p^∞-SELMER GROUPS AND RATIONAL POINTS ON CM ELLIPTIC CURVES

ASHAY BURUNGALE, FRANCESC CASTELLA, CHRISTOPHER SKINNER, AND YE TIAN

To Bernadette Perrin-Riou, with admiration

Abstract. Let $E/\mathbb{Q}$ be a CM elliptic curve and p a prime of good ordinary reduction for E. We show that if $\mathrm{Sel}_{p^\infty}(E/\mathbb{Q})$ has $\mathbb{Z}_p$-corank one, then $E(\mathbb{Q})$ has a point of infinite order. The non-torsion point arises from a Heegner point, and thus $\mathrm{ord}_{s=1} L(E,s) = 1$, yielding a p-converse to a theorem of Gross–Zagier, Kolyvagin, and Rubin in the spirit of [Ski20], [Zha14].

For $p > 3$, this gives a new proof of the main result of [BT20], which our approach extends to small primes. The approach generalizes to CM elliptic curves over totally real fields [BCST21].

Soit $E/\mathbb{Q}$ une courbe elliptique à multiplication complexe et p un nombre premier de bonne réduction ordinaire pour E. Nous montrons que si $\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) = 1$, *alors E a un point d'ordre infini. Le point de non-torsion provient d'un point de Heegner, et donc* $\mathrm{ord}_{s=1}L(E,s) = 1$, *ce qui donne un p-converse à un théorème de Gross–Zagier, Kolyvagin, et Rubin dans l'esprit de [Ski20], [Zha14].*

Pour $p > 3$, cela donne une nouvelle preuve du résultat principal de [BT20], que notre approche étend aux tous les nombres premiers. L'approche se généralise aux courbes elliptiques à multiplication complexe sur les corps totalement réels [BCST21].

Contents

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Introduction

Starting with [Ski20], [Zha14], several works have been devoted to p-converses to a celebrated theorem of Gross–Zagier, Kolyvagin, and Rubin: If the p^∞-Selmer group $\mathrm{Sel}_{p^\infty}(E/\mathbb{Q})$ has $\mathbb{Z}_p$-corank 1 for an elliptic curve $E/\mathbb{Q}$, then $\mathrm{ord}_{s=1}L(E,s)=1$. Besides being an evidence for the Birch and Swinnerton-Dyer conjecture, an important impetus for the p-converse theorems has come from recent developments in arithmetic statistics. For instance, such p-converse theorems have led to the proof [BSZ15] that a large proportion of elliptic curves over $\mathbb{Q}$—and conditionally, 100% of them—satisfy the Birch and Swinnerton-Dyer conjecture.

The p-converse theorems of [Ski20], [Zha14] are obtained by exhibiting a certain Heegner point on E with infinite order, and hold for primes $p>3$ of good ordinary reduction of E, and under certain hypotheses that excluded the CM elliptic curves.

Our main result is the following CM p-converse theorem. For primes $p>3$, the result was first proved in [BT20].

Theorem A. *Let $E/\mathbb{Q}$ be an elliptic curve with complex multiplication by an order of an imaginary quadratic field $\mathcal{K}$ of discriminant $-D_{\mathcal{K}}<0$. Assume that the Hecke character associated to E has conductor exactly divisible by $\mathfrak{d}_{\mathcal{K}}:=(\sqrt{-D_{\mathcal{K}}})$. Let p be a prime of good ordinary reduction for E. Then*

$$\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E/\mathbb{Q})=1 \quad\Longrightarrow\quad \mathrm{ord}_{s=1}L(E,s)=1.$$

In particular, if $\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E/\mathbb{Q})=1$ then $\mathrm{rank}_{\mathbb{Z}}E(\mathbb{Q})=1$ and $\#\text{Ш}(E/\mathbb{Q})<\infty$.

Note that the 'in particular' clause in Theorem A follows from combining its conclusion with the fundamental work of Gross–Zagier, Kolyvagin, and Rubin. In turn, this consequence yields the following mod p criterion for analytic rank one.

Corollary B. *Let $(E,\mathcal{K})$ be as in Theorem A, and let p be a prime of good ordinary reduction for E such that:*

(i) $E(\mathbb{Q})[p]=0$;
(ii) $\mathrm{Sel}_p(E/\mathbb{Q})\simeq\mathbb{Z}/p\mathbb{Z}$, *where* $\mathrm{Sel}_p(E/\mathbb{Q})\subset \mathrm{H}^1(\mathbb{Q},E[p])$ *is the p-Selmer group of E.*

Then $\mathrm{ord}_{s=1}L(E,s)=1$ *and* $\text{Ш}(E/\mathbb{Q})[p^\infty]=0$.

More generally, we prove a p-converse for CM abelian varieties $B_\lambda/\mathcal{K}$ associated with Hecke characters λ over $\mathcal{K}$ of infinity type $(-1,0)$ (see Theorem 7.0.2).

Our approach to the CM p-converse differs from [BT20]. A salient feature is that the approach generalizes to CM elliptic curves over totally real fields: It sidesteps the inherence of elliptic units in [BT20], and leads to the first p-converse theorems over general totally real fields. This note might thus be viewed as a prelude to [BCST21].

The conductor hypothesis in Theorem A arises from an appeal to [CH18] which supposes a classical Heegner hypothesis. This hypothesis may be removed (cf. [BCK21], [BCST21]) via the p-adic Waldspurger formula of Liu–Zhang–Zhang [LZZ18], whose habitat is the general Yuan–Zhang–Zhang framework [YZZ13]. (The hypothesis is not present in [BT20], thanks to Disegni's p-adic Gross–Zagier formula [Dis17], also in the framework of [YZZ13].)

Remark C. The p-converse as in Theorem A is independently due to Ressler and Yu [RY21], [Yu21]. Complementing the present note, their approach builds on [BT20] and does not require the conductor hypothesis. A key new element in their work is a counterpart of the main results of Agboola–Howard [AH06] for small primes.

Remark D. A spectacular result of Smith [Smi17] reduces Goldfeld's conjecture [Gol79] for CM elliptic curves $E/\mathbb{Q}$ with $E(\mathbb{Q})[2]\simeq(\mathbb{Z}/2\mathbb{Z})^2$ and admitting no rational cyclic 4-isogeny to the CM p-converse and its rank zero analogue for the prime $p=2$. The unconditional rank zero CM

p-converse is in fact proved in [BT19], [BT22]. Unfortunately, Theorem A falls short of providing the desired rank one CM p-converse: First, E should be allowed to have just *potentially* good ordinary reduction at p (this might be approachable by our strategy); the second—and the main—hindrance is that $\mathbb{Q}(\sqrt{-7})$ is the only imaginary quadratic field of class number 1 with 2 split, and incidentally $E(\mathbb{Q})[2] \simeq \mathbb{Z}/2\mathbb{Z}$ for all elliptic curves $E/\mathbb{Q}$ with CM by $\mathbb{Q}(\sqrt{-7})$ (see e.g. the table in [Ols74, p. 2]).

The approach. Assuming $\#\text{Ш}(E/\mathbb{Q})[p^\infty] < \infty$ and $p \nmid \#\mathcal{O}_\mathcal{K}^\times$, the p-converse as in Theorem A goes back to Rubin [Rub94, Thm. 4]. Around the same time, Rubin proved a striking formula [Rub92] which expresses the p-adic formal group logarithm of a point $P \in E(\mathbb{Q}) \otimes_\mathbb{Z} \mathbb{Z}_p$ in terms of the value of a Katz p-adic L-function *outside* its range of interpolation. Our approach to Theorem A is inspired by the p-adic Waldspurger formula of Bertolini–Darmon–Prasanna [BDP13], which is a remarkable generalization of Rubin's formula, and sheds new light on [Rub92] (cf. [BDP12]).

Let $E/\mathbb{Q}$ be an elliptic curve with CM by an imaginary quadratic field $\mathcal{K}$, and p a prime of good ordinary reduction. Let λ be the Hecke character over $\mathcal{K}$ associated with E so that

$$L(E, s) = L(\lambda, s).$$

In view of a non-vanishing result of Rohrlich [Roh84], we pick a pair (ψ, χ) of Hecke characters over $\mathcal{K}$ with χ of finite order such that

$$\psi\chi = \lambda, \quad L(\psi^*\chi, 1) \neq 0, \tag{0.1}$$

where $\psi^* := \psi \circ c$ is the composition of ψ with the non-trivial automorphism of $\mathcal{K}/\mathbb{Q}$. For $f = \theta_\psi$ the theta series associated to ψ, the main result of [BDP13] relates the p-adic formal group logarithm of a Heegner point $P_{\psi,\chi} \in B_{\psi,\chi}(\mathcal{K})$ to a value (outside the range of interpolation) of a p-adic Rankin L-series $\mathscr{L}_v(f, \chi)$. Here $B_{\psi,\chi}$ is a CM abelian variety over $\mathcal{K}$ endowed with a $\mathcal{K}$-rational map $i_\lambda : B_{\psi,\chi} \to E$. By the Gross–Zagier formula [YZZ13], $P_{\psi,\chi}$ is non-torsion if and only if $L'(f, \chi, 1) \neq 0$. Setting

$$P_\mathcal{K} := i_\lambda(P_{\psi,\chi}) \in E(\mathcal{K}),$$

one thus obtains a point on E which, in light of (0.1) and the factorization

$$L(f, \chi, s) = L(\lambda, s) \cdot L(\psi^*\chi, s), \tag{0.2}$$

is non-torsion if and only if $\mathrm{ord}_{s=1}L(E, s) = 1$.

Now, Theorem A is equivalent to: If $\mathrm{Sel}_{p^\infty}(E/\mathbb{Q})$ has $\mathbb{Z}_p$-corank 1, then $P_\mathcal{K}$ is non-torsion. In [BT20] the non-triviality is shown via the following.

(1) The anticyclotomic Iwasawa main conjecture (IMC) for (Hecke characters over) $\mathcal{K}$ in the root number $+1$ case [Rub91];
(2) The anticyclotomic IMC for $\mathcal{K}$ in the root number -1 case [AH06], [Arn07];
(3) The non-vanishing of the Λ-adic regulator appearing in (2) [Bur15];
(4) The Λ-adic Gross–Zagier formula [Dis17].

Here Λ denotes the anticyclotomic Iwasawa algebra over $\mathcal{K}$ (with certain coefficients).

Bypassing (2), (3), and (4), our approach builds on the explicit reciprocity law [CH18], which realizes $\mathscr{L}_v(f, \chi)$ as the image of a Λ-adic Heegner class $\mathbf{z}_{f,\chi}$ under a Perrin-Riou big logarithm map. Similarly as in [BDP12], we establish a factorization

$$\mathscr{L}_v(\theta_\psi, \chi)^2 \doteq \mathscr{L}_v(\psi^*\chi^*) \cdot \mathscr{L}_v(\psi\chi^*)$$

in §3 relating $\mathscr{L}_v(\theta_\psi, \chi)^2$ to the product of two Katz p-adic L-functions, mirroring (0.2). In particular, this leads to an expression relating the formal group logarithm of $P_\mathcal{K}$ to a value of a Katz p-adic L-function, as in Rubin's formula.

Along with an analogous decomposition for Selmer groups shown in §2, the Iwasawa–Greenberg main conjecture for $\mathscr{L}_v(\theta_\psi,\chi)^2$ is readily seen to be a consequence of the main results of [Rub91], [JLK11]. Building on the Λ-adic explicit reciprocity law, in §4 we prove the equivalence between the main conjecture for the p-adic L-function $\mathscr{L}_v(\theta_\psi,\chi)^2$ and a different main conjecture formulated in terms of the zeta element $\mathbf{z}_{f,\chi}$. Finally, the latter yields the implication

$$\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E/\mathbb{Q})=1 \quad\Longrightarrow\quad P_{\mathcal{K}}\neq 0\in E(\mathcal{K})\otimes_{\mathbb{Z}}\mathbb{Q}$$

via a variant of Mazur's control theorem.

Dedication. The p-converse for CM elliptic curves $E/\mathbb{Q}$ is due to Rubin if $\#\text{Ш}(E/\mathbb{Q})[p^\infty]<\infty$. The essence of our removal of this hypothesis is Iwasawa theory of Heegner points, as pioneered by Perrin-Riou [PR87]. The theory of big logarithm maps [PR94], another major contribution of Perrin-Riou, is also elemental to our approach. It is a great pleasure to dedicate this note to Bernadette Perrin-Riou as a humble gift on the occasion of her 65th birthday.

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1. Preliminaries

Fix throughout a prime p, an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and embeddings $\mathbb{C}\overset{\imath_\infty}{\hookleftarrow}\overline{\mathbb{Q}}\overset{\imath_p}{\hookrightarrow}\mathbb{C}_p$. Fix also an imaginary quadratic field $\mathcal{K}$ of discriminant $-D_{\mathcal{K}}<0$ and ring of integers $\mathcal{O}_{\mathcal{K}}$.

1.1. **CM abelian varieties.** We say that a Hecke character $\psi:\mathcal{K}^\times\backslash\mathbb{A}_{\mathcal{K}}^\times\to\mathbb{C}^\times$ has *infinity type* $(a,b)\in\mathbb{Z}^2$ if, writing $\psi=(\psi_v)_v$ with v running over the places of $\mathcal{K}$, the component ψ_∞ satisfies $\psi_\infty(z)=z^a\bar{z}^b$ for all $z\in(\mathcal{K}\otimes_{\mathbb{Q}}\mathbb{R})^\times\simeq\mathbb{C}^\times$, where the identification is made via $\imath_\infty$. Hence in particular the norm character $\mathbf{N}_{\mathcal{K}}$, given by $\mathfrak{q}\mapsto\#(\mathcal{O}_{\mathcal{K}}/\mathfrak{q})$ on ideals of $\mathcal{O}_{\mathcal{K}}$, has infinity type $(-1,-1)$. The *central character* of such ψ is the character ω_ψ on $\mathbb{A}^\times$ defined by

$$\psi|_{\mathbb{A}^\times}=\omega_\psi\cdot\mathbf{N}^{-(a+b)},$$

where $\mathbf{N}$ is the norm on $\mathbb{A}^\times$.

Our fixed embedding $\imath_p$ defines a natural map $\sigma:\mathcal{K}\otimes_{\mathbb{Q}}\mathbb{Q}_p\to\mathbb{C}_p$, and we let $\overline{\sigma}:\mathcal{K}\otimes_{\mathbb{Q}}\mathbb{Q}_p\to\mathbb{C}_p$ be the composition of σ with the non-trivial automorphism of $\mathcal{K}$. The *p-adic avatar* of a Hecke character ψ of infinity type (a,b) is the character $\hat{\psi}:\mathcal{K}^\times\backslash\mathbb{A}_{\mathcal{K},\mathrm{f}}^\times\to\mathbb{C}_p^\times$ given by

$$\hat{\psi}(x)=\imath_p\circ\imath_\infty^{-1}(\psi(x))\sigma(x_p)^a\overline{\sigma}(x_p)^b$$

for all $x\in\mathbb{A}_{\mathcal{K},\mathrm{f}}^\times$, where $x_p\in(\mathcal{K}\otimes\mathbb{Q}_p)^\times$ is the p-component of x.

Thoughout the following, we shall often omit the notational distinction between an algebraic Hecke character and its p-adic avatar, as it will be clear from the context which one is meant.

Let ψ be an algebraic Hecke character of $\mathcal{K}$ infinity type $(-1,0)$ with values in a number field $F_\psi\subset\overline{\mathbb{Q}}$ with ring of integer $\mathcal{O}_\psi$. Let $\mathfrak{P}$ be the prime of F_ψ above p induced by $\imath_p$, and denote by Φ_ψ the completion of F_ψ at $\mathfrak{P}$ and by $\mathscr{O}_\psi$ the ring of integers of Φ_ψ. By a well-known theorem of Casselman's (see [BDP12, Thm. 2.5] and the reference [Shi71, Thm. 6] therein), attached to ψ there is a CM abelian variety $B_{\psi/\mathcal{K}}$, unique up to isogeny over $\mathcal{K}$, with the property that

$$V_{\mathfrak{P}}B_\psi\simeq\hat{\psi}^{-1}$$

as one-dimensional Φ_ψ-representations of $G_{\mathcal{K}}$, where $V_{\mathfrak{P}}B_\psi=\big(\varprojlim B_\psi[\mathfrak{P}^j]\big)\otimes_{\mathscr{O}_\psi}\Phi_\psi$ is the rational $\mathfrak{P}$-adic Tate module of B_ψ.

1.2. **Heegner points.** Let $f \in S_2(\Gamma_1(N))$ be a normalized eigenform of weight 2, level N prime to p, and nebentypus ε_f. We assume that $\mathcal{K}$ satisfies the *Heegner hypothesis* relative to N:

$$\text{there is an ideal } \mathfrak{N} \subset \mathcal{O}_\mathcal{K} \text{ with } \mathcal{O}_\mathcal{K}/\mathfrak{N} \simeq \mathbb{Z}/N\mathbb{Z}, \tag{Heeg}$$

and fix once and for all an ideal $\mathfrak{N}$ as above. We assume also that

$$p\mathcal{O}_\mathcal{K} = v\overline{v} \text{ splits in } \mathcal{K}, \tag{spl}$$

with v the prime of $\mathcal{K}$ above p induced by our fixed embedding $\imath_p$. Let $F \subset \overline{\mathbb{Q}}$ be the number field generated by the Fourier coefficients of f. Denote by $\mathfrak{P}$ the prime of F above p induced by $\imath_p$, and assume that f is $\mathfrak{P}$-*ordinary*, i.e. $v_\mathfrak{P}(a_p(f)) = 0$, where $v_\mathfrak{P}$ is the $\mathfrak{P}$-adic valuation on F.

Let $A_f/\mathbb{Q}$ be the abelian variety of GL_2-type associated to f, determined up to isogeny over $\mathbb{Q}$ by the equality of L-functions

$$L(A_f, s) = \prod_{\tau : F \hookrightarrow \mathbb{C}} L(f^\tau, s),$$

where f^τ runs over all the conjugates of f. Denote by Φ the completion of F at $\mathfrak{P}$, and let $\mathscr{O}$ be the ring of integers of Φ. Let $T_\mathfrak{P} A_f := \varprojlim A_f[\mathfrak{P}^j]$ be the $\mathfrak{P}$-adic Tate module of A_f, which is free of rank two over $\mathscr{O}$.

For every positive integer c, let $\mathcal{K}_c$ be the ring class field of $\mathcal{K}$ of conductor c, so $\mathrm{Gal}(\mathcal{K}_c/\mathcal{K}) \simeq \mathrm{Pic}(\mathcal{O}_c)$ by class field theory, where $\mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_\mathcal{K}$ is the order of $\mathcal{K}$ of conductor c. For every $c > 0$ prime to N and every ideal $\mathfrak{a}$ of $\mathcal{O}_c$, we consider the CM point $x_\mathfrak{a} \in X_1(N)(\tilde{\mathcal{K}}_c)$ constructed in [CH18, §2.3], where $\tilde{\mathcal{K}}_c$ is the compositum of $\mathcal{K}_c$ and the ray class field of $\mathcal{K}$ of conductor $\mathfrak{N}$. Let $\Delta_\mathfrak{a}$ be the class of the degree 0 divisor $(x_\mathfrak{a}) - (\infty)$ in $J_1(N) = \mathrm{Jac}(X_1(N))$, and denote by $z_\mathfrak{a} = \delta(\Delta_\mathfrak{a})$ its image under the Kummer map

$$\delta : J_1(N)(\tilde{\mathcal{K}}_c) \to \mathrm{H}^1(\tilde{\mathcal{K}}_c, T_p J_1(N)).$$

Fix a parametrization $\pi : J_1(N) \to A_f$, and let $y_{f,\mathfrak{a}} \in \mathrm{H}^1(\tilde{\mathcal{K}}_c, T_\mathfrak{P} A_f)$ be the image of $y_\mathfrak{a}$ under the natural projection

$$\mathrm{H}^1(\tilde{\mathcal{K}}_c, T_p J_1(N)) \xrightarrow{\pi_*} \mathrm{H}^1(\tilde{\mathcal{K}}_c, T_p A_f) \to \mathrm{H}^1(\tilde{\mathcal{K}}_c, T_\mathfrak{P} A_f).$$

For the ease of notation, we set $y_{f,c} = y_{f,\mathfrak{a}}$ for $\mathfrak{a} = \mathcal{O}_c$. A standard calculation shows that if $p \nmid c$, then for every $n > 0$ we have

$$\mathrm{Cor}_{\tilde{\mathcal{K}}_{cp^n}/\tilde{\mathcal{K}}_{cp^{n-1}}}(y_{f,cp^n}) = \begin{cases} a_p(f) \cdot y_{f,cp^{n-1}} - \varepsilon_f(p) \cdot y_{f,cp^{n-2}} & \text{if } n > 1, \\ u_c^{-1}\big(a_p(f) - \sigma_v - \sigma_{\overline{v}}\big) \cdot y_{f,c} & \text{if } n = 1, \end{cases} \tag{1.1}$$

where $u_c := [\mathcal{O}_c^\times : \mathcal{O}_{cp}^\times]$ and $\sigma_v, \sigma_{\overline{v}} \in \mathrm{Gal}(\tilde{\mathcal{K}}_c/\mathcal{K})$ are Frobenius elements at the primes of $\mathcal{K}$ above p (see [CH18, Prop. 4.4]).

Let α be the $\mathfrak{P}$-adic unit root of $x^2 - a_p(f)x + \varepsilon_f(p)p$, and for any positive integer c prime to N define the α-*stabilized Heegner class* $y_{f,c,\alpha}$ by

$$y_{f,c,\alpha} := \begin{cases} y_{f,c} - \varepsilon_f(p)\alpha^{-1} \cdot y_{f,c/p} & \text{if } p \mid c, \\ u_c^{-1}\big(1 - \sigma_v\alpha^{-1} - \sigma_{\overline{v}}\alpha^{-1}\big) \cdot y_{f,c} & \text{if } p \nmid c. \end{cases}$$

This definition is motivated by the following result.

Lemma 1.2.1. *For all positive integers c prime to N, we have*

$$\mathrm{Cor}_{\tilde{\mathcal{K}}_{cp}/\tilde{\mathcal{K}}_c}(y_{f,cp,\alpha}) = \alpha \cdot y_{f,c,\alpha}.$$

Proof. This follows immediately from (1.1). □

1.3. **Heegner point main conjecture.** Fix a positive integer c prime to Np, and put $\tilde{\mathcal{K}}_{cp^\infty} = \bigcup_{m\geq 0} \tilde{\mathcal{K}}_{cp^m}$. The Galois group $\mathcal{G}_c = \mathrm{Gal}(\tilde{\mathcal{K}}_{cp^\infty}/\mathcal{K})$ decomposes as

$$\mathcal{G}_c \simeq \Delta_c \times \Gamma,$$

where Γ is the maximal torsion-free quotient of $\mathcal{G}_c$, giving the Galois group of the anticyclotomic $\mathbb{Z}_p$-extension $\mathcal{K}_\infty/\mathcal{K}$, and Δ_c is a finite abelian group.

Let χ be a finite order Hecke character of $\mathcal{K}$ with $\chi|_{\mathbb{A}^\times} = \varepsilon_f^{-1}$ and of conductor dividing $c\mathfrak{N}$. Upon enlarging F is necessary, assume that Φ contains the values of χ. For each n, take $m \gg 0$ so that $\tilde{\mathcal{K}}_{cp^m} \supset \mathcal{K}_n$, and set

$$z_{f,\chi,n} := \alpha^{-m} \sum_{\sigma\in\mathrm{Gal}(\tilde{\mathcal{K}}_{cp^m}/\mathcal{K}_n)} \chi(\sigma)\cdot y^\sigma_{f,cp^m,\alpha}. \tag{1.2}$$

In view of Lemma 1.2.1, the definition of $z_{f,\chi,n}$ does not depend on the choice of m. Moreover, letting $A_{f,\chi}$ be the Serre tensor $A_f \otimes \chi$, we see that $z_{f,\chi,n}$ defines a class

$$\mathbf{z}_{f,\chi,n} \in \mathrm{H}^1(\mathcal{K}_n, T_{\mathfrak{P}} A_{f,\chi}).$$

Let

$$\Lambda_0 = \mathscr{O}[\![\Gamma]\!], \qquad \Lambda = \Lambda_0 \otimes_{\mathscr{O}} \Phi \tag{1.3}$$

be the anticyclotomic Iwasawa algebras. From their construction, the classes $\mathbf{z}_{f,\chi,n}$ are contained in the pro-$\mathfrak{P}$ Selmer group $S_{\mathfrak{P}}(A_{f,\chi}/\mathcal{K}_n) \subset \mathrm{H}^1(\mathcal{K}_n, T_{\mathfrak{P}}A_{f,\chi})$, and by Lemma 1.2.1 they are norm-compatible, hence defining a class $\mathbf{z}_{f,\chi} = \{\mathbf{z}_{f,\chi,n}\}_n$ in the compact Λ_0-adic Selmer group

$$\mathscr{S}(A_{f,\chi}/\mathcal{K}_\infty) := \varprojlim_n S_{\mathfrak{P}}(A_{f,\chi}/\mathcal{K}_n).$$

On the other hand, let $A_{f,\chi}$ be the Serre tensor $A_f \otimes \chi$, and set

$$\mathscr{X}(A_{f,\chi}/\mathcal{K}_\infty) := \mathrm{Hom}_{\mathbb{Z}_p}\Big(\varinjlim_n \mathrm{Sel}_{\mathfrak{P}^\infty}(A_{f,\chi}/\mathcal{K}_n), \Phi/\mathscr{O}\Big),$$

where $\mathrm{Sel}_{\mathfrak{P}^\infty}(A_{f,\chi}/\mathcal{K}_n) \subset \mathrm{H}^1(\mathcal{K}_n, A_{f,\chi}[\mathfrak{P}^\infty])$ is the $\mathfrak{P}^\infty$-Selmer groups of $A_{f,\chi}$. Set also

$$\mathcal{S}(A_{f,\chi}/\mathcal{K}_\infty) = \mathscr{S}(A_{f,\chi}/\mathcal{K}_\infty)\otimes_{\mathscr{O}}\Phi, \quad \mathcal{X}(A_{f,\chi}/\mathcal{K}_\infty) = \mathscr{X}(A_{f,\chi}/\mathcal{K}_\infty)\otimes_{\mathscr{O}}\Phi,$$

which are finitely generated Λ-modules.

The following conjecture in a natural extension of Perrin-Riou's Heegner point main conjecture, [PR87, Conj. B].

Conjecture 1.3.1. *The modules $\mathcal{S}(A_{f,\chi}/\mathcal{K}_\infty)$ and $\mathcal{X}(A_{f,\chi}/\mathcal{K}_\infty)$ have both Λ-rank one, and*

$$\mathrm{char}_\Lambda(\mathcal{X}(A_{f,\chi}/\mathcal{K}_\infty)_{\Lambda\text{-tors}}) = \mathrm{char}_\Lambda\big(\mathcal{S}(A_{f,\chi}/\mathcal{K}_\infty)/\Lambda\cdot\mathbf{z}_{f,\chi}\big)^2,$$

where the subscript Λ-tors *denotes the maximal Λ-torsion submodule.*

In [BT20] a conjecture similar to Conjecture 1.3.1 is formulated in terms of a Λ-adic Heegner class deduced from work of Disegni [Dis17] (see [BT20, Conj. 2.2]). Similarly as in [BT20][1], our proof of Theorem A is based on a study of Conjecture 1.3.1. The novelty in our approach is in the proof of cases of this conjecture.

2. Selmer groups

In this section we introduce the different Selmer groups entering in our arguments. In particular, the decomposition in Proposition 2.3.1 will play a key role.

[1] As well as in other results on the p-converse theorem in rank 1 without a finiteness condition on the Tate–Shafarevich group that appeared after [Ski20]: [Wan21], [CW16], [CGLS22], etc.

2.1. **Selmer groups of certain Rankin–Selberg convolutions.** As in §1.2, let $f \in S_2(\Gamma_1(N))$ be a $\mathfrak{P}$-ordinary newform with nebentypus ε_f, and let $\mathcal{K}$ be an imaginary quadratic field satisfying (Heeg) and (spl).

Let $c > 0$ be a positive integer prime to N. Similarly as in [BDP12, Def. 3.10], we say that a Hecke character ξ of infinity type $(2+j, -j)$, with $j \in \mathbb{Z}$, has *finite type* $(c, \mathfrak{N}, \varepsilon_f)$ if it satisfies:

(a) $\omega_\xi \cdot \varepsilon_f = \mathbb{1}$, where ω_ξ is the central character of ξ;
(b) $\mathfrak{f}_\xi = c \cdot \mathfrak{N}'$, where $\mathfrak{N}'$ is the unique divisor of $\mathfrak{N}$ with norm equal to the conductor of ε_f;
(c) the local sign $\epsilon_q(f, \xi)$ is $+1$ for all finite primes q.

Condition (a) implies that the Rankin–Selberg L-function $L(f, \xi, s)$ is self-dual, with $s = 0$ as the central critical point, and by (c) the sign in the functional equation is $+1$ (resp. -1) when $j \geqslant 0$ (resp. $j < 0$). Denote by $\Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon_f)$ the set of such characters ξ, and put

$$\Sigma^{(1)}_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon_f) = \{\xi \in \Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon_f) \mid j < 0\}, \quad \Sigma^{(2)}_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon_f) = \{\xi \in \Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon_f) \mid j \geqslant 0\}.$$

Denote by $\rho_f : G_{\mathbb{Q}} \to \mathrm{Aut}_\Phi(V_f)$ the $\mathfrak{P}$-adic Galois representation associated to f, so that

$$V_f(1) \simeq \Phi \otimes_{\mathscr{O}} T_{\mathfrak{P}} A_f.$$

Let χ be a finite order character of $\mathcal{K}$ such that $\chi \mathbf{N}_{\mathcal{K}}^{-1} \in \Sigma^{(1)}_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon_f)$, and consider the conjugate self-dual $G_{\mathcal{K}}$-representation

$$V_{f,\chi} := V_f(1)|_{G_{\mathcal{K}}} \otimes \chi. \tag{2.1}$$

For any Λ_0-module M, let $M^\vee = \mathrm{Hom}_{\mathrm{cts}}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ be the Pontryagin dual. Fix a $G_{\mathcal{K}}$-stable lattice $T_{f,\chi} \subset V_{f,\chi}$, and define the $G_{\mathcal{K}}$-module

$$\mathbf{W}_{f,\chi} := T_{f,\chi} \otimes_{\mathscr{O}} \Lambda_0^\vee, \tag{2.2}$$

where the tensor product is endowed with the diagonal Galois action, with $G_{\mathcal{K}}$ acting on $\Lambda_0^\vee$ via the the inverse of the tautological character $\Psi : G_{\mathcal{K}} \twoheadrightarrow \mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K}) \hookrightarrow \Lambda_0^\times$.

Definition 2.1.1. Fix a finite set Σ of places of $\mathcal{K}$ containing ∞ and the primes dividing Np, and denote by $\mathcal{K}^\Sigma$ the maximal extension of $\mathcal{K}$ unramified outside Σ. The *Selmer group* $\mathscr{S}_v(\mathbf{W}_{f,\chi})$ is defined by

$$\mathscr{S}_v(\mathbf{W}_{f,\chi}) := \ker\left\{\mathrm{H}^1(\mathcal{K}^\Sigma/\mathcal{K}, \mathbf{W}_{f,\chi}) \to \mathrm{H}^1(\mathcal{K}_{\bar{v}}, \mathbf{W}_{f,\chi}) \times \prod_{w \in \Sigma, w \nmid p} \mathrm{H}^1(\mathcal{K}_w, \mathbf{W}_{f,\chi})\right\}.$$

We also set

$$\mathcal{X}_v(f, \chi) := \mathrm{Hom}_{\mathrm{cts}}(\mathscr{S}_v(\mathbf{W}_{f,\chi}), \mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\mathscr{O}} \Phi,$$

which is independent of the lattice $T_{f,\chi}$.

Note that $\mathcal{X}_v(f, \chi)$ and the Selmer group $\mathcal{X}(A_{f,\chi}/\mathcal{K}_\infty)$ defined in §1.3 differ only in their defining local conditions at the primes above p. More precisely, by $\mathfrak{P}$-ordinarity, for every prime w of $\mathcal{K}$ above p there is a $G_{\mathcal{K}_w}$-module exact sequence

$$0 \to \mathscr{F}_w^+ T_{f,\chi} \to T_{f,\chi} \to \mathscr{F}_w^- T_{f,\chi} \to 0 \tag{2.3}$$

with $\mathscr{F}_w^\pm T_{f,\chi}$ free of rank one over $\mathscr{O}$, and the quotient $\mathscr{F}_w^- T_{f,\chi}$ affording an unramified action of $G_{\mathcal{K}_w}$. Put

$$\mathscr{F}_w^\pm \mathbf{W}_{f,\chi} = \mathscr{F}_w^\pm T_{f,\chi} \otimes_{\mathscr{O}} \Lambda_0^\vee.$$

Then the Selmer group $\mathscr{S}_{\mathrm{ord}}(\mathbf{W}_{f,\chi})$ defined by

$$\mathscr{S}_{\mathrm{ord}}(\mathbf{W}_{f,\chi}) := \ker\left\{\mathrm{H}^1(\mathcal{K}^\Sigma/\mathcal{K}, \mathbf{W}_{f,\chi}) \to \prod_{w \mid p} \mathrm{H}^1(\mathcal{K}_w, \mathscr{F}_w^- \mathbf{W}_{f,\chi}) \times \prod_{w \in \Sigma, w \nmid p} \mathrm{H}^1(\mathcal{K}_w, \mathbf{W}_{f,\chi})\right\} \tag{2.4}$$

satisfies

$$\mathscr{S}_{\rm ord}(\mathbf{W}_{f,\chi})\otimes_{\mathscr{O}}\Phi\simeq\left(\varinjlim_{n}\mathrm{Sel}_{\mathfrak{P}^\infty}(A_{f,\chi^*}/\mathcal{K}_n)\right)\otimes_{\mathscr{O}}\Phi,$$

and so

$$\mathcal{X}_{\rm ord}(f,\chi):=\mathrm{Hom}_{\rm cts}(\mathscr{S}_{\rm ord}(\mathbf{W}_{f,\chi}),\mathbb{Q}_p/\mathbb{Z}_p)\otimes_{\mathscr{O}}\Phi\simeq\mathcal{X}(A_{f,\chi}/\mathcal{K}_\infty). \tag{2.5}$$

Letting $\mathbf{T}_{f,\chi}:=T_{f,\chi}\otimes_{\mathscr{O}}\Lambda_0$ with $G_\mathcal{K}$-action via $\rho_f\otimes\Psi$, and defining $\check{\mathscr{S}}_{\rm ord}(\mathbf{T}_{f,\chi})\subset\mathrm{H}^1(\mathcal{K}^\Sigma/\mathcal{K},\mathbf{T}_{f,\chi})$ in the same manner as in (2.4), we similarly have

$$\mathcal{S}_{\rm ord}(f,\chi):=\check{\mathscr{S}}_{\rm ord}(\mathbf{T}_{f,\chi})\otimes_{\mathscr{O}}\Phi\simeq\mathcal{S}(A_{f,\chi}/\mathcal{K}_\infty) \tag{2.6}$$

(see e.g. [CG96, §4]).

2.2. **Selmer groups of characters.** We keep the hypothesis that the imaginary quadratic field $\mathcal{K}$ satisfies (spl), and let ξ be a Hecke character of $\mathcal{K}$ of conductor $\mathfrak{f}_\xi$. Let F be a number field containing the values of ξ. Let Φ be the completion of F at the prime $\mathfrak{P}$ of F above p induced by $\imath_p$, and let $\mathscr{O}$ be the ring of integers of Φ. Denote by T_ξ the free $\mathscr{O}$-module of rank one on which $G_\mathcal{K}$ acts via $\hat{\xi}^{-1}$, and consider the $G_\mathcal{K}$-module

$$\mathbf{W}_\xi:=T_\xi\otimes_{\mathscr{O}}\Lambda_0^\vee,$$

where as before the Galois action on $\Lambda_0^\vee$ is given by the character Ψ^{-1}.

Definition 2.2.1. Let Σ be a finite set of places of $\mathcal{K}$ containing ∞ and the primes dividing p or $\mathfrak{f}_\xi$. The *Selmer group* $\mathscr{X}_v(\mathbf{W}_\xi)$ is defined by

$$\mathscr{X}_v(\mathbf{W}_\xi):=\ker\left\{\mathrm{H}^1(\mathcal{K}^\Sigma/\mathcal{K},\mathbf{W}_\xi)\to\mathrm{H}^1(\mathcal{K}_{\overline{v}},\mathbf{W}_\xi)\times\prod_{w\in\Sigma,w\nmid p}\mathrm{H}^1(\mathcal{K}_w,\mathbf{W}_\xi)\right\}.$$

We also set $\mathcal{X}_v(\xi)=\mathscr{X}_v(\mathbf{W}_\xi)\otimes_{\mathscr{O}}\Phi$.

Remark 2.2.2. Suppose ξ has infinity type $(-1,0)$, and denote by ξ^* the composition of ξ with the non-trivial automorphism of $\mathcal{K}/\mathbb{Q}$, so ξ^* has infinity type $(0,-1)$. Then from e.g. [AH06, §1.1] we see that $\mathcal{X}_v(\xi)$ corresponds to the Bloch–Kato Selmer group of ξ over $\mathcal{K}_\infty/\mathcal{K}$, whereas $\mathcal{X}_v(\xi^*)$ corresponds to the Selmer group obtained by *reversing* the local conditions at the primes above p in the corresponding Bloch–Kato Selmer group of ξ^*.

2.3. **Decomposition.** We now specialize the set-up in §2.1 to the case where $f=\theta_\psi$ is the theta series of a Hecke character ψ of $\mathcal{K}$ of infinity type $(-1,0)$. Then f has level $N=D_\mathcal{K}\cdot\mathbf{N}(\mathfrak{f}_\psi)$ and nebentypus $\varepsilon_f=\eta_\mathcal{K}\cdot\omega_\psi$, where $\eta_\mathcal{K}$ is the quadratic character associated to $\mathcal{K}/\mathbb{Q}$.

One easily checks (see [BDP12, Lem. 3.14]) that if $\mathfrak{f}_\psi$ is a cyclic ideal of norm $\mathbf{N}(\mathfrak{f}_\psi)$ prime to $D_\mathcal{K}$, then $\mathcal{K}$ satisfies the Heegner hypothesis (Heeg) relative to N, and one may take

$$\mathfrak{N}=\mathfrak{d}_\mathcal{K}\cdot\mathfrak{f}_\psi,\quad\text{where }\mathfrak{d}_\mathcal{K}:=(\sqrt{-D_\mathcal{K}}). \tag{2.7}$$

In the following, we assume that $\mathfrak{f}_\psi$ satisfies the above condition, and take $\mathfrak{N}$ as in (2.7). Fix an integer $c>0$ prime to Np, and let χ be a finite order character such that $\chi\mathbf{N}_\mathcal{K}^{-1}\in\Sigma_{\rm cc}^{(1)}(c,\mathfrak{N},\varepsilon_f)$.

The following decomposition will play an important role later.

Proposition 2.3.1. *Let ψ and χ be as above. There is a Λ-module isomorphism*

$$\mathcal{X}_v(\theta_\psi,\chi)\simeq\mathcal{X}_v(\psi^*\chi^*)\oplus\mathcal{X}_v(\psi\chi^*).$$

Proof. Put $f=\theta_\psi$, and note that there is a $G_\mathcal{K}$-module decomposition

$$V_{f,\chi}\simeq V_{\psi^*\chi^*}\oplus V_{\psi\chi^*}. \tag{2.8}$$

Since the module $\mathcal{X}_v(f,\chi) \subset \mathrm{H}^1(\mathcal{K}^\Sigma/\mathcal{K}, \mathbf{W}_{f,\chi}) \otimes_{\mathscr{O}} \Phi$ does not depend on the lattice $T_{f,\chi} \subset V_{f,\chi}$ chosen to define $\mathbf{W}_{f,\chi}$, by (2.8) we may assume that $T_{f,\chi} \simeq T_{\psi^*\chi^*} \oplus T_{\psi\chi^*}$ as $G_{\mathcal{K}}$-modules, and so

$$\mathbf{W}_{f,\chi} \simeq \mathbf{W}_{\psi^*\chi^*} \oplus \mathbf{W}_{\psi\chi^*}$$

as $G_{\mathcal{K}}$-modules. The result thus follows immediately by comparing the defining local conditions of the three Selmer groups involved at all places. □

3. p-adic L-functions

In this section we introduce the two p-adic L-functions needed for our arguments, and prove Proposition 3.3.1 relating the two.

3.1. **The BDP p-adic L-function.** Let $f = \sum_{n=1}^\infty a_n(f)q^n \in S_2(\Gamma_1(N))$ be an eigenform with $p \nmid N$ and nebentypus ε_f, let $\mathcal{K}$ be an imaginary quadratic field satisfying (Heeg) and (spl), and fix an ideal $\mathfrak{N} \subset \mathcal{O}_{\mathcal{K}}$ with cyclic quotient of order N. Let c be a positive integer prime to Np, and let χ be a finite order Hecke character of $\mathcal{K}$ such that $\chi\mathbf{N}_{\mathcal{K}}^{-1} \in \Sigma_{\mathrm{cc}}^{(1)}(c, \mathfrak{N}, \varepsilon_f)$.

Let F be a number field containing $\mathcal{K}$, the Fourier coefficients of f, and the values of χ, and let Φ be the completion of F at the prime of F above p induced by $\imath_p$, with ring of integers $\mathscr{O}$. Let Λ_0 and Λ be the anticyclotomic Iwasawa algebras as in (1.3), and set

$$\Lambda_0^{\mathrm{ur}} := \Lambda_0 \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{ur}} \simeq \mathscr{O}^{\mathrm{ur}}[\![\Gamma]\!], \quad \Lambda^{\mathrm{ur}} := \Lambda_0^{\mathrm{ur}} \otimes_{\mathscr{O}} \Phi,$$

where $\mathbb{Z}_p^{\mathrm{ur}}$ is the completion of the ring of integers of the maximal unramified extension of $\mathbb{Q}_p$.

The p-adic L-function in the next theorem was first constructed in [BDP13] as a continuous function on characters of Γ. Its realization as a measure in Λ_0^{ur} was given in [CH18] following an approach introduced in [Bra11]. As it will suffice for our purposes, we describe below a multiple of that p-adic L-function by an element in $\Phi^\times$.

As in [CH18, §2.3], define $\vartheta \in \mathcal{K}$ by

$$\vartheta := \frac{D' + \sqrt{-D_{\mathcal{K}}}}{2}, \quad \text{where } D' = \begin{cases} D_{\mathcal{K}} & \text{if } 2 \nmid D_{\mathcal{K}}, \\ D_{\mathcal{K}}/2 & \text{else,} \end{cases}$$

and let Ω_p and $\Omega_{\mathcal{K}}$ be CM periods attached to $\mathcal{K}$ as in [*op. cit.*, §2.5].

Theorem 3.1.1. *There exists an element $\mathscr{L}_v(f,\chi) \in \Lambda^{\mathrm{ur}}$ such that for every character ξ of Γ crystalline at both v and $\bar{v}$ and corresponding to a Hecke character of $\mathcal{K}$ of infinity type $(n,-n)$ with $n \in \mathbb{Z}_{\geqslant 1}$, we have*

$$\mathscr{L}_v(f,\chi)^2(\xi) = \frac{\Omega_p^{4n}}{\Omega_{\mathcal{K}}^{4n}} \cdot \frac{\Gamma(n)\Gamma(n+1)\xi(\mathfrak{N}^{-1})}{4(2\pi)^{2n+1}(\mathrm{Im}\,\vartheta)^{2n-1}} \cdot \big(1 - a_p(f)\chi\xi(\bar{v})p^{-1} + \varepsilon_f(p)\chi\xi(\bar{v})^2p^{-1}\big)^2 \cdot L(f,\chi\xi,1).$$

Proof. Let η be an anticyclotomic Hecke character of $\mathcal{K}$ of infinity type $(1,-1)$ and conductor dividing $c\mathcal{O}_{\mathcal{K}}$, and define $\mathfrak{L}_{v,\eta}(f,\chi) \in \Lambda_0^{\mathrm{ur}}$ by

$$\mathfrak{L}_{v,\eta}(f,\chi)(\phi) = \sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_c)} \eta\chi(\mathfrak{a})\mathbf{N}(\mathfrak{a})^{-1} \int_{\mathbb{Z}_p^\times} \eta_{\bar{v}}(\phi|[\mathfrak{a}])\, \mathrm{d}\mu_{f_{\mathfrak{a}}^\flat}$$

for all continuous characters $\phi : \Gamma \to \overline{\mathbb{Q}}_p^\times$, where:

- $f^\flat = \sum_{p \nmid n} a_n(f)q^n$ is the p-depletion of f,
- $\mu_{f_{\mathfrak{a}}^\flat}$ is the measure on $\mathbb{Z}_p^\times$ corresponding (under the Amice transform) to the power series

$$f^\flat(t_{\mathfrak{a}}^{\mathbf{N}(\mathfrak{a})c\sqrt{-D_{\mathcal{K}}}^{-1}}) \in \mathscr{O}^{\mathrm{ur}}[\![t_{\mathfrak{a}} - 1]\!]$$

with $t_{\mathfrak{a}}$ the Serre–Tate coordinate of the reduction of the point $x_{\mathfrak{a}}$ on the Igusa tower of tame level N constructed in [CH18, (2.5)],

- $\eta_v(x) := \eta(\mathrm{rec}_v(x))$ with $\mathrm{rec}_v : \mathbb{Q}_p^\times = \mathcal{K}_v^\times \to G_{\mathcal{K}}^{\mathrm{ab}} \twoheadrightarrow \Gamma$ the local reciprocity map at v,
- $\phi|[\mathfrak{a}] : \mathbb{Z}_p^\times \to \overline{\mathbb{Q}}_p^\times$ is defined by $(\phi|[\mathfrak{a}])(x) = \phi(\mathrm{rec}_v(x)\sigma_{\mathfrak{a}}^{-1})$ with $\sigma_{\mathfrak{a}}$ the Artin symbol of $\mathfrak{a}$.

The same calculation as in [CH18, Prop. 3.8] then shows that the element $\mathscr{L}_v(f,\chi) \in \Lambda^{\mathrm{ur}}$ defined by

$$\mathscr{L}_v(f,\chi)(\xi) := \mathfrak{L}_{v,\eta}(f,\chi)(\eta^{-1}\xi)$$

has, in view of the explicit Waldspurger formula in [Hsi14b, Thm. 3.14], the stated interpolation property up to fixed element in $\Phi^\times$. The result follows. □

Remark 3.1.2. We our later use, we note that the complex period $\Omega_{\mathcal{K}} \in \mathbb{C}^\times$ in Theorem 3.1.1 (which also agrees with that in [BDP13, (5.1.16)]) is *different* from the complex period $\Omega_\infty \in \mathbb{C}^\times$ defined in [dS87, p. 66] and [HT93, (4.4b)]. In fact, one has

$$\Omega_\infty = 2\pi i \cdot \Omega_{\mathcal{K}}.$$

In terms of Ω_∞, the interpolation formula in Theorem 6.0.1 reads

$$\mathscr{L}_v(f,\chi)^2(\xi) = \frac{\Omega_p^{4n}}{\Omega_\infty^{4n}} \cdot \frac{\Gamma(n)\Gamma(n+1)\xi(\mathfrak{N}^{-1})}{4(2\pi)^{1-2n}(\mathrm{Im}\,\vartheta)^{2n-1}} \cdot \left(1 - a_p(f)\chi\xi(\overline{v})p^{-1} + \varepsilon_f(p)\chi\xi(\overline{v})^2p^{-1}\right)^2 \cdot L(f,\chi\xi,1).$$

Specialized to the range of critical values for the representation $V_{f,\chi}$, the Iwasawa–Greenberg main conjecture [Gre94] predicts the following.

Conjecture 3.1.3. *The module $\mathcal{X}_v(f,\chi)$ is Λ-torsion, and*

$$\mathrm{char}_\Lambda(\mathcal{X}_v(f,\chi)) = (\mathscr{L}_v(f,\chi)^2).$$

In Theorem 4.0.2, we will explain the close link between Conjectures 1.3.1 and 3.1.3.

3.2. **Katz p-adic L-functions.**

We continue to assume that $\mathcal{K}$ satisfies (spl). Let $\mathfrak{c} \subset \mathcal{O}_{\mathcal{K}}$ be an ideal prime to p, and let $\mathcal{K}(\mathfrak{c}p^\infty)$ be the ray class field of $\mathcal{K}$ of conductor $\mathfrak{c}p^\infty$.

We say that a Hecke character ϕ of $\mathcal{K}$ is *self-dual* if it satisfies

$$\phi\phi^* = \mathbf{N}_{\mathcal{K}}.$$

Note that the infinity type of such ϕ is necessarily of the form $(-1+j,-j)$ for some $j \in \mathbb{Z}$.

The p-adic L-function in the next theorem follows from the work of Katz [Kat78], as extended by Hida–Tilouine [HT93] (see also [dS87]). Here we shall use the construction in [Hsi14a], and similarly as in Theorem 3.1.1, it will suffice for our purposes to describe a fixed $\Phi^\times$-multiple of the integral measure constructed in *op. cit.*.

For any Hecke character ξ of $\mathcal{K}$, we denote by $L_{\mathfrak{c}}(\xi,s)$ the Hecke L-function $L(\xi,s)$ with the Euler factors at the primes $\mathfrak{l}|\mathfrak{c}$ removed.

Theorem 3.2.1. *Let ϕ be a character of* $\mathrm{Gal}(\mathcal{K}(\mathfrak{c}p^\infty)/\mathcal{K})$ *corresponding to a self-dual Hecke character of infinity type $(-1+j,-j)$, with $j \in \mathbb{Z}_{\geqslant 0}$. Then there exists an element $\mathscr{L}_v(\phi) \in \Lambda^{\mathrm{ur}}$ such that for every character ξ of Γ crystalline at both v and $\bar{v}$ and corresponding to a Hecke character of infinity type $(n,-n)$ with $n \geqslant j$, we have*

$$\mathscr{L}_v(\phi)(\xi) = \frac{\Omega_p^{2n-2j+1}}{\Omega_\infty^{2n-2j+1}} \cdot \Gamma(n+1-j) \cdot \frac{(2\pi)^{n-j}}{(\mathrm{Im}\,\vartheta)^{n-j}} \cdot (1-\phi^{-1}\xi(\overline{v}))^2 \cdot L_{\mathfrak{c}}(\phi^{-1}\xi,0).$$

Proof. Let $\mathfrak{L}_v$ be the integral p-adic measure on $\mathrm{Gal}(\mathcal{K}(\mathfrak{c}p^\infty)/\mathcal{K})$ constructed in [Hsi14a, §4.8], so for every character χ of $\mathrm{Gal}(\mathcal{K}(\mathfrak{c}p^\infty)/\mathcal{K})$ corresponding to a Hecke character of $\mathcal{K}$ of infinity type $(k+\ell,-\ell)$ with $k>\ell\geqslant 0$ we have

$$\mathfrak{L}_v(\chi)=\frac{\Omega_p^{k+2\ell}}{\Omega_\infty^{k+2\ell}}\cdot\Gamma(k+\ell)\cdot\frac{(2\pi)^\ell}{(\operatorname{Im}\vartheta)^\ell}\cdot(1-\chi^{-1}(v)p^{-1})(1-\chi(\overline{v}))\cdot L_{\mathfrak{c}}(\chi,0).$$

Setting

$$\mathscr{L}_v(\phi)(\xi):=\mathfrak{L}_v(\phi^{-1}\cdot\pi^*\xi)$$

for all characters ξ of Γ, where $\pi^*\xi$ is the pullback of ξ under the projection $\pi:\mathrm{Gal}(\mathcal{K}(\mathfrak{c}p^\infty)/\mathcal{K})\to\Gamma$, the result follows immediately from [Hsi14a, Prop. 4.9], noting that the condition $n\geqslant j$ assures that the infinity type of $\phi^{-1}\xi$, namely $(1+n-j,j-n)$, is of the form $(1+\ell,-\ell)$ with $\ell\geqslant 0$, and the p-adic multiplier that appears is

$$(1-\phi\xi^{-1}(v)p^{-1})(1-\phi^{-1}\xi(\overline{v}))=(1-\phi^{-1}\xi(\overline{v}))^2,$$

since ϕ is self-dual and ξ is anticyclotomic. □

3.3. **Factorization.** As in §2.3, we now specialize to the case where $f=\theta_\psi$ for a Hecke character ψ of $\mathcal{K}$ of infinity type $(-1,0)$ and conductor $\mathfrak{f}_\psi$ with cyclic quotient of norm prime to $D_{\mathcal{K}}$, so that $\mathcal{K}$ satisfies hypothesis (Heeg) relative to $N=D_{\mathcal{K}}\cdot\mathbf{N}(\mathfrak{f}_\psi)$.

Fix an integer $c>0$ prime to Np, and let χ be a finite order Hecke character of $\mathcal{K}$ such that $\chi\mathbf{N}_{\mathcal{K}}^{-1}\in\Sigma_{\mathrm{cc}}^{(1)}(c,\mathfrak{N},\varepsilon_f)$. Then we have a $G_{\mathcal{K}}$-module decomposition

$$V_{f,\chi}\simeq V_{\psi^*\chi^*}\oplus V_{\psi\chi^*},\tag{3.1}$$

where $V_{f,\chi}$ is as in (2.1). Note that each of the characters $\psi\chi$ and $\psi^*\chi$ are self-dual (see [BDP12, Rem. 3.7]).

For the rest of this paper, we shall write $\mathscr{L}_v(\phi)$ for the p-adic L-function in Theorem 3.2.1 constructed with the auxiliary tame conductor $\mathfrak{c}=c\mathfrak{N}$ used in the proof.

The following result is a manifestation of the Artin formalism arising from the decomposition (3.1). A similar result in shown in [BDP12, Thm. 3.17]. As we shall see in §6, this is a counterpart on the analytic side of the Selmer group decomposition in Proposition 2.3.1.

Proposition 3.3.1. *Suppose that $f=\theta_\psi$ and χ are as above. Then*

$$\mathscr{L}_v(f,\chi)^2=u\cdot\mathscr{L}_v(\psi^*\chi^*)\cdot\mathscr{L}_v(\psi\chi^*),$$

where u is a unit in $(\Lambda^{\mathrm{ur}})^\times$.

Proof. This will follow by comparing the values interpolated by each side of the desired equality, using that an element in Λ^{ur} is uniquely determined by its values at infinitely many characters.

Let ξ be a character of Γ of infinity type $(n,-n)$ with $n\in\mathbb{Z}_{\geqslant 1}$ as in the statement of Theorem 3.1.1. The decomposition (3.1) yields

$$L(f,\chi\xi,1)=L(\psi\chi\xi\mathbf{N}_{\mathcal{K}}^{-1},0)\cdot L(\psi^*\chi\xi\mathbf{N}_{\mathcal{K}}^{-1},0)=L((\psi^*\chi^*)^{-1}\xi,0)\cdot L((\psi\chi^*)^{-1}\xi,0),\tag{3.2}$$

using that $\psi\chi$ and $\psi^*\chi$ are self-dual. The factors in (3.2) are interpolated by $\mathscr{L}_v(\psi^*\chi^*)(\xi)$ and $\mathscr{L}_v(\psi\chi^*)(\xi)$, respectively. Noting that

$$\big(1-(\psi^*\chi^*)^{-1}\xi(\overline{v})\big)\cdot\big(1-(\psi\chi^*)^{-1}\xi(\overline{v})\big)=\big(1-a_p(f)\chi\xi(\overline{v})p^{-1}+\varepsilon_f(p)\chi\xi(\overline{v})^2p^{-1}\big),$$

in light of Theorem 3.2.1 for $\mathscr{L}_v(\psi^*\chi^*)$ and $\mathscr{L}_v(\psi\chi^*)$ (with $j=1$ and $j=0$, respectively), we thus find

$$\mathscr{L}_v(\psi\chi)(\xi)\cdot\mathscr{L}_v(\psi^*\chi)(\xi)=\frac{\Omega_p^{2n-1}}{\Omega_\infty^{2n-1}}\cdot\frac{\Omega_p^{2n+1}}{\Omega_\infty^{2n+1}}\cdot\Gamma(n)\Gamma(n+1)\cdot\frac{(2\pi)^{n-1}}{(\operatorname{Im}\vartheta)^{n-1}}\cdot\frac{(2\pi)^n}{(\operatorname{Im}\vartheta)^n}$$
$$\times\big(1-a_p(f)\chi\xi(\overline{v})p^{-1}+\varepsilon_f(p)\chi\xi(\overline{v})^2p^{-1}\big)^2\cdot L(f,\chi\xi,1).$$

The result now follows from Theorem 3.1.1 and Remark 3.1.2. □

Remark 3.3.2. Note that the trivial character is in the range of interpolation for $\mathscr{L}_v(\psi\chi^*)$, but lies outside the range of interpolation for both $\mathscr{L}_v(\psi^*\chi^*)$ and $\mathscr{L}_v(f,\chi)$.

4. Explicit reciprocity law

In this section we explain a variant of the explicit reciprocity law proved in [CH18] relating the Λ-adic Heegner class $\mathbf{z}_{f,\chi}$ to the p-adic L-function $\mathscr{L}_v(f,\chi)$ via a Perrin-Riou big logarithm map, and record a key consequence. We let $f=\theta_\psi$ and χ be as in §2.3.

For every $w|p$, the natural map $\mathrm{H}^1(\mathcal{K}_w,\mathscr{F}_w^+\mathbf{T}_{f,\chi})\to\mathrm{H}^1(\mathcal{K}_w,\mathbf{T}_{f,\chi})$ induced by (2.3) is injective, since its kernel is $\mathrm{H}^0(\mathcal{K}_w,\mathscr{F}_w^-\mathbf{T}_{f,\chi})=0$. Therefore, in view of (2.6) the image of $\mathbf{z}_{f,\chi}$ under the restriction map

$$\mathrm{loc}_w:\mathrm{H}^1(\mathcal{K},\mathbf{T}_{f,\chi})\to\mathrm{H}^1(\mathcal{K}_w,\mathbf{T}_{f,\chi})$$

is naturally contained in $\mathrm{H}^1(\mathcal{K}_w,\mathscr{F}_w^+\mathbf{T}_{f,\chi})$. Let Φ^{ur} the compositum of Φ and $\mathbb{Q}_p^{\mathrm{ur}}$.

Theorem 4.0.1. *There is a Λ^{ur}-linear isomorphism $\mathrm{Log}_v:\mathrm{H}^1(\mathcal{K}_v,\mathscr{F}_v^+\mathbf{T}_{f,\chi})\otimes\Lambda^{\mathrm{ur}}\to\Lambda^{\mathrm{ur}}$ such that*

$$\mathrm{Log}_v\big(\mathrm{loc}_v(\mathbf{z}_{f,\chi})\big)=c\cdot\mathscr{L}_v(f,\chi)$$

for some $c\in(\Phi^{\mathrm{ur}})^\times$.

Proof. The existence of the map Log_v (with coefficients in Λ_0^{ur}, rather than Λ^{ur}) follows from the two-variable extension by Loeffler–Zerbes [LZ14] of Perrin-Riou's big logarithm map [PR94], and the proof of the explicit reciprocity law (integrally) is given in [CH18, §5.3]. That the Λ^{ur}-linear map Log_v is injective follows from [LZ14, Prop. 4.11], and so it becomes an isomorphism after extending scalars to $\Lambda^{\mathrm{ur}}=\Lambda_0^{\mathrm{ur}}\otimes_{\mathscr{O}}\Phi$. □

Similarly as observed in [Cas13] and [Wan21], the equivalence between Conjectures 1.3.1 and 3.1.3 can be deduced from Theorem 4.0.1 using Poitou–Tate global duality.

Theorem 4.0.2. *Assume that the class $\mathbf{z}_{f,\chi}$ is not Λ-torsion. Then the following are equivalent:*

(a) $\mathrm{rank}_\Lambda\mathcal{S}_{\mathrm{ord}}(f,\chi)=\mathrm{rank}_\Lambda\mathcal{X}_{\mathrm{ord}}(f,\chi)=1$,

(a') $\mathcal{X}_v(f,\chi)$ *is Λ-torsion;*

and the following are equivalent:

(b) $\mathrm{char}_\Lambda\big(\mathcal{X}_{\mathrm{ord}}(f,\chi)_{\Lambda\text{-tors}}\big)\subset\mathrm{char}_\Lambda\big(\mathcal{S}_{\mathrm{ord}}(f,\chi)/\Lambda\cdot\mathbf{z}_{f,\chi}\big)^2$,

(b') $\mathrm{char}_\Lambda\big(\mathcal{X}_v(f,\chi)\big)\subset\big(\mathscr{L}_v(f,\chi)^2\big)$,

and similarly for the opposite divisibilities. In particular, Conjectures 1.3.1 and 3.1.3 are equivalent.

Remark 4.0.3. Note that for the last claim in the theorem we are using the isomorphisms (2.5) and (2.6).

Proof of Theorem 4.0.2. This can be extracted from the arguments in [Cas17, App. A], but since our setting is slightly different (in particular, $E(K)[p]$ is reducible) we provide the necessary details for the convenience of the reader. We explain the implications (a) ⇒ (a') and (b') ⇒ (b) (the only implication we will need later), and note that the other implication follows from the same ideas.

Following [Cas17, §2.1], below we denote by $\mathcal{S}_{\mathrm{str,rel}}(f,\chi)$ (resp. $\mathcal{S}_{\mathrm{ord,rel}}(f,\chi)$, etc.) the Selmer group defined as in §2.1 but with the strict at v and relaxed at $\overline{v}$ (resp. ordinary at v and relaxed at $\overline{v}$, etc.) local conditions, so in particular $\mathcal{S}_{\mathrm{ord,ord}}(f,\chi)=\mathcal{S}_{\mathrm{ord}}(f,\chi)$ by definition.

Assume (a), and consider the exact sequence from global duality

$$0 \to \mathcal{S}_{\mathrm{str,ord}}(f,\chi) \to \mathcal{S}_{\mathrm{ord}}(f,\chi) \xrightarrow{\mathrm{loc}_v} \mathrm{H}^1(\mathcal{K}_v, \mathscr{F}_v^+\mathbf{T}_{f,\chi}) \to \mathcal{X}_{\mathrm{rel,ord}}(f,\chi) \to \mathcal{X}_{\mathrm{ord}}(f,\chi) \to 0. \tag{4.1}$$

Since $\mathbf{z}_{f,\chi}$ is not Λ-torsion by hypothesis, by Theorem 4.0.1 it follows from (4.4) that $\mathcal{X}_{\mathrm{rel,ord}}(f,\chi)$ has Λ-rank one and $\mathcal{S}_{\mathrm{str,ord}}(f,\chi)$ is Λ-torsion. Since

$$\mathrm{rank}_\Lambda \mathcal{X}_{\mathrm{rel,ord}}(f,\chi) = 1 + \mathrm{rank}_\Lambda \mathcal{X}_{\mathrm{ord,str}}(f,\chi)$$

(*cf.* [Cas17, Lem. 2.3]), we conclude that $\mathcal{X}_{\mathrm{ord,str}}(f,\chi)$ is Λ-torsion, and from the exact sequence

$$0 \to \mathcal{S}_{\mathrm{str,rel}}(f,\chi) \to \mathcal{S}_{\mathrm{ord,rel}}(f,\chi) \xrightarrow{\mathrm{loc}_v} \mathrm{H}^1(\mathcal{K}_v, \mathscr{F}_v^+\mathbf{T}_{f,\chi}) \to \mathcal{X}_{\mathrm{rel,str}}(f,\chi) \to \mathcal{X}_{\mathrm{ord,str}}(f,\chi) \to 0 \tag{4.2}$$

we conclude that $\mathcal{X}_{\mathrm{rel,str}}(f,\chi) = \mathcal{X}_v(f,\chi)$ is Λ-torsion, i.e., (a') holds.

Now, in addition to (a), assume (b'). Then $\mathcal{S}_{\mathrm{str,rel}}(f,\chi)$ is Λ-torsion (since so is $\mathcal{X}_v(f,\chi)$, as we just showed), and since $\mathrm{H}^1(K^\Sigma/K, \mathbf{T}_{f,\chi})$ is Λ-torsion free as a consequence of (2.8) and [AH06, Prop. 1.1.6], it follows that in fact

$$\mathcal{S}_{\mathrm{str,rel}}(f,\chi) = 0. \tag{4.3}$$

Thus (4.2) reduces to the exact sequence

$$0 \to \mathcal{S}_{\mathrm{ord,rel}}(f,\chi) \xrightarrow{\mathrm{loc}_v} \mathrm{H}^1(\mathcal{K}_v, \mathscr{F}_v^+\mathbf{T}_{f,\chi}) \to \mathcal{X}_v(f,\chi) \to \mathcal{X}_{\mathrm{ord,str}}(f,\chi) \to 0. \tag{4.4}$$

Since $\mathrm{H}^1(\mathcal{K}_v, \mathscr{F}_v^+\mathbf{T}_{f,\chi})$ has Λ-rank one, the assumption that $\mathbf{z}_{f,\chi}$ is not Λ-torsion together with Theorem 4.0.1 implies that $\mathcal{S}_{\mathrm{ord,rel}}(f,\chi)$ has Λ-rank one. Since $\mathbf{z}_{f,\chi} \in \mathcal{S}_{\mathrm{ord}}(f,\chi) \subset \mathcal{S}_{\mathrm{ord,rel}}(f,\chi)$, it follows that $\mathcal{S}_{\mathrm{ord}}(f,\chi)$ also has Λ-rank one, and by [Cas17, Lem. 2.3(1)] so does $\mathcal{X}_{\mathrm{ord}}(f,\chi)$.

Hence the quotient $\mathcal{S}_{\mathrm{ord,rel}}(f,\chi)/\mathcal{S}_{\mathrm{ord}}(f,\chi)$ is Λ-torsion, and since it injects in $\mathrm{H}^1(\mathcal{K}_{\overline{v}}, \mathscr{F}_{\overline{v}}^-\mathbf{T}_{f,\chi})$ which is Λ-torsion-free, this shows the equality $\mathcal{S}_{\mathrm{ord}}(f,\chi) = \mathcal{S}_{\mathrm{ord,rel}}(f,\chi)$. Therefore the first two terms in the exact sequence (4.4) agree with the first two terms in the exact sequence

$$0 \to \mathcal{S}_{\mathrm{ord}}(f,\chi) \xrightarrow{\mathrm{loc}_v} \mathrm{H}^1(\mathcal{K}_v, \mathscr{F}_v^+\mathbf{T}_{f,\chi}) \to \mathcal{X}_{\mathrm{rel,ord}}(f,\chi) \to \mathcal{X}_{\mathrm{ord}}(f,\chi) \to 0 \tag{4.5}$$

(note that $\mathcal{S}_{\mathrm{str,ord}}(f,\chi)$ as a consequence of (4.3)), and this yields

$$0 \to \frac{\mathcal{S}_{\mathrm{ord}}(f,\chi)}{\Lambda\cdot\mathbf{z}_{f,\chi}} \xrightarrow{\mathrm{loc}_v} \frac{\mathrm{H}^1(\mathcal{K}_v, \mathscr{F}_v^+\mathbf{T}_{f,\chi})}{\Lambda\cdot\mathrm{loc}_v(\mathbf{z}_{f,\chi})} \to \mathrm{coker}(\mathrm{loc}_v) \to 0.$$

In view of Theorem 4.0.1, it follows that

$$\mathrm{char}_\Lambda\left(\frac{\mathcal{S}_{\mathrm{ord}}(f,\chi)}{\Lambda\cdot\mathbf{z}_{f,\chi}}\right)\cdot\mathrm{char}_\Lambda\big(\mathrm{coker}(\mathrm{loc}_v)\big)\Lambda^{\mathrm{ur}} = (\mathscr{L}_v(f,\chi)). \tag{4.6}$$

Next, from (4.4) and (4.5) we can extract the short exact sequences

$$0 \to \mathrm{coker}(\mathrm{loc}_v) \to \mathcal{X}_v(f,\chi) \to \mathcal{X}_{\mathrm{ord,str}}(f,\chi) \to 0,$$
$$0 \to \mathrm{coker}(\mathrm{loc}_v) \to \mathcal{X}_{\mathrm{rel,ord}}(f,\chi) \to \mathcal{X}_{\mathrm{ord}}(f,\chi) \to 0,$$

from which we readily obtain (taking Λ-torsion in the first exact sequence and using a straightforward variant of [Cas17, Lem. 2.3]) the relations

$$\begin{aligned}\mathrm{char}_\Lambda\big(\mathcal{X}_v(f,\chi)\big) &= \mathrm{char}_\Lambda\big(\mathcal{X}_{\mathrm{ord,str}}(f,\chi)\big)\cdot\mathrm{char}_\Lambda\big(\mathrm{coker}(\mathrm{loc}_v)\big)\\ &= \mathrm{char}_\Lambda\big(\mathcal{X}_{\mathrm{rel,ord}}(f,\chi)_{\Lambda\text{-tors}}\big)\cdot\mathrm{char}_\Lambda\big(\mathrm{coker}(\mathrm{loc}_v)\big)\\ &= \mathrm{char}_\Lambda\big(\mathcal{X}_{\mathrm{ord}}(f,\chi)_{\Lambda\text{-tors}}\big)\cdot\mathrm{char}_\Lambda(\mathrm{coker}(\mathrm{loc}_v))^2.\end{aligned}$$

Combined with (4.6), we thus obtain

$$\mathrm{char}_\Lambda\big(\mathcal{X}_v(f,\chi)\big)\cdot\mathrm{char}_\Lambda\left(\frac{\mathcal{S}_{\mathrm{ord}}(f,\chi)}{\Lambda\cdot\mathbf{z}_{f,\chi}}\right)^2\Lambda^{\mathrm{ur}} = \mathrm{char}_\Lambda\big(\mathcal{X}_{\mathrm{ord}}(f,\chi)_{\Lambda\text{-tors}}\big)\cdot\big(\mathscr{L}_v(f,\chi)^2\big).$$

The result follows. □

5. Twisted anticyclotomic main conjectures for $\mathcal{K}$

Let $\mathcal{K}$ be an imaginary quadratic field satisfying (spl). The Iwasawa main conjecture for $\mathcal{K}$ was proved by Rubin [Rub91] under some restrictions on p (including $p \nmid \mathcal{O}_{\mathcal{K}}^{\times}$) that were removed in subsequent work by Johnson-Leung–Kings [JLK11] and Oukhaba–Viguié [OV16]. In this section we record a consequence of these results for the anticyclotomic $\mathbb{Z}_p$-extension.

Note that if ξ is a self-dual Hecke character in the sense of §3.2, then the Hecke L-function $L(\xi^{-1}, s)$ is self-dual, with a functional equation relating its values at s and $-s$. In the following, by the *sign* of ξ we refer to the sign appearing in the functional equation for $L(\xi^{-1}, s)$.

Theorem 5.0.1. *Let ψ be a Hecke character of $\mathcal{K}$ of infinity type $(-1, 0)$, and let χ be a finite order of character of such that the product $\psi\chi$ is self-dual. Assume that $\psi\chi^*$ has sign $+1$. Then:*

(i) *$\mathcal{X}_v(\psi\chi^*)$ is Λ-torsion and the following equality holds:*

$$\mathrm{char}_{\Lambda}\big(\mathcal{X}_v(\psi\chi^*)\big) = \big(\mathscr{L}_v(\psi\chi^*)\big).$$

(ii) *The following divisibility holds:*

$$\mathrm{char}_{\Lambda}\big(\mathcal{X}_v(\psi^*\chi^*)\big) \subset \big(\mathscr{L}_v(\psi^*\chi^*)\big).$$

Proof. As noted in Remark 2.2.2, the Iwasawa module $\mathcal{X}_v(\psi\chi^*)$ recovers the Bloch–Kato Selmer group for $\psi\chi^*$ over the anticyclotomic $\mathbb{Z}_p$-extension $\mathcal{K}_\infty/\mathcal{K}$, and so the result of part (i) follows from [AH06, Thm. 2.4.17], as extended in [Arn07, Thm. 3.9]. (In these references, the hypothesis $p > 3$ arises from their appearance in [Rub91], but as already mentioned this restriction can be removed thanks to [JLK11, OV16].)

For (ii), put $\widetilde{\Lambda}_0 = \mathscr{O}[\![\mathrm{Gal}(\mathcal{K}(\mathfrak{c}p^\infty)/\mathcal{K})]\!]$, $\widetilde{\Lambda} = \widetilde{\Lambda}_0 \otimes_{\mathscr{O}} \Phi$, and define $\widetilde{\mathcal{X}}_v(\psi^*\chi^*), \widetilde{\mathcal{X}}_v(\psi\chi^*)$ similarly as $\mathcal{X}_v(\psi^*\chi^*), \mathcal{X}_v(\psi\chi^*)$ in §2.2 but with $\widetilde{\Lambda}_0$ in place of Λ_0. By the Iwasawa main conjecture for $\mathcal{K}$, the module $\mathcal{X}_v(\psi\chi^*)$ is $\widetilde{\Lambda}_0$-torsion, with

$$\mathrm{char}_{\widetilde{\Lambda}}\big(\widetilde{\mathcal{X}}_v(\psi\chi^*)\big) = \big(\mathfrak{L}_v(\psi\chi^*)\big), \tag{5.1}$$

where $\mathfrak{L}_v$ is the integral p-adic measure appearing in the proof of Theorem 3.2.1, and $\mathfrak{L}_v(\psi\chi^*)$ denotes its twist by $\psi^{-1}\chi^{*-1}$. Noting that $\widetilde{\mathcal{X}}_v(\psi^*\chi^*)$ is the twist (in the sense of [Rub00, §6.1]) of $\widetilde{\mathcal{X}}_v(\psi\chi^*)$ by $\psi\psi^{*-1}$, from Corollary 6.2.2 and Lemma 6.1.2 in *loc. cit.* we deduce from (5.1) that the module $\widetilde{\mathcal{X}}_v(\psi^*\chi^*)$ is $\widetilde{\Lambda}_0$-torsion, with

$$\mathrm{char}_{\widetilde{\Lambda}}\big(\widetilde{\mathcal{X}}_v(\psi^*\chi^*)\big) = \big(\mathfrak{L}_v(\psi^*\chi^*)\big). \tag{5.2}$$

The divisibility in (ii) now follows from (5.2) after descent. □

6. The main results

Recall that $\mathcal{K}$ is an imaginary quadratic field of discriminant $-D_{\mathcal{K}} < 0$ satisfying (spl), with v the prime of $\mathcal{K}$ above p induced by our fixed embedding $\imath_p$.

Theorem 6.0.1. *Let ψ be a Hecke character of $\mathcal{K}$ of infinity type $(-1, 0)$ and conductor $\mathfrak{f}_\psi$ with cyclic quotient of norm prime to $D_{\mathcal{K}}$, and set*

$$f = \theta_\psi, \quad N = D_{\mathcal{K}} \cdot \mathbf{N}(\mathfrak{f}_\psi), \quad \mathfrak{N} = \mathfrak{d}_{\mathcal{K}} \cdot \mathfrak{f}_\psi.$$

Let c be a positive integer prime to Np, and let χ be a finite order character such that $\chi\mathbf{N}_{\mathcal{K}}^{-1} \in \Sigma^{(1)}_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon_f)$. Assume that $\psi\chi$ has sign -1. Then:

(i) *The class $\mathbf{z}_{f,\chi}$ is not Λ-torsion.*

(ii) *The module* $\mathcal{X}_v(f,\chi)$ *is* Λ*-torsion, and*

$$\mathrm{char}_\Lambda\big(\mathcal{X}_v(f,\chi)\big)\Lambda^{\mathrm{ur}} \subset \big(\mathscr{L}_v(f,\chi)^2\big).$$

Proof. Part (i) follows from [BD20, Thm. 1.1], so we focus on (ii). By the Gross–Zagier formula, the non-triviality of $\mathbf{z}_{f,\chi}$ implies that for all but finitely many finite order characters $\xi:\Gamma\to\mu_{p^\infty}$ we have

$$\mathrm{ord}_{s=1}L(f,\chi\xi,s)=1. \tag{6.1}$$

Fix any such ξ, and note that $L(f,\chi\xi,s)$ factors as

$$L(f,\chi\xi,s)=L(\psi\chi\xi,s)\cdot L(\psi^*\chi\xi,s) \tag{6.2}$$

and has sign -1, since (Heeg) holds in our setting (see §3.3). By our sign assumption on $\psi\chi$, it follows that $L(\psi^*\chi,s)$ has sign $+1$ and from (6.1) and (6.2) we conclude

$$\mathrm{ord}_{s=1}L(\psi\chi\xi,s)=1,\quad L(\psi^*\chi\xi,1)\neq 0.$$

By an application of the Gross-Zagier formula [YZZ13] and [Nek07, Thm. 3.2] we have

$$\mathrm{ord}_{s=1}L(\psi\chi\xi,s)=1 \quad\Longrightarrow\quad \mathrm{corank}_{\mathscr{O}}\mathrm{H}^1_{\mathrm{f}}(\mathcal{K},W_{\psi^*\chi^*\xi^*})=1,$$

and by [Nek12, Thm. B] we have

$$L(\psi^*\chi\xi,1)\neq 0 \quad\Longrightarrow\quad \mathrm{corank}_{\mathscr{O}}\mathrm{H}^1_{\mathrm{f}}(\mathcal{K},W_{\psi\chi^*\xi^*})=0,$$

where $\mathrm{H}^1_{\mathrm{f}}(\mathcal{K},W_{\psi^*\chi^*\xi^*})$ and $\mathrm{H}^1_{\mathrm{f}}(\mathcal{K},W_{\psi\chi^*\xi^*})$ are the Bloch–Kato Selmer groups for $\psi^*\chi^*\xi^*$ and $\psi\chi^*\xi^*$, respectively, whose definition is recalled in §7 below.

By the analogue of the decomposition (7.1) below, it follows that $\mathrm{H}^1_{\mathrm{f}}(K,W_{f,\chi\xi})$ has $\mathscr{O}$-corank one, and therefore so does $\mathrm{Sel}_{\mathfrak{P}^\infty}(A_{f,\chi\xi}/\mathcal{K})$. Varying ξ, by a variant of Mazur's control theorem it follows that $\mathcal{X}(A_{f,\chi}/\mathcal{K}_\infty)$ (or equivalently, $\mathcal{S}_{\mathrm{ord}}(f,\chi)$ and $\mathcal{X}_{\mathrm{ord}}(f,\chi)$) has Λ-rank one, and so by Theorem 4.0.2 we conclude that $\mathcal{X}_v(f,\chi)$ is Λ-torsion.

Finally, by the decomposition in Proposition 2.3.1 and the factorization in Proposition 3.3.1, the divisibility in part (ii) of the theorem follows from Theorem 5.0.1. □

Corollary 6.0.2. *Let* $f=\theta_\psi$ *and* χ *be an in Theorem 6.0.1, and assume that* $\psi\chi$ *has sign* -1*. Then the modules* $\mathcal{S}(A_{f,\chi}/\mathcal{K}_\infty)$ *and* $\mathcal{X}(A_{f,\chi}/\mathcal{K}_\infty)$ *have both* Λ*-rank one, and*

$$\mathrm{char}_\Lambda(\mathcal{X}(A_{f,\chi}/\mathcal{K}_\infty)_{\Lambda\text{-tors}})\subset \mathrm{char}_\Lambda\big(\mathcal{S}(A_{f,\chi}/\mathcal{K}_\infty)/\Lambda\cdot\mathbf{z}_{f,\chi}\big)^2.$$

Proof. That $\mathcal{S}(A_{f,\chi}/\mathcal{K}_\infty)$ and $\mathcal{X}(A_{f,\chi}/\mathcal{K}_\infty)$ have both Λ-rank one has been shown in the course of the proof of Theorem 6.0.1, and the divisibility in the statement of Corollary 6.0.2 follows from Theorem 4.0.2 and the divisibility in part (ii) of Theorem 6.0.1. □

7. The p-converse

In this section we deduce from our main results the proof of Theorem A in the Introduction. Let λ be a self-dual Hecke character of infinity type $(-1,0)$ and conductor $\mathfrak{f}_\psi$, and suppose that:

(a) λ has sign -1;
(b) λ has central character $\omega_\lambda=\eta_{\mathcal{K}}$;
(c) $\mathfrak{d}_{\mathcal{K}}\|\mathfrak{f}_\lambda$.

Note that $\mathfrak{f}_\lambda$ is divisible by $\mathfrak{d}_{\mathcal{K}}=(\sqrt{-D_{\mathcal{K}}})$ by condition (b). Since λ is self-dual, $\mathfrak{f}_\lambda$ is invariant under complex conjugation, so by condition (c) we can write $\mathfrak{f}_\lambda=(c)\mathfrak{d}_{\mathcal{K}}$ for a unique $c>0$.

We shall apply Corollary 6.0.2 for a pair (ψ,χ) which is *good for* λ in the following sense:

(G1) ψ has infinity type $(-1,0)$ and conductor $\mathfrak{f}_\psi$ with cyclic quotient of norm prime to $pD_{\mathcal{K}}$;
(G2) χ is a finite order character such that $\chi\mathbf{N}_{\mathcal{K}}^{-1}\in\Sigma^{(1)}_{\mathrm{cc}}(c,\mathfrak{N},\varepsilon_f)$, where $f=\theta_\psi$ and $\mathfrak{N}=\mathfrak{f}_\psi\mathfrak{d}_{\mathcal{K}}$;
(G3) $\psi\chi=\lambda$;

(G4) $L(\psi^{*-1}\chi^{-1},0)\neq 0$.

The existence of good pairs for λ is shown in [BDP12, Lem. 3.29] building on the non-vanishing results of Greenberg [Gre85] and Rohrlich [Roh84].

Fix a good pair (ψ,χ) for λ, and let F be a number field of containing the values of ψ and χ. Let $\mathfrak{P}$ be the prime of F above p induced by our fixed embedding $\imath_p$, let Φ be the completion of F at $\mathfrak{P}$, and let $\mathscr{O}$ be the ring of integers of Φ. Similarly as in (2.2), for any Hecke character ξ put

$$W_\xi := T_\xi\otimes_{\mathscr{O}}\mathscr{D},$$

where $\mathscr{D}=\Phi/\mathscr{O}$. Let Σ a finite set of places of $\mathcal{K}$ containing ∞, p, and the primes of $\mathcal{K}$ dividing the conductor of λ. Denote by $\mathrm{H}^1_{\mathrm{f}}(\mathcal{K},W_{\psi^*\chi^*})$ the *Bloch–Kato Selmer group* for $\psi^*\chi^*$:

$$\mathrm{H}^1_{\mathrm{f}}(\mathcal{K},W_{\psi^*\chi^*})=\ker\left\{\mathrm{H}^1(\mathcal{K}^\Sigma/\mathcal{K},W_{\psi^*\chi^*})\rightarrow\frac{\mathrm{H}^1(\mathcal{K}_{\overline{v}},W_{\psi^*\chi^*})}{\mathrm{H}^1(\mathcal{K}_{\overline{v}},W_{\psi^*\chi^*})_{\mathrm{div}}}\times\mathrm{H}^1(\mathcal{K}_v,W_{\psi^*\chi^*})\times\prod_{w\in\Sigma,w\nmid p}\mathrm{H}^1(\mathcal{K}_w^{\mathrm{ur}},W_{\psi^*\chi^*})\right\},$$

where $\mathrm{H}^1(\mathcal{K}_{\overline{v}},W_{\psi^*\chi^*})_{\mathrm{div}}\subset\mathrm{H}^1(\mathcal{K}_{\overline{v}},W_{\psi^*\chi^*})$ is the maximal divisible submodule and $\mathcal{K}_w^{\mathrm{ur}}$ denotes the maximal unramified extension of $\mathcal{K}_w$. Similarly, let

$$\mathrm{H}^1_{\mathrm{f}}(\mathcal{K},W_{\psi\chi^*})=\ker\left\{\mathrm{H}^1(\mathcal{K}^\Sigma/\mathcal{K},W_{\psi\chi^*})\rightarrow\mathrm{H}^1(\mathcal{K}_{\overline{v}},W_{\psi\chi^*})\times\frac{\mathrm{H}^1(\mathcal{K}_v,W_{\psi\chi^*})}{\mathrm{H}^1(\mathcal{K}_v,W_{\psi\chi^*})_{\mathrm{div}}}\times\prod_{w\in\Sigma,w\nmid p}\mathrm{H}^1(\mathcal{K}_w^{\mathrm{ur}},W_{\psi\chi^*})\right\}$$

be the Bloch–Kato Selmer group for $\psi\chi^*$ (see also [AH06, §1.1]). Finally, let $V_{f,\chi}$ be as in (2.1).

Lemma 7.0.1. *In the above setting, we have*

$$\mathrm{corank}_{\mathscr{O}}\mathrm{Sel}_{\mathfrak{P}^\infty}(A_{f,\chi}/\mathcal{K})=\mathrm{corank}_{\mathscr{O}}\mathrm{H}^1_{\mathrm{f}}(\mathcal{K},W_{\psi^*\chi^*})+\mathrm{corank}_{\mathscr{O}}\mathrm{H}^1_{\mathrm{f}}(\mathcal{K},W_{\psi\chi^*}).$$

Proof. It is a standard fact (see e.g. [BK90]), $\mathrm{Sel}_{\mathfrak{P}^\infty}(A_{f,\chi}/\mathcal{K})$ agrees with the Bloch–Kato Selmer group

$$\mathrm{H}^1_{\mathrm{f}}(\mathcal{K},W_{f,\chi})\subset\mathrm{H}^1(\mathcal{K},W_{f,\chi}),$$

where $W_{f,\chi}:=T_{f,\chi}\otimes_{\mathscr{O}}\mathscr{D}$ for the $G_{\mathcal{K}}$-stable $\mathscr{O}$-lattice $T_{f,\chi}\subset V_{f,\chi}$ coming from $T_{\mathfrak{P}}A_f$. In turn (see e.g. [Gre99, Prop. 2.2]), the local conditions defining $\mathrm{H}^1_{\mathrm{f}}(\mathcal{K},W_{f,\chi})$ at the primes w of $\mathcal{K}$ above p can be described in terms of the filtration (2.3), namely:

$$\mathrm{H}^1_{\mathrm{f}}(\mathcal{K},W_{f,\chi})=\ker\left\{\mathrm{H}^1(\mathcal{K}^\Sigma/\mathcal{K},W_{f,\chi})\rightarrow\prod_{w\mid p}\frac{\mathrm{H}^1(\mathcal{K}_w,W_{f,\chi})}{\mathrm{H}^1(\mathcal{K}_w,\mathscr{F}_w^+W_{f,\chi})_{\mathrm{div}}}\times\prod_{w\in\Sigma,w\nmid p}\mathrm{H}^1(\mathcal{K}_w^{\mathrm{ur}},W_{f,\chi}),\right\}$$

where $\mathscr{F}_w^{\pm}W_{f,\chi}:=\mathscr{F}_w^{\pm}T_{f,\chi}\otimes_{\mathscr{O}}\mathscr{D}$. Note that since $f=\theta_\psi$ we have

$$\mathrm{H}^1(\mathcal{K}_v,\mathscr{F}_v^+W_{f,\chi})=\mathrm{H}^1(\mathcal{K}_v,W_{\psi\chi^*}),\quad \mathrm{H}^1(\mathcal{K}_{\overline{v}},\mathscr{F}_{\overline{v}}^+W_{f,\chi})=\mathrm{H}^1(\mathcal{K}_{\overline{v}},W_{\psi^*\chi^*}).$$

Since different lattices $T_{f,\chi}$ give rise to Selmer groups $\mathrm{H}^1_{\mathrm{f}}(\mathcal{K},W_{f,\chi})$ having the same $\mathscr{O}$-corank, taking $T_{f,\chi}$ so that $W_{f,\chi}\simeq W_{\psi^*\chi^*}\oplus W_{\psi\chi^*}$, comparing the local conditions we thus find

$$\mathrm{H}^1_{\mathrm{f}}(\mathcal{K},W_{f,\chi})\simeq\mathrm{H}^1_{\mathrm{f}}(\mathcal{K},W_{\psi^*\chi^*})\oplus\mathrm{H}^1_{\mathrm{f}}(\mathcal{K},W_{\psi\chi^*})\tag{7.1}$$

and the result follows. $\square$

The following recovers Theorem A in the introduction as a special case.

Theorem 7.0.2. *Let λ be a self-dual Hecke character of $\mathcal{K}$ of infinity type $(-1,0)$ with central character $\omega_\lambda=\eta_{\mathcal{K}}$ and whose conductor $\mathfrak{f}_\lambda$ satisfies $\mathfrak{d}_{\mathcal{K}}\Vert\mathfrak{f}_\lambda$. Then*

$$\mathrm{corank}_{\mathscr{O}}\mathrm{Sel}_{\mathfrak{P}^\infty}(B_\lambda/\mathcal{K})=1\quad\Longrightarrow\quad\mathrm{ord}_{s=1}L(\lambda,s)=1.$$

Proof. By the p-parity conjecture [Nek01], if $\mathrm{corank}_{\mathscr{O}}\mathrm{Sel}_{\mathfrak{p}^\infty}(B_\lambda/\mathcal{K}) = 1$ then λ has sign -1. Let (ψ, χ) be a good pair for λ, i.e., satisfying conditions (G1)–(G4) above, so in particular

$$L(\psi^{*-1}\chi^{-1}, 0) \neq 0. \tag{7.2}$$

By Theorem 3.2.1, the nonvanishing (7.2) implies that the p-adic L-function $\mathscr{L}_v(\psi\chi^*)$ does not vanish at trivial character, so by Theorem 3.2.1 it follows that

$$\#\big(\mathscr{X}_v(\psi\chi^*)/(\gamma - 1)\mathscr{X}_v(\psi\chi^*)\big) < \infty,$$

where $\gamma \in \Gamma$ is any topological generator. Since $\mathscr{X}_v(\psi\chi^*)$ corresponds to the Bloch–Kato Selmer group for $\psi\chi^*$ over $\mathcal{K}_\infty/\mathcal{K}$ (see Remark 2.2.2), it follows that $\mathrm{corank}_{\mathscr{O}}\mathrm{H}^1_{\mathrm{f}}(\mathcal{K}, W_{\psi\chi^*}) = 0$. Since $\mathrm{Sel}_{\mathfrak{p}^\infty}(B_\lambda/\mathcal{K}) \simeq \mathrm{H}^1_{\mathrm{f}}(\mathcal{K}, W_{\psi^*\chi^*})$, from Lemma 7.0.1 we thus obtain

$$\mathrm{corank}_{\mathscr{O}}\mathrm{Sel}_{\mathfrak{p}^\infty}(B_\lambda/\mathcal{K}) = 1 \quad \Longrightarrow \quad \mathrm{corank}_{\mathscr{O}}\mathrm{Sel}_{\mathfrak{p}^\infty}(A_{f,\chi}/\mathcal{K}) = 1. \tag{7.3}$$

Now, Corollary 6.0.2 together with a variant of Mazur's control theorem immediately yields the implication

$$\mathrm{corank}_{\mathscr{O}}\mathrm{Sel}_{\mathfrak{p}^\infty}(A_{f,\chi}/\mathcal{K}) = 1 \quad \Longrightarrow \quad \mathbf{z}_{f,\chi,0} \neq 0 \in S_{\mathfrak{P}}(A_{f,\chi}/\mathcal{K}) \otimes_{\mathscr{O}} \Phi. \tag{7.4}$$

where $\mathbf{z}_{f,\chi,0}$ is the image of $\mathbf{z}_{f,\chi}$ under the specialization map $\mathcal{S}(A_{f,\chi}/\mathcal{K}_\infty) \to \mathrm{H}^1(\mathcal{K}, V_{f,\chi})$ at the trivial character. By definition, the class $\mathbf{z}_{f,\chi,0}$ is a nonzero multiple of

$$\sum_{\sigma \in \mathrm{Gal}(\tilde{\mathcal{K}}_c)/\mathcal{K}} \chi(\sigma) \cdot y_{f,c}^\sigma,$$

where $y_{f,c}$ is the Heegner class introduced in §1.2, and so

$$\mathbf{z}_{f,\chi,0} \neq 0 \quad \Longleftrightarrow \quad \mathrm{ord}_{s=1} L(f, \chi, s) = 1 \tag{7.5}$$

by virtue of the general Gross–Zagier formula, [YZZ13, CST14]. Finally, we note once more that (2.8) yields the factorization

$$L(f, \chi, s) = L(\psi\chi, s) \cdot L(\psi^*\chi, s).$$

Combining (7.3), (7.4), and (7.5) we thus obtain

$$\begin{aligned} \mathrm{corank}_{\mathscr{O}}\mathrm{Sel}_{\mathfrak{p}^\infty}(B_\lambda/\mathcal{K}) = 1 \quad &\Longrightarrow \quad \mathrm{ord}_{s=1} L(f, \chi, s) = 1 \\ &\Longrightarrow \quad \mathrm{ord}_{s=1} L(\psi\chi, s) = 1, \end{aligned}$$

using (7.2) and the above factorization for the last implication. Since $\psi\chi = \lambda$, this concludes the proof. □

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Ashay Burungale: California Institute of Technology, 1200 E California Blvd, Pasadena CA 91125, and The University of Texas at Austin, 2515 Speedway, Austin, TX 78712, USA

Email address: ashayburungale@gmail.com

Francesc Castella: University of California Santa Barbara, South Hall, Santa Barbara, CA 93106, USA

Email address: castella@ucsb.edu

Christopher Skinner: Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544-1000, USA

Email address: cmcls@princeton.edu

Ye Tian: Academy of Mathematics and Systems Science, MCM, HLM, Chinese Academy of Sciences, Beijing 100190, and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

Email address: ytian@math.ac.cn