathbf{Q}_p$$

*is nonzero. Moreover, in that case $\kappa_p(E)$ spans the strict Selmer group $\mathrm{Sel}_0(\mathbf{Q}, V_pE) = \ker(\mathrm{loc}_p)$.*

**Remark 5.2.4.** Theorem 5.2.3 recovers [CH22, Thm. A] under slightly weaker hypotheses on $\bar{\rho}_f$, and moreover, shows that the condition $\mathrm{Sel}(\mathbf{Q}, V_pE) \neq \ker(\mathrm{loc}_p)$ is *necessary* for the nonvanishing of $\kappa_p(E)$ when $\mathrm{Sel}(\mathbf{Q}, V_pE)$ is 2-dimensional.

**Remark 5.2.5.** When $\mathrm{Sel}(\mathbf{Q}, V_pE)$ is 2-dimensional and different from $\ker(\mathrm{loc}_p)$, it follows from Theorem 5.2.3 that

$$\kappa_p(E) = \lambda \cdot \big(\log_p(Q)P - \log_p(P)Q\big),$$

for some $\lambda \in \mathbf{Q}_p^\times$, where $(P, Q)$ is any basis for $\mathrm{Sel}(\mathbf{Q}, V_pE)$ and $\log_p$ is the composition of $\mathrm{loc}_p$ with the formal group logarithm on $E(\mathbf{Q}_p)\hat{\otimes}\mathbf{Q}_p$. This is consistent with the refined conjecture in [DR16, Conj. 3.12] specialised to the setting of [*op. cit.*,§4.5.3], which further predicts a rationality statement for $\lambda$ (see also [BSV22a, Conj. 3.4, Rem. 3.5.2]).

The rest of this section is devoted to the proof of Theorem 5.2.3.

5.3. **Some Galois cohomology.** Put $K_p = K \otimes_{\mathbf{Q}} \mathbf{Q}_p \simeq K_{\mathfrak{p}} \oplus K_{\overline{\mathfrak{p}}}$. Let

$$\mathrm{loc}_p = \mathrm{loc}_{\mathfrak{p}} \oplus \mathrm{loc}_{\overline{\mathfrak{p}}} : \mathrm{H}^1(G_{K,\Sigma}, V_pE) \to \mathrm{H}^1(K_p, V_pE) \tag{5.3}$$

be the restriction map, and denote by $X_f$ (resp. $X_{\mathfrak{p}}$) the image of $\mathrm{Sel}(K, V_pE)$ (resp. $\mathrm{Sep}_{\emptyset,0}(K, V_pE)$) under $\mathrm{loc}_p$.

**Lemma 5.3.1.** *We have $\dim_{\mathbf{Q}_p} \mathrm{Sel}_{\emptyset,0}(K, V_pE) = \dim_{\mathbf{Q}_p} \mathrm{Sel}(K, V_pE) + \delta$, where*

$$\delta = \begin{cases} -1 & \text{if } X_f \neq 0, \\ 0 & \text{if } X_f = X_{\mathfrak{p}} = 0, \\ 1 & \text{if } X_f = 0 \text{ and } X_{\mathfrak{p}} \neq 0. \end{cases}$$

*Proof.* Since $X_f$ is invariant under the action of complex conjugation, if $X_f$ is nonzero then the restriction maps $\mathrm{Sel}(K, V_pE) \to \mathrm{H}^1(K_{\mathfrak{q}}, V_pE)$ for $\mathfrak{q} \in \{\mathfrak{p}, \overline{\mathfrak{p}}\}$ are both nonzero, and so in this case the result follows from [CHK+23, Lem. 2.2].

Now suppose $X_f = 0$, so that $\mathrm{Sel}(K, V_pE)$ is the same as the strict Selmer group $\mathrm{Sel}_{\mathrm{str}}(K, V_pE) := \ker(\mathrm{loc}_p)$, and consider the exact sequence

$$0 \to \mathrm{Sel}_{\mathrm{str}}(K, V_pE) \to \mathrm{Sel}_{\emptyset,0}(K, V_pE) \xrightarrow{\mathrm{loc}_{\mathfrak{p}}} \mathrm{H}^1(K_{\mathfrak{p}}, V_pE). \tag{5.4}$$

If $X_{\mathfrak{p}} = 0$, then (5.4) yields the result, so it remains to consider the case $X_{\mathfrak{p}} \neq 0$. If $\dim_{\mathbf{Q}_p} X_{\mathfrak{p}} = 2$, then the image of (5.3) contains $X_{\mathfrak{p}} \oplus X_{\overline{\mathfrak{p}}}$, where $X_{\overline{\mathfrak{p}}}$ is the image of $X_{\mathfrak{p}}$ under complex conjugation (equivalently, the image of $\mathrm{Sel}_{0,\emptyset}(K, V_pE)$), but this contradicts [Ski20, Lem. 2.3.1]. Therefore if $X_{\mathfrak{p}} \neq 0$ then it is one-dimensional, and the result in this case follows again from (5.4). □

**Remark 5.3.2.** Standard conjectures predict that $X_f \neq 0$ unless $\mathrm{Sel}(K, V_pE) = 0$ (indeed, the vanishing of $X_f$ when $\mathrm{Sel}(K, V_pE) \neq 0$ would imply that $\text{Ш}(E/K)[p^\infty]$ is infinite).

Recall that we set $\Lambda = \mathcal{O}[\![T]\!]$, and for any $\Lambda$-module $M$ denote by $M_{/T} = M/TM$ the cokernel of multiplication by $T$. The following is a variant of Mazur's control theorem [Maz72].

**Proposition 5.3.3.** *Multiplication by $T$ induces natural maps*

$$r^* : \mathrm{Sel}_{0,\emptyset}(K, A_f) \to \mathrm{Sel}_{0,\emptyset}(K, A_f \otimes \Psi_T^{\mathbf{c}-1})[T],$$
$$r : \mathrm{Sel}_{\emptyset,0}(K, T_f \otimes \Psi_T^{1-\mathbf{c}})_{/T} \to \mathrm{Sel}_{\emptyset,0}(K, T_f)$$

*with finite kernel and cokernel.*

*Proof.* Letting $\mathrm{Sel}_{\mathrm{rel}}(K, A_f)$ and $\mathrm{Sel}_{\mathrm{rel}}(L, A_f \otimes \Psi_T^{\mathbf{c}-1})$ be the Selmer groups defined in the same manner as $\mathrm{Sel}_{0,\emptyset}(K, A_f)$ and $\mathrm{Sel}_{\emptyset,0}(L, A_f \otimes \Psi_T^{\mathbf{c}-1})$, respectively, but with the (propagated in the case of $A_f$) relaxed local condition at the primes above $p$, the map $r^*$ fits into the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Sel}_{0,\emptyset}(K, A_f) & \longrightarrow & \mathrm{Sel}_{\mathrm{rel}}(K, A_f) & \longrightarrow & \mathrm{H}^1(K_{\mathfrak{p}}, A_f) \times \frac{\mathrm{H}^1(K_{\overline{\mathfrak{p}}}, A_f)}{\mathrm{H}^1(K_{\overline{\mathfrak{p}}}, A_f)_{\mathrm{div}}} \\ & \downarrow{\scriptstyle r^*} & & \downarrow{\scriptstyle s^*} & & \downarrow{\scriptstyle t^*} \\ 0 \longrightarrow & \mathrm{Sel}_{0,\emptyset}(K, A_f \otimes \Psi_T^{\mathbf{c}-1})[T] & \longrightarrow & \mathrm{Sel}_{\mathrm{rel}}(K, A_f \otimes \Psi_T^{\mathbf{c}-1})[T] & \longrightarrow & \mathrm{H}^1(K_{\mathfrak{p}}, A_f \otimes \Psi_T^{\mathbf{c}-1})[T] \times \{0\}, \end{array}$$

where $\mathrm{H}^1(K_{\overline{\mathfrak{p}}}, A_f)_{\mathrm{div}}$ denotes the maximal divisible submodule of $\mathrm{H}^1(K_{\overline{\mathfrak{p}}}, A_f)$. The map $t^*$ arises from the cohomology long exact sequence associated to

$$0 \to A_f \to A_f \otimes \Psi_T^{\mathbf{c}-1} \xrightarrow{\times T} A_f \otimes \Psi_T^{\mathbf{c}-1} \to 0,$$

and therefore is surjective with kernel $\mathrm{H}^0(K_{\mathfrak{p}}, A_f \otimes \Psi_T^{\mathbf{c}-1})_{/T}$. Since [KO20, Lem. 2.7] implies that $\#\mathrm{H}^0(K_{\mathfrak{p}}, A_f \otimes \Psi_T^{\mathbf{c}-1}) < \infty$, by the Snake Lemma to prove the stated property for $r^*$ it suffices to show that $s^*$ has finite kernel and cokernel.

The latter map fits into the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Sel}_{\mathrm{rel}}(K, A_f) & \longrightarrow & \mathrm{H}^1(G_{K,\Sigma}, A_f) & \longrightarrow & \bigoplus_{w \in \Sigma, w \nmid p} \frac{\mathrm{H}^1(K_w, A_f)}{\mathrm{H}^1_f(K_w, A_f)} \\ & \downarrow{\scriptstyle s^*} & & \downarrow{\scriptstyle u^*} & & \downarrow{\scriptstyle v^*} \\ 0 \longrightarrow & \mathrm{Sel}_{\mathrm{rel}}(K, A_f \otimes \Psi_T^{\mathbf{c}-1})[T] & \longrightarrow & \mathrm{H}^1(G_{K,\Sigma}, A_f \otimes \Psi_T^{\mathbf{c}-1})[T] & \longrightarrow & \bigoplus_{w \in \Sigma, w \nmid p} \mathrm{H}^1(K_w, A_f \otimes \Psi_T^{\mathbf{c}-1})[T], \end{array}$$

where $\mathrm{H}^1_f(K_w, A_f)$ is the natural image of $\ker\{\mathrm{H}^1(K_w, V_f) \to \mathrm{H}^1(K_w^{\mathrm{ur}}, V_f)\}$ (see Definition 3.2.1). The finiteness of $\mathrm{H}^0(K_{\mathfrak{p}}, A_f \otimes \Psi_T^{\mathbf{c}-1})$ implies that of $\mathrm{H}^0(G_{K,\Sigma}, A_f \otimes \Psi_T^{\mathbf{c}-1})$, and so the map $u^*$ has finite kernel. Since $u^*$ is clearly surjective and by [Gre99, Lem. 3.3] (see also [JSW17, §3.3.6]) the kernel of the map $v^*$ is finite, by the Snake Lemma it follows that $s^*$ has the desired properties. This shows the result for $r^*$ and the case of $r$ is shown similarly. ☐

## 5.4. **Proof of the main result.**

*Proof of Theorem 5.2.3.* The decomposition in Corollary 3.3.2 gives

$$X^{\boldsymbol{f}}(\mathbf{Q}, \mathbb{A}^\dagger_{\boldsymbol{f}\underline{\boldsymbol{g}\boldsymbol{g}^*}}) \simeq \mathrm{Sel}(K, A_f \otimes \Psi_T^{\mathbf{c}-1})^\vee \oplus \big(\mathrm{Sel}(K, A_f \otimes \chi\Psi_T^{\mathbf{c}-1})[T]\big)^\vee. \tag{5.5}$$

The same argument as in the proof of Proposition 5.3.3 shows that the natural restriction map

$$\mathrm{Sel}(K, A_f \otimes \chi) \to \mathrm{Sel}(K, A_f \otimes \chi\Psi_T^{\mathbf{c}-1})[T]$$

has finite kernel and cokernel, and therefore from [BD05, Cor. 4] (a consequence of the divisibility in the anticyclotomic main conjecture proved in *op. cit.* with $p = \ell$), as extended in [PW11, CH15, KPW17]), it follows that

$$L(E/K, \chi, 1) \neq 0 \quad \Longrightarrow \quad \#\mathrm{Sel}(K, A_f \otimes \chi\Psi_T^{\mathbf{c}-1})[T] < \infty.$$

Together with Theorem 4.2.1 and Proposition 2.4.1, in light of (5.5) this shows that $X^{\boldsymbol{f}}(\mathbf{Q}, \mathbb{A}^\dagger_{f\underline{\boldsymbol{gg}}^*})$ is $\Lambda$-torsion, with

$$\mathrm{char}_\Lambda\big(X^{\boldsymbol{f}}(\mathbf{Q}, \mathbb{A}^\dagger_{f\underline{\boldsymbol{gg}}^*})\big) = \big(\mathscr{L}_p^{\boldsymbol{f}}(f, \underline{\boldsymbol{gg}}^*)^2\big)$$

in $\Lambda\otimes\mathbf{Q}_p$. By Proposition 4.1.2, it follows that $\kappa(f, \underline{\boldsymbol{gg}}^*)$ is not $\Lambda$-torsion, that both $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^\dagger_{f\underline{\boldsymbol{gg}}^*})$ and $X^{\mathrm{bal}}(\mathbf{Q}, \mathbb{A}^\dagger_{f\underline{\boldsymbol{gg}}^*})$ have $\Lambda$-rank one, and that

$$\mathrm{char}_\Lambda\big(X^{\mathrm{bal}}(\mathbf{Q}, \mathbb{A}^\dagger)_{\mathrm{tors}}\big) = \mathrm{char}_\Lambda\big(\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^\dagger)/(\kappa(f, \boldsymbol{gg}^*))\big)^2$$

in $\Lambda \otimes \mathbf{Q}_p$. Denote by $\kappa_1(f, \underline{\boldsymbol{gg}}^*) \in \mathrm{Sel}_{\emptyset,0}(K, T_f \otimes \Psi_T^{1-\mathbf{c}})$ the projection of $\kappa(f, \underline{\boldsymbol{gg}}^*)$ onto the first direct summand in (4.2). By Proposition 3.3.1 and Corollary 3.3.2, the above shows that both $\mathrm{Sel}_{\emptyset,0}(K, T_f \otimes \Psi_T^{1-\mathbf{c}})$ and $X_{0,\emptyset}(K, A_f \otimes \Psi_T^{\mathbf{c}-1})$ have $\Lambda$-rank one, with

$$\mathrm{char}_\Lambda\big(X_{0,\emptyset}(K, A_f \otimes \Psi_T^{\mathbf{c}-1})_{\mathrm{tors}}\big) = \mathrm{char}_\Lambda\big(\mathrm{Sel}_{\emptyset,0}(K, T_f \otimes \Psi_T^{1-\mathbf{c}})/(\kappa_1(f, \underline{\boldsymbol{gg}}^*))\big)^2$$

in $\Lambda \otimes \mathbf{Q}_p$. By the control theorem of Proposition 5.3.3, this implies that

$$\mathrm{corank}_{\mathbf{Z}_p}\mathrm{Sel}_{0,\emptyset}(K, A_f) = \mathrm{rank}_{\mathbf{Z}_p}\big(X_{0,\emptyset}(K, A_f \otimes \Psi_T^{\mathbf{c}-1})_{/T}\big) = 1 + 2\,\mathrm{rank}_{\mathbf{Z}_p}(\mathfrak{Z}_{/T}), \tag{5.6}$$

where $\mathfrak{Z} = \mathrm{Sel}_{\emptyset,0}(K, T_f \otimes \Psi_T^{1-\mathbf{c}})/(\kappa_1(f, \underline{\boldsymbol{gg}}^*))$. Thus we conclude that

$$\mathrm{corank}_{\mathbf{Z}_p}\mathrm{Sel}_{0,\emptyset}(K, A_f) = 1 \quad \Longleftrightarrow \quad (\kappa_1(f, \underline{\boldsymbol{gg}}^*) \bmod T) \neq 0,$$

where $(\kappa_1(f, \underline{\boldsymbol{gg}}^*) \bmod T)$ is the image of $\kappa_1(f, \underline{\boldsymbol{gg}}^*)$ in $\mathrm{Sel}_{\emptyset,0}(K, T_f \otimes \Psi_T^{1-\mathbf{c}})_{/T}$. Since the natural map

$$\mathrm{Sel}_{\emptyset,0}(K, T_f \otimes \Psi_T^{1-\mathbf{c}})_{/T} \to \mathrm{Sel}_{\emptyset,0}(K, T_f)$$

has finite kernel by Proposition 5.3.3, and it sends $(\kappa_1(f, \underline{\boldsymbol{gg}}^*) \bmod T)$ to $\kappa_p(E)$ by construction, we arrive at

$$\kappa_p(E) \neq 0 \quad \Longleftrightarrow \quad \mathrm{rank}_{\mathbf{Z}_p}\mathrm{Sel}_{\emptyset,0}(K, T_f) = 1, \tag{5.7}$$

using the action of complex conjugation to reverse the roles of $\mathfrak{p}$ and $\overline{\mathfrak{p}}$. Now, assuming $\kappa_p(E) \neq 0$, by Theorem 4.2.1 (resp. the $p$-parity conjecture [Nek01]) the case $\delta = 0$ (resp. $\delta = 1$) is excluded in Lemma 5.3.1, and so $\dim_{\mathbf{Q}_p}\mathrm{Sel}(K, V_pE) = 2$; since the nonvanishing of $L(E^K, 1)$ implies that $\mathrm{Sel}(\mathbf{Q}, V_pE) = \mathrm{Sel}(K, V_pE)$ by Kato's work, the first implication in Theorem 5.2.3 follows.

Conversely, if $\dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) = 2$ (and so $\dim_{\mathbf{Q}_p}\mathrm{Sel}(K, V_pE) = 2$), from (5.6) we see that the case $\delta = 0$ is excluded in Lemma 5.3.1 and the case $\delta = -1$ holds (i.e., $\dim_{\mathbf{Q}_p}\mathrm{Sel}_{\emptyset,0}(K, V_pE) = 1$) if and only if $\mathrm{Sel}(\mathbf{Q}, V_pE) \neq \ker(\mathrm{loc}_p)$. By (5.7), this concludes the proof. □

**Remark 5.4.1.** Even though the approach in this paper recovers a strenghtened form of [CH22, Thm. A] under slightly weaker hypotheses, a noteworthy advantage of the approach in *op. cit.* is that — by relating the derived $p$-adic heights against $\kappa_p(E)$ to the leading coefficient of $\Theta_{f/K}(T)$ (see [CH22, Thm. 5.3]) — it provides a method to numerically verity the nonvanishing of $\kappa_p(E)$, and to relate its rationality properties predicted by [DR16, Conj. 3.12] to (the leading coefficient part of) the anticyclotomic $p$-adic Birch–Swinnerton-Dyer conjectures formulated in [BD96]. (See also [BSV22a] for a more general setting.)

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