AN ANTICYCLOTOMIC EULER SYSTEM FOR ADJOINT MODULAR GALOIS REPRESENTATIONS

RAÚL ALONSO, FRANCESC CASTELLA, AND ÓSCAR RIVERO

Abstract. Let K be an imaginary quadratic field and p a prime split in K. In this paper we construct an anticyclotomic Euler system for the adjoint representation attached to elliptic modular forms base changed to K. We also relate our Euler system to a p-adic L-function deduced from the construction by Eischen–Wan and Eischen–Harris–Li–Skinner of p-adic L-functions for unitary groups. This allows us to derive new cases of the Bloch–Kato conjecture in rank zero, and a divisibility towards an Iwasawa main conjecture.

Contents

1. Introduction

The goal of this paper is to study the Bloch–Kato conjecture and the anticyclotomic Iwasawa theory of certain twists of the adjoint Galois representation attached to elliptic modular forms base changed to an imaginary quadratic field.

Our main result is the construction of an anticyclotomic Euler system in this setting, which we relate to an analogue of the Hida–Schmidt p-adic L-function for the symmetric square. By Kolyvagin's method for anticyclotomic split Euler systems, as developed by Jetchev–Nekovář–Skinner, our results yield new cases of the Bloch–Kato conjecture in rank zero and a divisibility towards an Iwasawa main conjecture.

1.1. **The set-up.** Let $g \in S_l(N_g, \chi_g)$ be an ordinary newform of weight $l \geq 2$, level N_g, and nebentypus χ_g. Let $K/\mathbb{Q}$ be an imaginary quadratic field, and let ψ be a Hecke character of K of infinity type $(1-k, 0)$ for some even integer $k \geq 2$. We assume that the associated theta series $\theta_\psi \in S_k(N_\psi)$ has trivial nebentypus. Fix an odd prime $p \nmid 2N_gN_\psi$ and an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and for simplicity in this Introduction assume that the Hecke field of g and the values of ψ are contained in a number field L with a prime $\mathfrak{P}$ above p such that $L_{\mathfrak{P}} = \mathbb{Q}_p$. We assume that p splits in K and is a prime of ordinary reduction for g and, again for simplicity, that $p \nmid h_K$, the class number of K. We will also assume that g is not of CM-type.

Date: May 15, 2023.
2010 *Mathematics Subject Classification.* 11R23; 11F85, 14G35.

Let V_g be the (dual to Deligne's) p-adic Galois representation attached to g, and denote by $\mathrm{ad}^0(V_g) \subset \mathrm{End}_{\mathbb{Q}_p}(V_g)$ the adjoint representation on the trace-zero endomorphisms of V_g. We consider the conjugate self-dual G_K-representation

$$V := \mathrm{ad}^0(V_g)(\psi^{-1})(1 - k/2),$$

where (ψ^{-1}) denotes the twist by the inverse of ψ and $(1-k/2)$ is the twist by the $(1-k/2)$-th power of the p-adic cyclotomic character.

1.2. **Euler systems and p-adic L-functions.** In this paper we construct an anticyclotomic Euler system for V and relate it to an associated anticyclotomic p-adic L-function.

For a positive integer m we write $K[m]$ for the maximal p-extension inside the ring class field of K of conductor m. Denote by $\mathcal{S}'$ the set of all squarefree products of primes q in the positive density set $\mathcal{P}'$ of Definition 5.2; in particular, these primes split in K. For any p-adic G_K-representation W and a prime $\mathfrak{q}$ of K, put

$$P_{\mathfrak{q}}(W; X) = \det(1 - \mathrm{Fr}_{\mathfrak{q}}^{-1} X | W^\vee(1)),$$

where $\mathrm{Fr}_{\mathfrak{q}}$ denotes an arithmetic Frobenius element for the prime $\mathfrak{q}$ and $W^\vee$ denotes the contragredient representation of W. A natural lattice $T_g \subset V_g$ described in §2.1 defines a lattice in V denoted by T. Finally, let $H^1_{\mathrm{Iw}}(K[mp^\infty], T) = \varprojlim_r H^1(K[mp^r], T)$.

Theorem A (Theorem 5.4)**.** *Assume that $H^1(K[mp^s], T)$ is torsion-free for all $m \in \mathcal{S}'$ and $s \geq 0$. There exists a collection of classes*

$$\left\{\kappa_{\psi,\mathrm{ad}^0(g),m,\infty} \in H^1_{\mathrm{Iw}}(K[mp^\infty], T) \,:\, m \in \mathcal{S}'\right\}$$

such that whenever $m, mq \in \mathcal{S}'$ with q a prime, we have

$$\mathrm{cor}_{K[mq]/K[m]}(\kappa_{\psi,\mathrm{ad}^0(g),mq,\infty}) = P_{\mathfrak{q}}(V; \mathrm{Fr}_{\mathfrak{q}}^{-1})\, \kappa_{\psi,\mathrm{ad}^0(g),m,\infty},$$

where $\mathfrak{q}$ is any of the primes of K above q.

We obtain the Euler system classes $\kappa_{\psi,\mathrm{ad}^0(g),m,\infty}$ from a suitable modification of the diagonal Euler system classes $\kappa_{\psi,g,g^*,m,\infty}$ for

$$V^\psi_{\mathrm{ad}(g)} := V_g \otimes V_{g^*}(\psi^{-1})(1 - c)$$

constructed in [ACR21], where $g^* = g \otimes \chi_g^{-1}$ is the twist of g by the inverse of its nebentypus, and $c = (k + 2l - 2)/2$. It follows from our construction (and the results of [BSV21] that it builds upon) that $\kappa_{\psi,\mathrm{ad}^0(g),m,\infty}$ lands in the balanced Selmer groups $\mathrm{Sel}_{\mathrm{bal}}(K[mp^\infty], T)$ introduced in §2.3.

Next we are interested in the non-triviality of our Euler system in terms of L-values. To this end, in §4 we use some basic instances of Langlands functoriality to deduce from the work of Eischen–Harris–Li–Skinner [EHLS20] the construction of a p-adic L-function

$$L_p(\mathrm{ad}^0(g_K) \otimes \psi) \in \mathrm{Frac}\, \Lambda^{\mathrm{ac}}$$

interpolating the central L-value $L(V, 0)$ and its twists by a p-adic family of anticyclotomic Hecke characters. Here Λ^{ac} is the Iwasawa algebra of the Galois group Γ^{ac} of the anticyclotomic $\mathbb{Z}_p$-extension K_∞/K. Denoting by $\kappa_{\psi,\mathrm{ad}^0(g),\infty}$ the image of $\kappa_{\psi,\mathrm{ad}^0(g),1,\infty}$ in $\mathrm{Sel}_{\mathrm{unb}}(K_\infty, T)$, we can then prove the following.

Write $(p) = \mathfrak{p}\overline{\mathfrak{p}}$, with $\mathfrak{p}$ the prime of K above p induced by ι_p. For the following result, let $K_{\infty,\overline{\mathfrak{p}}}$ be the $\mathbb{Z}_p$-extension of K unramified outside $\overline{\mathfrak{p}}$, and let $\mathscr{F}^{\mathrm{bal}}_{\overline{\mathfrak{p}}}(T^\psi_{\mathrm{ad}(g)})$ denote the subspace of $T^\psi_{\mathrm{ad}(g)}$ defined in Section 2.3.

Theorem B (Corollary 5.7)**.** *Under some technical hypotheses on ψ, there is a Perrin-Riou big logarithm map $\mathfrak{Log} : H^1_{\mathrm{Iw}}(K_{\infty,\bar{\mathfrak{p}}}, \mathscr{F}^{\mathrm{bal}}_{\bar{\mathfrak{p}}}(T^{\psi}_{\mathrm{ad}(g)})) \longrightarrow \mathbb{Z}_p^{\mathrm{ur}}[\![\Gamma^{\mathrm{ac}}]\!]$ such that*

$$\mathfrak{Log}(\mathrm{res}_{\bar{\mathfrak{p}}}(\kappa_{\psi,\mathrm{ad}^0(g),\infty}))^2 = L_p(\mathrm{ad}^0(g_K) \otimes \psi) \cdot \mathscr{L}_{\mathfrak{p}}^{\mathrm{Katz}}(\psi)^{-,\iota}$$

up to multiplication by an element in $\overline{\mathbb{Q}}_p^{\times}$, where $\mathscr{L}_{\mathfrak{p}}^{\mathrm{Katz}}(\psi)^{-,\iota}$ is an anticyclotomic projection of Katz's p-adic L-function.

The proof of this result builds on the explicit reciprocity law of [BSV21] and a factorization formula for Hsieh's triple product p-adic L-function (see Theorem 4.9). This factorization is a p-adic manifestation of the Artin formalism arising from the decomposition

$$V^{\psi}_{\mathrm{ad}(g)} \simeq V \oplus V', \tag{1.1}$$

where $V' = \mathbb{Q}_p(\psi^{-1})(1 - k/2)$, and may be seen as an anticyclotomic analogue of Dasgupta's factorization [Das16]. However, the proof in our case is largely simplified by the fact that the p-adic L-functions involved have overlapping ranges of p-adic interpolation.

The technical hypotheses on ψ are used to ensure that the congruence ideal of a Hida family attached to ψ is generated by a second anticyclotomic projection of Katz's p-adic L-function, which in turn interpolates the ratio between two different types of periods.

1.3. **Applications.** Using Kolyvagin's methods, as developed by Jetchev–Nekovář–Skinner [JNS] in the split anticyclotomic setting, we can deduce bounds on Selmer groups from the non-triviality of our Euler system. Our main result in this direction is the proof of new cases of the Bloch–Kato conjecture [BK90] in rank zero.

For the statement, we denote by ε_ℓ the epsilon factor attached to the Weil–Deligne representation associated with the restriction of $\mathrm{Ind}_K^{\mathbb{Q}}(V^{\psi}_{\mathrm{ad}(g)})$ to $G_{\mathbb{Q}_\ell}$. It is then known that the sign $\varepsilon(V^{\psi}_{\mathrm{ad}(g)})$ in the functional equation for $L(V^{\psi}_{\mathrm{ad}(g)}, s)$ is given by

$$\varepsilon(V^{\psi}_{\mathrm{ad}(g)}) = \prod_{\ell \leq \infty} \varepsilon_\ell,$$

where $\varepsilon_\infty = +1$ if $k \geq 2l$ and -1 if $2 \leq k < 2l$. On the other hand, here we say that V has "big image" if it safisfies the explicit conditions in Proposition 6.3.

Theorem C (Theorem 7.4)**.** *In addition to the above hypotheses, assume that:*

(a) $\varepsilon_\ell = +1$ *for all primes* $\ell \mid N_g N_\psi$*,*
(b) $\gcd(N_g, N_\psi)$ *is squarefree,*
(c) g *is non-Eisenstein mod* p*,*
(d) V *has big image,*
(e) $L(\theta_\psi, k/2) \neq 0$.

If $k \geq 2l$ then the following implication holds:

$$L(V, 0) \neq 0 \quad \Longrightarrow \quad \mathrm{Sel}(K, V) = 0,$$

where $\mathrm{Sel}(K, V)$ *is the Bloch–Kato Selmer group.*

Note that the hypotheses in Theorem C imply that $L(V, s)$ has sign $+1$ in its functional equation, and so the nonvanishing of $L(V, 0)$ is expected to hold generically.

We can also deduce applications to the Iwasawa main conjecture for V. More precisely, under certain hypotheses, Greenberg's general formulation of the Iwasawa main conjecture for motives [Gre94] leads to the prediction that the unbalanced Selmer group $\mathrm{Sel}_{\mathrm{unb}}(K_\infty, A)$ defined in §2, where $A = V/T$, is Λ^{ac}-cotorsion, with characteristic ideal generated by $L_p(\mathrm{ad}^0(g_K) \otimes \psi)$. In the direction of this conjecture we can prove the following, where we let $\mathbb{Z}_p^{\mathrm{ur}}$ denote the completion of the ring of integers of the maximal unramified extension of $\mathbb{Q}_p$.

Theorem D (Theorem 7.6)**.** *In addition to the above hypotheses, assume that:*

(a) $\varepsilon_\ell = +1$ *for all primes* $\ell \mid N_g N_\psi$,
(b) $\gcd(N_g, N_\psi)$ *is squarefree,*
(c) g *is non-Eisenstein mod* p,
(d) V *has big image,*
(e) θ_ψ *has global root number* $\varepsilon(\theta_\psi) = +1$.

If the p*-adic* L*-function* $L_p(\mathrm{ad}^0(g_K)\otimes\psi)$ *is nonzero, then the Pontryagin dual of* $\mathrm{Sel}_{\mathrm{unb}}(K_\infty, A)$ *is* Λ^{ac}*-torsion, with*

$$\mathrm{Char}_{\Lambda^{\mathrm{ac}}}\big(\mathrm{Sel}_{\mathrm{unb}}(K_\infty, A)^\vee\big) \supset \big(L_p(\mathrm{ad}^0(g_K)\otimes\psi)\cdot \mathscr{L}_{\mathfrak{p}}^{\mathrm{Katz}}(\psi)^{-,\iota}\big)$$

in $\mathbb{Z}_p^{\mathrm{ur}}\hat{\otimes}_{\mathbb{Z}_p}\Lambda^{\mathrm{ac}}[1/p]$.

Note that the presence of $\mathscr{L}_{\mathfrak{p}}^{\mathrm{Katz}}(\psi)^{-,\iota}$ in the divisibility of Theorem D is analogous to the appearance of the Kubota–Leopoldt p-adic L-function in the divisibility towards the Iwasawa main conjecture for the Galois representation attached to the symmetric square of a modular form in [LZ19, Thm. B].

In fact, the present work originated from an attempt to develop anticyclotomic analogues of the results in [*op. cit.*]. In particular, the idea of modifying the diagonal Euler system classes of [ACR21] to obtain the correct norm relations (see §5.1) was adopted from their work.

1.4. **Outline of the paper.** We begin by introducing in §2 our set-up and Galois representation of interest, and various Selmer groups associated with it. In §3 we describe in detail the construction of the diagonal cycle class giving rise to the bottom class of our Euler system, and study its behaviour according to a certain sign (given by $\varepsilon(\theta_\psi)$ in the notations of Theorem D). The results of this section, which are developed in a slightly more general setting than the rest of the paper, are unnecessary for the proof of our main results, but they are included here for completeness (in particular, Proposition 3.2 might be of independent interest). In §4 we introduce the different p-adic L-functions that appear in our picture, including an analogue of the Hida–Schmidt p-adic L-function deduced from the work of Eischen *et. al.* on p-adic L-functions for unitary groups, and prove the aforementioned analogue of Dasgupta's factorization. Finally, in §5 we give the construction of our Euler system by suitably modifying the diagonal cycle Euler system classes constructed in our previous work [ACR21], and in §6 and §7 we apply this to deduce the arithmetic applications highlighted in the Introduction.

1.5. **Acknowledgements.** It is a pleasure to thank David Loeffler and Chris Skinner for their very valuable advice in connection with this work. We are also grateful to Ellen Eischen and Xin Wan for correspondence regarding the subject of this note, and to Shilin Lai and Sam Mundy for several helpful conversations. Finally, we thank the anonymous referees for a careful reading of the text, whose comments notably contributed to improve the exposition of the article.

During the preparation of this paper, F.C. was partially supported by the NSF grant DMS-1946136 and DMS-2101458; O.R. was supported by the Royal Society Newton International Fellowship NIF\R1\202208.

2. Galois representations and Selmer groups

In this section we introduce our Galois representations of interest and the Selmer groups associated with them that we shall be studying.

2.1. **Galois representations.** Let $g = \sum_{n=1}^{\infty} a_n(g)q^n \in S_l(N_g, \chi_g)$ be an ordinary newform of weight $l \geq 2$, level N_g, and nebentypus χ_g. Let $p > 2$ be a prime and let $E = L_{\mathfrak{P}}$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ arising as the completion of the Hecke field L of g at a prime $\mathfrak{P}$ of L above p. By work of Eichler–Shimura and Deligne, there is a two-dimensional representation

$$\rho_g : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_E(V_g) \simeq \mathrm{GL}_2(E)$$

unramified outside pN_g and characterized by the property

$$\operatorname{trace} \rho_g(\mathrm{Fr}_q) = a_q(g)$$

for all primes $q \nmid pN_g$, where Fr_q denotes an arithmetic Frobenius element at q. Let $Y_1(N_g)$ be the open modular curve over $\mathbb{Q}$ parameterizing pairs (A, P) consisting of an elliptic curve A and a point $P \in A$ of order N_g. Let $\mathscr{L}_{l-2}$ is the sheaf introduced in [BSV21, §2.3]. As in [ACR21], we shall work with the geometric realization of V_g arising as the maximal quotient of

$$H^1_{\mathrm{et}}(Y_1(N_g)_{\overline{\mathbb{Q}}}, \mathscr{L}_{l-2}(1)) \otimes_{\mathbb{Z}_p} E$$

on which the dual Hecke operators T'_q and $\langle d \rangle'$ act as multiplication by $a_q(g)$ and $\chi_g(d)$ for all primes $q \nmid N_g$ and all $d \in (\mathbb{Z}/N_g\mathbb{Z})^\times$. We also let $T_g \subset V_g$ be the $\mathcal{O}$-lattice defined by the natural image of

$$H^1_{\mathrm{et}}(Y_1(N_g)_{\overline{\mathbb{Q}}}, \mathscr{L}_{l-2}(1)) \otimes_{\mathbb{Z}_p} \mathcal{O}$$

under the quotient map $H^1_{\mathrm{et}}(Y_1(N_g)_{\overline{\mathbb{Q}}}, \mathscr{L}_{l-2}(1)) \otimes_{\mathbb{Z}_p} E \twoheadrightarrow V_g$.

Throughout the following, we shall assume that g is not of CM-type.

2.2. **The adjoint representation.** Let K be an imaginary quadratic field of discriminant $-D_K < 0$. Let ψ be a Hecke character of K of infinity type $(1-k, 0)$ for some even integer $k \geq 2$ and central character equal to ε_K, the quadratic character attached to $K/\mathbb{Q}$ (thus the associated theta series θ_ψ has trivial nebentypus). We assume that ψ has conductor $\mathfrak{c} \subset \mathcal{O}_K$ prime to p and, upon enlarging $\mathcal{O}$ if necessary, that its p-adic avatar $\psi_{\mathfrak{P}}$ takes values in $\mathcal{O}$.

Definition 2.1. Let V be the E-valued G_K-representation given by

$$V := \mathrm{ad}^0(V_g)(\psi_{\mathfrak{P}}^{-1})(1 - k/2),$$

where $\mathrm{ad}^0(V_g) \subset \mathrm{End}_E(V_g)$ denotes the adjoint representation on the trace-zero endomorphisms of V_g.

Let $g^* = g \otimes \chi_g^{-1}$ be the twist of g by the inverse of its nebentypus. We shall study the arithmetic of V by exploiting the decomposition

$$V^{\psi}_{\mathrm{ad}(g)} := V_g \otimes V_{g^*}(\psi_{\mathfrak{P}}^{-1})(1-c) \simeq V \oplus V', \tag{2.1}$$

where $c = (k + 2l - 2)/2$ and $V' = E(\psi_{\mathfrak{P}}^{-1})(1 - k/2)$.

2.3. **Selmer groups.** From now on, we assume that p is a prime of good ordinary reduction for g such that

$$(p) = \mathfrak{p}\overline{\mathfrak{p}} \text{ splits in } K, \tag{2.2}$$

with $\mathfrak{p}$ the prime of K above p determined by our fixed embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

By p-ordinarity, the Galois representation V_g is equipped with a $G_{\mathbb{Q}_p}$-stable filtration

$$0 \longrightarrow V_g^+ \longrightarrow V_g \longrightarrow V_g^- \longrightarrow 0$$

with $V_g^{\pm}$ one-dimensional and the $G_{\mathbb{Q}_p}$-action on V_g^- given by the unramified character sending an arithmetic Frobenius Fr_p to α_g, the p-adic unit root of $x^2 - a_p(g)x + \chi_g(p)p^{l-1}$. Of course, twisting these by χ_g^{-1} we obtain $V_{g^*}^{\pm} = V_g^{\pm} \otimes \chi_g^{-1}$.

Let F/K be any finite extension and, for $v \mid p$ any prime of F above p, define

$$\mathscr{F}_v^{\rm bal}(V_{{\rm ad}(g)}^{\psi}) := \begin{cases} (V_g^+ \otimes V_{g^*} + V_g \otimes V_{g^*}^+)(\psi_{\mathfrak{P}}^{-1})(1-c) & \text{if } v \mid \mathfrak{p}, \\ V_g^+ \otimes V_{g^*}^+(\psi_{\mathfrak{P}}^{-1})(1-c) & \text{if } v \mid \overline{\mathfrak{p}}, \end{cases} \tag{2.3}$$

and

$$\mathscr{F}_v^{\rm unb}(V_{{\rm ad}(g)}^{\psi}) := \begin{cases} V_{{\rm ad}(g)}^{\psi} & \text{if } v \mid \mathfrak{p}, \\ \{0\} & \text{if } v \mid \overline{\mathfrak{p}}, \end{cases} \tag{2.4}$$

and, for $? \in \{{\rm bal}, {\rm unb}\}$, put $\mathscr{F}_v^?(V) = \mathscr{F}_v^?(V_{{\rm ad}(g)}^{\psi}) \cap V$ and $\mathscr{F}_v^?(V') = \mathscr{F}_v^?(V_{{\rm ad}(g)}^{\psi}) \cap V'$.

Fix Σ any finite set of places of K containing ∞ and the primes dividing pN_gN_ψ. With a slight abuse of notation, for any finite extension of F/K we also denote by Σ the set of places of F lying over the places in Σ, and denote by $G_{F,\Sigma}$ the Galois group of the maximal extension of F unramified outside Σ. Further, for any non-archimedean field F_v, we write $F_v^{\rm nr}$ for the maximal unramified extension of F_v.

Definition 2.2. Let F/K be a finite extension, and for $M \in \{V_{{\rm ad}(g)}^{\psi}, V, V'\}$ and $? \in \{{\rm bal}, {\rm unb}\}$ define the Selmer group ${\rm Sel}_?(F, M)$ by

$${\rm Sel}_?(F,M) = \ker\left(H^1(G_{F,\Sigma}, M) \longrightarrow \prod_{v \mid p} \frac{H^1(F_v, M)}{H^1_?(F_v, M)} \times \prod_{v \in \Sigma, v \nmid p\infty} H^1(F_v^{\rm nr}, M)\right),$$

where

$$H^1_?(F_v, M) = {\rm im}(H^1(F_v, \mathscr{F}_v^?(M)) \longrightarrow H^1(F_v, M)).$$

We call ${\rm Sel}_{\rm bal}(F,M)$ (resp. ${\rm Sel}_{\rm unb}(F,M)$) the *balanced* (resp. *unbalanced*) Selmer group.

Remark 2.3. Let $f = \theta_\psi$ be the weight k eigenform associated with ψ, and denote by $V_{fgg^*} := V_f \otimes V_g \otimes V_{g^*}(1-c)$ the Kummer self-dual twist of the Galois representation attached to (f, g, g^*). Since $V_f = {\rm Ind}_K^{\mathbb{Q}}\, \psi$, one can easily check that the isomorphism given by Shapiro's lemma

$$H^1(\mathbb{Q}, V_{fgg^*}) \simeq H^1(K, V_{{\rm ad}(g)}^{\psi})$$

identifies the Selmer groups ${\rm Sel}_{\rm bal}(\mathbb{Q}, V_{fgg^*})$ and ${\rm Sel}_f(\mathbb{Q}, V_{fgg^*})$ considered in [ACR21, Def. 7.5] with the above ${\rm Sel}_{\rm bal}(K, V_{{\rm ad}(g)}^{\psi})$ and ${\rm Sel}_{\rm unb}(K, V_{{\rm ad}(g)}^{\psi})$, respectively.

Put $T_{{\rm ad}(g)}^{\psi} = T_g \otimes T_{g^*}(\psi_{\mathfrak{P}}^{-1})(1-c)$. Then the decomposition (2.1) induces a decomposition

$$T_{{\rm ad}(g)}^{\psi} \simeq T \oplus T',$$

where T and T' are lattices in V and V', respectively. We also set

$$A_{{\rm ad}(g)}^{\psi} = V_{{\rm ad}(g)}^{\psi}/T_{{\rm ad}(g)}^{\psi}, \quad A = V/T, \quad A' = V'/T'.$$

Then, for $? \in \{{\rm bal}, {\rm unb}\}$ and $M \in \{T_{{\rm ad}(g)}^{\psi}, T, T', A_{{\rm ad}(g)}^{\psi}, A, A'\}$, we define the local conditions $H^1_?(F_v, M)$ from the local conditions above by propagation, and use them to define the Selmer groups ${\rm Sel}_?(K, M)$ using the same recipe as in Definition 2.2. Finally, for $M_1 \in \{T_{{\rm ad}(g)}^{\psi}, T, T'\}$ and $M_2 \in \{A_{{\rm ad}(g)}^{\psi}, A, A'\}$, we put

$${\rm Sel}_?(K_\infty, M_1) := \varprojlim_n {\rm Sel}_?(K_n, M_1), \quad {\rm Sel}_?(K_\infty, M_2) := \varinjlim_n {\rm Sel}_?(K_n, M_2),$$

where the limits are with respect to corestriction and restriction, respectively.

To help orient the reader, we note the following simple relation between the different Selmer groups introduced above.

Proposition 2.4. *The decomposition* $V^{\psi}_{\mathrm{ad}(g)} = V \oplus V'$ *induces isomorphisms*

$$\mathrm{Sel}_{\mathrm{bal}}(K_\infty, T^{\psi}_{\mathrm{ad}(g)}) \simeq \mathrm{Sel}_{\mathrm{bal}}(K_\infty, T) \oplus \mathrm{Sel}(K_\infty, T'),$$
$$\mathrm{Sel}_{\mathrm{unb}}(K_\infty, T^{\psi}_{\mathrm{ad}(g)}) \simeq \mathrm{Sel}_{\mathrm{unb}}(K_\infty, T) \oplus \mathrm{Sel}(K_\infty, T'),$$

where $\mathrm{Sel}(K_\infty, T')$ *is the Bloch–Kato Selmer group for* T'.

Proof. It suffices to show that for any finite extension F/K we have

$$\mathrm{Sel}_{\mathrm{bal}}(F, V') \simeq \mathrm{Sel}_{\mathrm{unb}}(F, V') \simeq \mathrm{Sel}(F, V'),$$

where $\mathrm{Sel}(F, V')$ is the Bloch–Kato Selmer group of $V' = E(\psi_{\mathfrak{P}}^{-1})$, which is given by

$$\mathrm{Sel}(F, V') = \ker\left(H^1(G_{F,\Sigma}, V') \to \prod_{v \mid \bar{\mathfrak{p}}} H^1(F_v, V') \times \prod_{v \in \Sigma, v \nmid p} H^1(F_v^{\mathrm{nr}}, V') \right)$$

(see [AH06, §1.1] or [Arn07, §1.2]). For $\mathrm{Sel}_{\mathrm{unb}}(F, V')$ this is clear from (2.4); for $\mathrm{Sel}_{\mathrm{bal}}(F, V')$ it follows by noting that the subspace $\mathscr{F}_v^{\mathrm{bal}}(V^{\psi}_{\mathrm{ad}(g)}) \subset V^{\psi}_{\mathrm{ad}(g)}$ in (2.3) contains V' for $v \mid \mathfrak{p}$ and intersects trivially with it for $v \mid \bar{\mathfrak{p}}$. □

3. Construction of the bottom class

In this section, we recall the construction of a Λ-adic cohomology class associated with the triple product of three modular forms as explained in [BSV21]. We follow the exposition in [*op. cit.*] with slight modifications and specializing the discussion to the case of interest in this paper. At the end of this section we analyze the behaviour of this cohomology class depending on the sign of one of the modular forms.

This section is independent of the rest of the paper, and the reader solely interested in the results stated in the Introduction can proceed to Section 4.

Let f and g be newforms of weight $k = r_1 + 2$ and $l = r_2 + 2$, level N_f and N_g and character $\chi_f = 1$ and χ_g, respectively. We assume that $p \nmid 2N_f N_g$ and that both f and g are ordinary at p. We denote by $h = g^*$ the newform obtained by conjugating the Fourier coefficients of g. Let L be a finite extension of $\mathbb{Q}$ containing the Fourier coefficients of f and g and let $E = L_{\mathfrak{P}}$ be its completion at a prime $\mathfrak{P}$ above p, with ring of integers $\mathcal{O}$. Define $N = \mathrm{lcm}(N_f, N_g)$.

Consider the Iwasawa algebra $\Lambda = \mathbb{Z}_p[\![1 + p\mathbb{Z}_p]\!]$. There exist finite flat Λ-modules $\Lambda_{\mathbf{f}}$ and $\Lambda_{\mathbf{g}}$ and primitive Hida families $\mathbf{f} \in \Lambda_{\mathbf{f}}[\![q]\!]$ and $\mathbf{g} \in \Lambda_{\mathbf{g}}[\![q]\!]$ passing through the ordinary p-stabilizations f_α and g_α of f and g, respectively. Let $\mathbf{h} = \mathbf{g}^*$ be the Hida family $\mathbf{g} \otimes \chi_g^{-1}$, which passes through the ordinary p-stabilization g_α^* of g^*. Our conventions for Hida families are those described in [ACR21, §5.1].

Let μ_{p-1} denote the group of roots of unity in $\mathbb{Z}_p^\times$ and consider the decomposition $\mathbb{Z}_p^\times \cong \mu_{p-1} \times (1 + p\mathbb{Z}_p)$. Let $\omega : \mathbb{Z}_p^\times \to \mu_{p-1} \subset \mathbb{Z}_p^\times$ be the map defined by projection onto the first factor, according to the previous decomposition. Also, for an element $z \in \mathbb{Z}_p^\times$, we denote by $\langle z \rangle$ its projection onto the second factor (alternatively, $\langle z \rangle = z/\omega(z)$).

Let $\mathrm{Cont}(\mathbb{Z}_p, \Lambda)$ be the Λ-module of continuous functions on $\mathbb{Z}_p$ with values in Λ. To make notation less cumbersome, we denote by $[z]$ the group-like element $[\langle z \rangle]$ in Λ. For each integer i, let $\kappa_i : \mathbb{Z}_p^\times \to \Lambda^\times$ be the character defined by $z \mapsto \omega^i(z)[z]$. We also define the sets $\mathsf{T} = \mathbb{Z}_p^\times \times \mathbb{Z}_p$ and $\mathsf{T}' = p\mathbb{Z}_p \times \mathbb{Z}_p^\times$. Then, we can define the Λ-modules

$$\mathcal{A}_i = \left\{ f : \mathsf{T} \to \Lambda \mid f(1, z) \in \mathrm{Cont}(\mathbb{Z}_p, \Lambda) \text{ and } f(a \cdot t) = \kappa_i(a) \cdot f(t) \text{ for all } a \in \mathbb{Z}_p^\times,\, t \in \mathsf{T} \right\},$$
$$\mathcal{A}'_i = \left\{ f : \mathsf{T}' \to \Lambda \mid f(pz, 1) \in \mathrm{Cont}(\mathbb{Z}_p, \Lambda) \text{ and } f(a \cdot t) = \kappa_i(a) \cdot f(t) \text{ for all } a \in \mathbb{Z}_p^\times,\, \gamma \in \mathsf{T}' \right\},$$
$$\mathcal{D}_i = \mathrm{Hom}_{\mathrm{cont},\Lambda}(\mathcal{A}_i, \Lambda), \quad \mathcal{D}'_i = \mathrm{Hom}_{\mathrm{cont},\Lambda}(\mathcal{A}'_i, \Lambda).$$

We define in addition characters $\kappa_f^*, \kappa_g^*, \kappa_{g^*}^*, \kappa^* : \mathbb{Z}_p^\times \to \Lambda\hat{\otimes}\Lambda\hat{\otimes}\Lambda$ by

$$\begin{aligned}\kappa_f^*(z) &= \omega^{r_2-r_1/2}(z)[z]^{-1/2}\otimes[z]^{1/2}\otimes[z]^{1/2}\\ \kappa_g^*(z) &= \omega^{r_1/2}(z)[z]^{1/2}\otimes[z]^{-1/2}\otimes[z]^{1/2}\\ \kappa_h^*(z) &= \omega^{r_1/2}(z)[z]^{1/2}\otimes[z]^{1/2}\otimes[z]^{-1/2}\\ \kappa^*(z) &= \omega^{r_1/2+r_2}(z)[z]^{1/2}\otimes[z]^{1/2}\otimes[z]^{1/2}.\end{aligned}$$

We denote by $\boldsymbol{\kappa}^*$ the character of the Galois group $G_\mathbb{Q}$ defined by $\boldsymbol{\kappa}^* = \kappa^*\circ\epsilon_{\mathrm{cyc}}$, and similarly for the other characters introduced above.

Let $Y = Y_1(N,p)$ denote the same modular curve as in [BSV21, §8.1] and let $\Gamma = \Gamma_1(N,p)$ be the corresponding modular group. The function

$$\mathbf{Det} : \mathsf{T}'\times\mathsf{T}\times\mathsf{T} \longrightarrow \Lambda\hat{\otimes}\Lambda\hat{\otimes}\Lambda,$$

defined as in [*loc. cit.*], yields an element in the group

$$H^0_{\mathrm{et}}(Y, \boldsymbol{\mathcal{A}}'_{r_1}\otimes\boldsymbol{\mathcal{A}}_{r_2}\otimes\boldsymbol{\mathcal{A}}_{r_2}(-\kappa^*)).$$

Here $\boldsymbol{\mathcal{A}}'_{r_1}$ and $\boldsymbol{\mathcal{A}}_{r_2}$ denote the étale sheaves associated with $\mathcal{A}'_{r_1}$ and $\mathcal{A}_{r_2}$, respectively, as explained in [BSV21, §4.2]. Then, with essentially the same notations as in [*op. cit.*, §8.1], we define the class

$$\kappa^{(1)} = \frac{1}{a_p(\mathbf{f})}\mathtt{s}_{\mathbf{fgh}}\circ(e_{\mathrm{ord}}\otimes e_{\mathrm{ord}}\otimes e_{\mathrm{ord}})\circ(w_p\otimes 1\otimes 1)\circ\mathtt{K}\circ\mathtt{HS}\circ d_*(\mathbf{Det})$$

inside the group

$$H^1\big(\mathbb{Q}, H^1(\Gamma,\mathcal{D}'_{r_1})^{\mathrm{ord}}\hat{\otimes}H^1(\Gamma,\mathcal{D}'_{r_2})^{\mathrm{ord}}\hat{\otimes}H^1(\Gamma,\mathcal{D}'_{r_2})^{\mathrm{ord}}(2-\boldsymbol{\kappa}^*)\big).$$

For a $\mathbb{Z}_p$-algebra A, let $L_{r_2}(A)$ be defined as in [*op. cit.*, p. 17]. We will sometimes denote $L_{r_2}(\mathbb{Z}_p)$ simply by L_{r_2}. Let $\kappa_f^{1/2} : \mathbb{Z}_p^\times \to \Lambda_\mathbf{f}^\times$ denote the map $z\mapsto \omega^{r_1/2}(z)[z]^{1/2}$ and let $\boldsymbol{\kappa}_f^{1/2} = \kappa_f^{1/2}\circ\epsilon_{\mathrm{cyc}}$. According to [*op. cit.*, eq. (90)], the Λ-module $H^1(\Gamma,\mathcal{D}'_{r_2})^{\mathrm{ord}}$ specializes to $H^1(\Gamma, L_{r_2})^{\mathrm{ord}}$ at weight $l = r_2+2$. Therefore, the class $\kappa^{(1)}$ yields a class

$$\kappa^{(2)} \in H^1\big(\mathbb{Q}, H^1(\Gamma,\mathcal{D}'_{r_1})^{\mathrm{ord}}\otimes H^1(\Gamma,L_{r_2})^{\mathrm{ord}}\otimes H^1(\Gamma,L_{r_2})^{\mathrm{ord}}(2-r_2-\boldsymbol{\kappa}_f^{1/2})\big).$$

We define Hecke operators T'_q, $[d]'_N$ acting on group cohomology as in [ACR21, §5.3]. Let $\mathbb{V}_\mathbf{f}(N)$ be the maximal quotient of $H^1(\Gamma,\mathcal{D}'_{r_1})^{\mathrm{ord}}(1)\otimes_\Lambda\Lambda_\mathbf{f}$ on which the Hecke operators T'_q for primes $q\nmid N$ act as multiplication by $a_q(\mathbf{f})$ and the diamond operators $[d]'_N$ act as multiplication by $\chi_f(d)$ (actually, the character χ_f is trivial in our case). We define $T_g(N)$ and $T_{g^*}(N)$ in a similar way as quotients of $H^1(\Gamma, L_{r_2}(\mathcal{O}))^{\mathrm{ord}}(1)$. Also, let $\mathbb{V}_\mathbf{f}$ be the maximal quotient of $H^1(\Gamma_1(N_f,p),\mathcal{D}'_{r_1})^{\mathrm{ord}}(1)\otimes_\Lambda\Lambda_\mathbf{f}$ on which the Hecke operators T'_q act as multiplication by $a_q(\mathbf{f})$ and the diamond operators $[d]'_N$ act as multiplication by $\chi_f(d)$ and define T_g and T_{g^*} in a similar way as quotients of $H^1(\Gamma_1(N_g,p), L_{r_2}(\mathcal{O}))^{\mathrm{ord}}(1)$.

To shorten notation, we define

$$\begin{aligned}\mathbb{V}(\mathbf{f},g,g^*) &= \mathbb{V}_\mathbf{f}\otimes T_g\otimes T_{g^*}(-1-r_2-\boldsymbol{\kappa}_f^{1/2}),\\ \mathbb{V}(\mathbf{f},g,g^*)(N) &= \mathbb{V}_\mathbf{f}(N)\otimes T_g(N)\otimes T_{g^*}(N)(-1-r_2-\boldsymbol{\kappa}_f^{1/2}),\\ \mathbb{V}(\mathbf{f},g,g^*)_f &= \mathbb{V}_\mathbf{f}^-\otimes T_g^+\otimes T_{g^*}^+(-1-r_2-\boldsymbol{\kappa}_f^{1/2}),\\ \mathbb{V}(\mathbf{f},g,g^*)_f(N) &= \mathbb{V}_\mathbf{f}^-(N)\otimes T_g^+(N)\otimes T_{g^*}^+(N)(-1-r_2-\boldsymbol{\kappa}_f^{1/2}).\end{aligned}$$

We also introduce

$$M(\mathbf{f},g,g^*)_f = \mathbb{V}_{\mathbf{f}}^-\hat{\otimes}T_g^+\hat{\otimes}T_{g^*}^+(-2-2r_2)\hat{\otimes}\Lambda(\boldsymbol{\kappa}^{-1})[1/p],$$
$$M(\mathbf{f},g,g^*)_f(N) = \mathbb{V}_{\mathbf{f}}^-(N)\hat{\otimes}T_g^+(N)\hat{\otimes}T_{g^*}^+(N)(-2-2r_2)\hat{\otimes}\Lambda(\boldsymbol{\kappa}^{-1})[1/p],$$

where $\boldsymbol{\kappa} : G_{\mathbb{Q}} \to \Lambda^\times$ is defined by $\boldsymbol{\kappa}(\sigma) = \omega^{r_1/2-r_2-1}(\epsilon_{\rm cyc}(\sigma))[\epsilon_{\rm cyc}(\sigma)]$.

The class $\kappa^{(2)}$ yields a class

$$\kappa^{(2)}(\mathbf{f},g,g^*) \in H^1\left(\mathbb{Q},\mathbb{V}(\mathbf{f},g,g^*)(N)\right).$$

This is the class defined in [BSV21, eq. 155] specialized to weight l in the second and third factors. It follows from [*op. cit.*, Cor. 8.2] that the restriction at p of this class belongs to the group

$$H^1_{\rm bal}\left(\mathbb{Q}_p,\mathbb{V}(\mathbf{f},g,g^*)(N)\right).$$

Let $S^{\rm ord}_{\Lambda_{\mathbf{f}}}(N,\omega^{r_1})$ denote the space of Hida families of tame level N, character ω^{r_1} and with coefficients in $\Lambda_{\mathbf{f}}$. We denote by $S^{\rm ord}_{\Lambda_{\mathbf{f}}}(N,\omega^{r_1})[\mathbf{f}]$ the subspace of $S^{\rm ord}_{\Lambda_{\mathbf{f}}}(N,\omega^{r_1})$ on which the Hecke operators U_p and T_ℓ for $\ell\nmid N$ act with the same eigenvalues as on $\mathbf{f}$. Let $S_l(\Gamma_1(N,p),\chi_g)[g_\alpha]$ denote the space of modular forms of weight l, level $\Gamma_1(N,p)$ and nebentypus χ_g which are eigenforms for the Hecke operators U_p and T_ℓ for $\ell\nmid N$ with the same eigenvalues as g_α. We similarly define $S_l(\Gamma_1(N,p),\chi_g^{-1})[g_\alpha^*]$. Then, a choice of level-N test vectors $\breve{\mathbf{f}}$, $\breve{g}$ and $\breve{h}$ for $\mathbf{f}$, g_α and g_α^*, respectively, is a choice of elements

$$\breve{\mathbf{f}} \in S^{\rm ord}_{\Lambda}(N,\omega^{r_1})[\mathbf{f}],\quad \breve{g}\in S_l(\Gamma_1(N,p),\chi_g)[g_\alpha],\quad \breve{h}\in S_l(\Gamma_1(N,p),\chi_g^{-1})[g_\alpha^*],$$

each of which can be written, in terms of their q-expansions, as

$$\breve{\mathbf{f}}(q) = \sum_{0<d|N/N_f} r_d^{\breve{\mathbf{f}}}\cdot \mathbf{f}(q^d),$$
$$\breve{g}(q) = \sum_{0<d|N/N_g} r_d^{\breve{g}}\cdot g_\alpha(q^d),$$
$$\breve{h}(q) = \sum_{0<d|N/N_g} r_d^{\breve{h}}\cdot g_\alpha^*(q^d),$$

with $r_d^{\breve{\mathbf{f}}}\in\Lambda_{\mathbf{f}}$ and $r_d^{\breve{g}}, r_d^{\breve{h}}\in\mathcal{O}$. Let

$$\varpi_{\breve{\mathbf{f}}}^* : S^{\rm ord}_{\Lambda_{\mathbf{f}}}(N_f,\omega^{r_1}) \longrightarrow S^{\rm ord}_{\Lambda_{\mathbf{f}}}(N,\omega^{r_1})$$

denote the map defined by

$$\Phi(q)\mapsto \sum_{0<d|N/N_f} r_d^{\breve{\mathbf{f}}}\cdot\Phi(q^d).$$

Similarly, we define

$$\varpi_{\breve{g}}^* : S_l(\Gamma_1(N_g,p),\chi_g) \longrightarrow S_l(\Gamma_1(N,p),\chi_g),$$
$$\varpi_{\breve{h}}^* : S_l(\Gamma_1(N_g,p),\chi_g^{-1}) \longrightarrow S_l(\Gamma_1(N,p),\chi_g^{-1}).$$

Therefore, we can write $\breve{\mathbf{f}} = \varpi_{\breve{\mathbf{f}}}^*(\mathbf{f})$, $\breve{g} = \varpi_{\breve{g}}^*(g_\alpha)$ and $\breve{h} = \varpi_{\breve{h}}^*(g_\alpha^*)$. At the same time, for each $d\mid N/N_f$, the map $v_d : Y_1(N,p)\to Y_1(N_f,p)$, corresponding to multiplication by d on the complex upper half-plane under the standard complex uniformizations, yields a pushforward map

$$v_{d*} : H^1(\Gamma_1(N,p),\mathcal{D}'_{r_1}) \longrightarrow H^1(\Gamma_1(N_f,p),\mathcal{D}'_{r_1})$$

which induces a map

$$v_{d*} : \mathbb{V}_{\mathbf{f}}(N) \longrightarrow \mathbb{V}_{\mathbf{f}}.$$

Let $\varpi_{\breve{\mathbf{f}}*} = \sum_{0<d|N/N_f} r_d^{\breve{\mathbf{f}}} v_{d*}$. Let $\eta_{\breve{\mathbf{f}}}$ and $\eta_{\mathbf{f}}$ denote the differentials attached to $\breve{\mathbf{f}}$ and $\mathbf{f}$, respectively, in [BSV21, eq. (122)]. Then,

$$\langle x, \eta_{\breve{\mathbf{f}}} \rangle = \langle \varpi_{\breve{\mathbf{f}}*}(x), \eta_{\mathbf{f}} \rangle \quad \text{for all } x \in \mathbb{V}_{\mathbf{f}}^-(N).$$

Similarly, we can define maps

$$\varpi_{\breve{g}*}, \varpi_{\breve{h}*} : H^1(\Gamma_1(N,p), L_{r_2}) \longrightarrow H^1(\Gamma(N_g,p), L_{r_2}),$$

which induce maps

$$\varpi_{\breve{g}*} : T_g(N) \longrightarrow T_g, \quad \varpi_{\breve{h}*} : T_{g^*}(N) \longrightarrow T_{g^*}.$$

Let $\omega_{\breve{g}}$, ω_{g_α}, $\omega_{\breve{h}}$ and $\omega_{g_\alpha^*}$ denote the differentials attached to $\breve{g}$, g_α, $\breve{h}$ and g_α^*, respectively, in [BSV21, eq. (30)]. Then,

$$\langle x, \omega_{\breve{g}} \rangle = \langle \varpi_{\breve{g}*}(x), \omega_{g_\alpha} \rangle \quad \text{for all } x \in T_g^+(N),$$
$$\langle x, \omega_{\breve{h}} \rangle = \langle \varpi_{\breve{g}*}(x), \omega_{g_\alpha^*} \rangle \quad \text{for all } x \in T_{g^*}^+(N)$$

For a choice of level-N test vectors $\breve{\mathbf{f}} = \varpi_{\breve{\mathbf{f}}}^*(\mathbf{f})$, $\breve{g} = \varpi_{\breve{g}}^*(g_\alpha)$, $\breve{h} = \varpi_{\breve{h}}^*(g_\alpha^*)$, we have a map

$$\mathfrak{Log}(\breve{\mathbf{f}}, \breve{g}, \breve{h}) : H^1_{\mathrm{bal}}(\mathbb{Q}_p, \mathbb{V}(\mathbf{f}, g, g^*)(N)) \longrightarrow \Lambda_{\mathbf{f}}[1/p]$$

obtained from the map defined in [BSV21, Prop. 7.3] by specializing to weight l the second and third variables. It follows from [*op. cit.*, Thm. A] that the image of $\mathrm{res}_p(\kappa^{(2)}(\mathbf{f}, g, h))$ under the map above is an element $\mathscr{L}_p(\breve{\mathbf{f}}, \breve{g}, \breve{h}) \in \Lambda_{\mathbf{f}}[1/p]$ such that, for all $k' \geq 2l$ satisfying $k' \equiv k \pmod{2(p-1)}$,

$$\mathscr{L}_p(\breve{\mathbf{f}}, \breve{g}, \breve{h})(k') = \frac{\langle \breve{\mathbf{f}}_{k'}^w, \delta^t \breve{g} \times \breve{h} \rangle_{Np}}{\langle \breve{\mathbf{f}}_{k'}^w, \breve{\mathbf{f}}_{k'}^w \rangle_{Np}}.$$

Let $\bar{\mathcal{L}}_{\breve{\mathbf{f}}\breve{g}\breve{h}}$ be the map defined in [BSV21, Prop. 7.1] specialized to weight l in the second and third variables. Then, we obtain a map

$$\langle \bar{\mathcal{L}}_{\breve{\mathbf{f}}\breve{g}\breve{h}}(-), \eta_{\breve{\mathbf{f}}} \omega_{\breve{g}} \omega_{\breve{h}} \rangle : H^1(\mathbb{Q}_p, M(\mathbf{f}, g, g^*)_f(N)) \longrightarrow \Lambda_{\mathbf{f}} \hat{\otimes} \Lambda[1/p].$$

The map $\mathfrak{Log}(\breve{\mathbf{f}}, \breve{g}, \breve{h})$ is obtained by composing the natural projection

$$H^1_{\mathrm{bal}}(\mathbb{Q}_p, \mathbb{V}(\mathbf{f}, g, g^*)(N)) \longrightarrow H^1(\mathbb{Q}_p, \mathbb{V}(\mathbf{f}, g, g^*)_f(N))$$

with a suitable specialization of the map above.

Now, from the previous discussion, we have that

$$\langle \bar{\mathcal{L}}_{\breve{\mathbf{f}}\breve{g}\breve{h}}(-), \eta_{\breve{\mathbf{f}}} \omega_{\breve{g}} \omega_{\breve{h}} \rangle = \langle \bar{\mathcal{L}}_{\mathbf{f} g_\alpha g_\alpha^*}((\varpi_{\breve{\mathbf{f}},*} \otimes \varpi_{\breve{g},*} \otimes \varpi_{\breve{h},*})(-)), \eta_{\mathbf{f}} \omega_{g_\alpha} \omega_{g_\alpha^*} \rangle,$$

where the map $\bar{\mathcal{L}}_{\mathbf{f} g_\alpha g_\alpha^*}$ is defined in a way analogous to the way in which the map $\bar{\mathcal{L}}_{\breve{\mathbf{f}}\breve{g}\breve{h}}$ is defined in [BSV21, Prop. 7.1]. Therefore, as before, the composition of the natural projection

$$H^1_{\mathrm{bal}}(\mathbb{Q}_p, \mathbb{V}(\mathbf{f}, g, g^*)) \longrightarrow H^1(\mathbb{Q}_p, \mathbb{V}(\mathbf{f}, g, g^*)_f)$$

with a suitable specialization of the map

$$\left\langle \bar{\mathcal{L}}_{\mathbf{f} g_\alpha g_\alpha^*}(-), \eta_{\mathbf{f}} \omega_{g_\alpha} \omega_{g_\alpha^*} \right\rangle : H^1(\mathbb{Q}_p, M(\mathbf{f}, g, g^*)_f) \longrightarrow \Lambda_{\mathbf{f}} \hat{\otimes} \Lambda[1/p]$$

yields a map

$$\mathfrak{Log}(\mathbf{f}, g_\alpha, g_\alpha^*) : H^1_{\mathrm{bal}}(\mathbb{Q}_p, \mathbb{V}(\mathbf{f}, g, g^*)) \longrightarrow \Lambda_{\mathbf{f}}[1/p].$$

Moreover, for any choice of test vectors $\breve{\mathbf{f}}, \breve{g}, \breve{h}$ as above, we have

$$\mathfrak{Log}(\mathbf{f}, g_\alpha, g_\alpha^*)\big(\mathrm{res}_p\big((\varpi_{\breve{\mathbf{f}},*} \otimes \varpi_{\breve{g},*} \otimes \varpi_{\breve{h},*})(\kappa^{(2)}(\mathbf{f}, g, g^*))\big)\big) = \mathscr{L}_p(\breve{\mathbf{f}}, \breve{g}, \breve{h}).$$

It follows from [Hsi21, §3.5-6] and [*op. cit.*, Thm. 7.1] that there exist level-N test vectors $\breve{\mathbf{f}}, \breve{g}, \breve{h}$ for which, under some technical assumptions, we have a precise formula for the specializations of $\mathscr{L}_p(\breve{\mathbf{f}}, \breve{g}, \breve{h})$ at even weights $k' \geq 2l$. We fix such test vectors. Then, we define

$$\kappa^{(3)} = (\varpi_{\breve{\mathbf{f}},*} \otimes \varpi_{\breve{g},*} \otimes \varpi_{\breve{h},*})\kappa^{(2)}$$

in the group

$$H^1\big(\mathbb{Q}, H^1(\Gamma_1(N_f,p), \mathcal{D}'_{r_1})^{\mathrm{ord}} \hat{\otimes} H^1(\Gamma_1(N_g,p), L_{r_2})^{\mathrm{ord}} \hat{\otimes} H^1(\Gamma_1(N_g,p), L_{r_2})^{\mathrm{ord}}(2 - r_2 - \boldsymbol{\kappa}_f^{1/2})\big)$$

and let

$$\kappa^{(3)}(\mathbf{f}, g, g^*) \in H^1\big(\mathbb{Q}, \mathbb{V}_{\mathbf{f}}(-\boldsymbol{\kappa}_f^{1/2}) \hat{\otimes} \operatorname{ad}(T_g)\big)$$

be the class obtained from $\kappa^{(3)}$ by projection to the isotypic quotients for $\mathbf{f}$, g and g^*. Then

$$\mathfrak{Log}(\mathbf{f}, g_\alpha, g_\alpha^*)\big(\operatorname{res}_p\big(\kappa^{(3)}(\mathbf{f}, g, g^*)\big)\big) = \mathscr{L}_p(\breve{\mathbf{f}}, \breve{g}, \breve{h}).$$

Observe that the map $w_{N_g} : H^1(\Gamma_1(N_g,p), L_{r_2}(\mathcal{O}))^{\mathrm{ord}} \to H^1(\Gamma_1(N_g,p), L_{r_2}(\mathcal{O}))^{\mathrm{ord}}$ defined in [BSV21, §4.1.2] descends to a map $w_{N_g} : T_g \to T_{g^*}$. Taking the Galois action into account, this is actually a map $T_g \to T_{g^*}(\chi_g)$. Similarly, we have a map $w_{N_g} : T_{g^*} \to T_g(\chi_g^{-1})$. (We are denoting all these maps in the same way in the hope that this will not cause any confusion.)

Let $s : T_g \otimes T_{g^*} \to T_{g^*} \otimes T_g$ be the map which interchanges the two factors. Then, the composition $\tilde{s} = (-N_g)^{-r_2}\, s \circ (w_{N_g}, w_{N_g})$ defines an endomorphism of $\operatorname{ad}(T_g) = T_g \otimes T_{g^*}(-1-r_2)$. This endomorphism is in fact an involution.

Lemma 3.1. *Consider the direct sum decomposition* $\operatorname{ad}(T_g) = \operatorname{ad}^0(T_g) \oplus 1$. *Then:*

(1) $\operatorname{ad}^0(T_g)$ *is the* 1*-eigenspace for* $\tilde{s}$*;*
(2) 1 *is the* -1*-eigenspace for* $\tilde{s}$*.*

Proof. As in [BSV21, p. 19], there is a bilinear form $L_{r_2}(\mathcal{O}) \otimes L_{r_2}(\mathcal{O}) \to \mathcal{O} \otimes \det^{-r_2}$. Via cup-product and the isomorphism $H^2_{\mathrm{par}}(\Gamma_1(N_g,p), \mathcal{O}) \simeq \mathcal{O}(1)$, we obtain a pairing

$$H^1_{\mathrm{par}}(\Gamma_1(N_g,p), L_{r_2}(\mathcal{O}))^{\mathrm{ord}} \times H^1_{\mathrm{par}}(\Gamma_1(N_g,p), L_{r_2}(\mathcal{O}))^{\mathrm{ord}} \longrightarrow \mathcal{O}(r_2+1),$$

where H^1_{par} stands for parabolic cohomology as defined in [GS93, p. 427]. Since cup-product is anti-commutative in degree 1, the pairing above satisfies $\langle \alpha, \beta \rangle = (-1)^{r_2+1} \langle \beta, \alpha \rangle$ for any α, β in $H^1_{\mathrm{par}}(\Gamma_1(N_g,p), L_{r_2}(\mathcal{O}))^{\mathrm{ord}}$. On the other hand, the operator w_{N_g} acting on $H^1_{\mathrm{par}}(\Gamma_1(N_g,p), L_{r_2}(\mathcal{O}))^{\mathrm{ord}}$ satisfies $w_{N_g}^2 = (-N_g)^{r_2}$ and $\langle w_{N_g}\alpha, w_{N_g}\beta \rangle = N_g^{r_2} \langle \alpha, \beta \rangle$ for any elements $\alpha, \beta \in H^1_{\mathrm{par}}(\Gamma_1(N_g,p), L_{r_2}(\mathcal{O}))^{\mathrm{ord}}$. Therefore we have

$$\langle \alpha, w_{N_g}\beta \rangle = \frac{1}{N_g^{r_2}} \langle w_{N_g}\alpha, w_{N_g}^2\beta \rangle = (-1)^{r_2} \langle w_{N_g}\alpha, \beta \rangle = -\langle \beta, w_{N_g}\alpha \rangle.$$

In particular, we deduce that $\langle \alpha, w_{N_g}\alpha \rangle = 0$.

We can realize T_g (resp. T_{g^*}) as the maximal quotient of $H^1_{\mathrm{par}}(\Gamma_1(N_g,p), L_{r_2}(\mathcal{O}))^{\mathrm{ord}}$ on which the Hecke operators T'_q act as multiplication by $a_q(g)$ (resp. $a_q(g^*)$) and the diamond operators $[d]'_{N_g}$ act as multiplication by $\chi_g(d)$ (resp. $\chi_g(d)^{-1}$). Thus we obtain a commutative diagram

$$\begin{array}{ccc} H^1_{\mathrm{par}}(\Gamma_1(N_g,p), L_{r_2}(\mathcal{O}))^{\mathrm{ord}} \times H^1_{\mathrm{par}}(\Gamma_1(N_g,p), L_{r_2}(\mathcal{O}))^{\mathrm{ord}} & \longrightarrow & \mathcal{O}(r_2+1) \\ \downarrow & & \| \\ T_g \times T_{g^*} & \longrightarrow & \mathcal{O}(r_2+1). \end{array}$$

Therefore, for any elements $\alpha, \beta \in T_g$, we have $\langle \alpha, w_{N_g}\beta \rangle = -\langle \beta, w_{N_g}\alpha \rangle$. The lemma follows easily from this. □

We will assume in the remaining of this section that $N_g \mid N_f$, so that $N = N_f$. Under this assumption, our test vectors are $\breve{\mathbf{f}} = \mathbf{f}$, $\breve{g}(q) = \pi_1^*(g_\alpha)$ and $\breve{h} = \pi_2^*(g_\alpha^*)$, up to multiplication by some constants in $\mathrm{Frac}\,\Lambda_{\mathbf{f}}$ which do not affect the discussion that follows.

Let s_{N_g} denote the operator which acts on the group

$$H^1\big(\mathbb{Q}, H^1(\Gamma, \mathcal{D}'_{r_1})^{\mathrm{ord}} \hat{\otimes} H^1(\Gamma_1(N_g,p), L_{r_2})^{\mathrm{ord}} \hat{\otimes} H^1(\Gamma_1(N_g,p), L_{r_2})^{\mathrm{ord}}(2 - r_2 - \boldsymbol{\kappa}_f^{1/2})\big)$$

by interchanging the second and third factors and define $\tilde{s}_{N_g} = (-N_g)^{-r_2} s_{N_g} \circ (1 \otimes w_{N_g} \otimes w_{N_g})$.

Proposition 3.2. *The class $\kappa^{(3)}(\mathbf{f}, g, g^*)$ satisfies*

$$\tilde{s}_{N_g}\big(\kappa^{(3)}(\mathbf{f}, g, g^*)\big) = -[N]^{-1/2}(w_N \otimes 1 \otimes 1)\kappa^{(3)}(\mathbf{f}, g, g^*).$$

In particular, when we consider the direct sum decomposition

$$H^1\big(\mathbb{Q}, \mathbb{V}_{\mathbf{f}}(-\boldsymbol{\kappa}_f^{1/2}) \hat{\otimes}\, \mathrm{ad}(T_g)\big) = H^1\big(\mathbb{Q}, \mathbb{V}_{\mathbf{f}}(-\boldsymbol{\kappa}_f^{1/2}) \hat{\otimes}\, \mathrm{ad}^0(T_g)\big) \oplus H^1\big(\mathbb{Q}, \mathbb{V}_{\mathbf{f}}(-\boldsymbol{\kappa}_f^{1/2})\big),$$

the class $\kappa^{(3)}(\mathbf{f}, g, g^*)$ lies in the summand

(1) $H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{f}}(-\boldsymbol{\kappa}_f^{1/2}) \hat{\otimes}\, \mathrm{ad}^0(T_g))$, if $\varepsilon(f) = 1$;
(2) $H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{f}}(-\boldsymbol{\kappa}_f^{1/2}))$, if $\varepsilon(f) = -1$.

Proof. We have the following commutative diagram

$$\begin{array}{ccc} H^0_{\mathrm{et}}(Y, \boldsymbol{\mathcal{A}}'_{r_1} \otimes \boldsymbol{\mathcal{A}}_{r_2} \otimes \boldsymbol{\mathcal{A}}_{r_2}(-\kappa^*)) & \xrightarrow{d_*} & H^4_{\mathrm{et}}(Y^3, \boldsymbol{\mathcal{A}}'_{r_1} \boxtimes \boldsymbol{\mathcal{A}}_{r_2} \boxtimes \boldsymbol{\mathcal{A}}_{r_2}(-\kappa^*) \otimes \mathbb{Z}_p(2)) \\ \big\downarrow{\scriptstyle w_N} & & \big\downarrow{\scriptstyle (w_N, w_N, w_N)} \\ H^0_{\mathrm{et}}(Y, \boldsymbol{\mathcal{A}}'_{r_1} \otimes \boldsymbol{\mathcal{A}}_{r_2} \otimes \boldsymbol{\mathcal{A}}_{r_2}(-\kappa^*)) & \xrightarrow{d_*} & H^4_{\mathrm{et}}(Y^3, \boldsymbol{\mathcal{A}}'_{r_1} \boxtimes \boldsymbol{\mathcal{A}}_{r_2} \boxtimes \boldsymbol{\mathcal{A}}_{r_2}(-\kappa^*) \otimes \mathbb{Z}_p(2)), \end{array}$$

where w_N stands here for the operator defined in [BSV21, §2.3.1] and (w_N, w_N, w_N) is defined in a similar way for the cohomology of Y^3. It follows from the definition of **Det** that $w_N(\mathbf{Det}) = \mathbf{Det}$. Since $w_p w_N = [p]_N w_N w_p$ and $\mathbf{s_{fgh}} \circ ([p]_N w_N \otimes w_N \otimes w_N) = [p]'_N (w_N \otimes w_N \otimes w_N) \circ \mathbf{s_{fgh}}$, it follows that

$$(w_N \otimes w_N \otimes w_N)\kappa^{(1)} = \kappa^*_{\mathbf{fgh}}(N)([p]_N \otimes 1 \otimes 1)\kappa^{(1)}.$$

Since $(w_N^2 \otimes 1 \otimes 1)$ acts as multiplication by $[-N] \otimes 1 \otimes 1$, we deduce that

$$(1 \otimes w_N \otimes w_N)\kappa^{(1)} = \kappa_f^*(N)([p]'_N w_N \otimes 1 \otimes 1)\kappa^{(1)}$$

and therefore that

$$N^{-r_2}(1 \otimes w_N \otimes w_N)\kappa^{(2)} = \kappa_f^{-1/2}(N)([p]'_N w_N \otimes 1 \otimes 1)\kappa^{(2)}.$$

Let s_N denote the operator which acts on the group

$$H^1\big(\mathbb{Q}, H^1(\Gamma, \mathcal{D}'_{r_1})^{\mathrm{ord}} \hat{\otimes} H^1(\Gamma, L_{r_2})^{\mathrm{ord}} \hat{\otimes} H^1(\Gamma, L_{r_2})^{\mathrm{ord}}(2 - r_2 - \boldsymbol{\kappa}_f^{1/2})\big)$$

by interchanging the second and third factors. Then, we have that

$$s_N \circ (1 \otimes w_N \otimes w_N) = (1 \otimes w_N \otimes w_N) \circ s_N,$$

and, taking into account the definition of **Det** and the fact that the Künneth isomorphism

$$H^3_{\mathrm{et}}(Y^3_{\overline{\mathbb{Q}}}, \boldsymbol{\mathcal{A}}'_{r_1} \boxtimes \boldsymbol{\mathcal{A}}_{r_2} \boxtimes \boldsymbol{\mathcal{A}}_{r_2}) \cong H^1_{\mathrm{et}}(Y_{\overline{\mathbb{Q}}}, \boldsymbol{\mathcal{A}}'_{r_1}) \otimes H^1_{\mathrm{et}}(Y_{\overline{\mathbb{Q}}}, \boldsymbol{\mathcal{A}}_{r_2}) \otimes H^1_{\mathrm{et}}(Y_{\overline{\mathbb{Q}}}, \boldsymbol{\mathcal{A}}_{r_2})$$

is given by cup-product, which is anti-commutative in degree 1 (*cf.* the proof of [LZ19, Prop. 4.1.2]), we deduce that $s_N(\kappa^{(2)}) = (-1)^{r_1/2 + r_2 + 1}\kappa^{(2)}$. Define $\tilde{s}_N = (-N)^{-r_2} s_N \circ (1 \otimes w_N \otimes w_N)$. Then, we have that

$$\tilde{s}_N(\kappa^{(2)}) = (-1)^{r_1/2+1}\kappa_f^{-1/2}(N)([p]'_N w_N \otimes 1 \otimes 1)\kappa^{(2)}.$$

Since $(1 \otimes \pi_{1*} \otimes \pi_{2*}) \circ \tilde{s}_N = \tilde{s}_{N_g} \circ (1 \otimes \pi_{1*} \otimes \pi_{2*})$, it follows that

$$\tilde{s}_{N_g}(\kappa^{(3)}) = -[N]^{-1/2}([p]'_N w_N \otimes 1 \otimes 1)\kappa^{(3)}$$

and therefore that

$$\tilde{s}_{N_g}\big(\kappa^{(3)}(\mathbf{f}, g, g^*)\big) = -[N]^{-1/2}(w_N \otimes 1 \otimes 1)\kappa^{(3)}(\mathbf{f}, g, g^*).$$

Finally, it follows from [How07, Prop. 2.3.6] that $-[N]^{-1/2} w_N$ acts on $\mathbb{V}_{\mathbf{f}}$ as multiplication by $\varepsilon(f)$, so the last part of the proposition follows from the previous lemma. □

Remark 3.3. From the definition of the map

$$\mathfrak{Log}(\mathbf{f}, g_\alpha, g_\alpha^*) : H^1_{\mathrm{bal}}(\mathbb{Q}_p, \mathbb{V}(\mathbf{f}, g, g^*)) \longrightarrow \Lambda_{\mathbf{f}}[1/p],$$

one can see that it factors through the cohomology of $\mathbb{V}(\mathbf{f}, g, g^*)_f$, and therefore that it factors through

$$H^1_{\mathrm{bal}}(\mathbb{Q}_p, \mathbb{V}(\mathbf{f}, g, g^*)) \longrightarrow H^1\big(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}}(-\boldsymbol{\kappa}_f^{1/2}) \otimes \mathrm{ad}^0(T_g)\big) \longrightarrow H^1(\mathbb{Q}_p, \mathbb{V}(\mathbf{f}, g, g^*)_f).$$

Therefore, *without* the need to appeal to the reciprocity law, it follows from Proposition 3.2 that when $\varepsilon(f) = -1$ we have

$$\mathfrak{Log}(\mathbf{f}, g_\alpha, g_\alpha^*)(\kappa) = 0.$$

Of course, this can also be seen from the reciprocity law: Since $\varepsilon(f) = -1$ forces the vanishing of $L(\mathbf{f}_{k'}, k'/2)$ for all $k' \equiv k \pmod{2(p-1)}$, and this is a factor of $L(\mathbf{f}_{k'} \otimes g \otimes g^*, c')$, it follows form the interpolation formula that $\mathscr{L}_p(\breve{\mathbf{f}}, \breve{g}, \breve{h})$ is identically zero.

Remark 3.4. As noted above, the discussion in this section is unnecessary for the applications that we will discuss. Indeed, as observed in the previous remark, the reciprocity law factors through $H^1\big(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}}(-\boldsymbol{\kappa}_f^{1/2}) \otimes \mathrm{ad}^0(T_g)\big)$. Therefore, the nonvanishing of the triple product p-adic L-function at some point (necessarily when $\varepsilon(f) = +1$) implies that the image of $\kappa^{(3)}$ in this group is nontrivial, which is what we will actually need. However, it is interesting that we can already see from the geometric construction that the class lies where it is expected.

Remark 3.5. Let us discuss the sign in a little bit more detail. In order to construct the p-adic L-function attached to (f, g, g^*), it is required in [Hsi21] that the local signs at finite primes of the arithmetic specializations of the representation $\mathbb{V}(\mathbf{f}, \mathbf{g}, \mathbf{g}^*) = \mathbb{V}_{\mathbf{f}} \hat{\otimes} \mathbb{V}_{\mathbf{g}} \hat{\otimes} \mathbb{V}_{\mathbf{g}^*}(-1 - \boldsymbol{\kappa}^*)$ are all equal to 1. In particular, in our case, this imposes the condition that $\varepsilon_\ell(\mathbf{f}_{k'}) = \varepsilon_\ell(\mathbf{f}_{k'} \otimes \mathrm{ad}^0(g))$ for all $\ell \mid N$ and for all $k' \equiv k \pmod{2(p-1)}$. The corresponding signs at infinity can be computed from the Hodge types $\{(p,q), (q,p)\}$ as in [Del79, §5.3]. For the representation $V_{\mathbf{f}_{k'}} \otimes \mathrm{ad}^0(V_g)$, the Hodge types are as follows:

(i) $\{(k'/2 + l - 2, -k'/2 - l + 1), (-k'/2 - l + 1, k'/2 + l - 2)\}$;
(ii) $\{(k'/2 - 1, -k'/2), (-k'/2, k'/2 - 1)\}$;
(iii) $\{(k'/2 - l, -k'/2 + l - 1), (-k'/2 + l - 1, k'/2 - l)\}$.

After that, and following the results of [*loc. cit.*], we get that the sign $\varepsilon_\infty(f \otimes \mathrm{ad}^0(g))$ is $(-1)^{k'/2}$ if $k' \geq 2l$ and $(-1)^{1+k'/2}$ if $k' < 2l$. The sign of $\varepsilon_\infty(\mathbf{f}_{k'})$, however, is always equal to $(-1)^{k'/2}$. Therefore, in the balanced region (i.e. for $k' < 2l$), the motives attached to $\mathbf{f}_{k'}$ and $\mathbf{f}_{k'} \otimes \mathrm{ad}^0(g)$ have opposite global signs. Since it is in this region that the corresponding specializations of the class $\kappa^{(3)}(\mathbf{f}, g, g^*)$ belong to the Bloch-Kato Selmer group, we expect the behaviour that was shown in Proposition 3.2.

4. The p-adic L-function

In this section, we keep the assumption that $(p)=\mathfrak{p}\overline{\mathfrak{p}}$ splits in K. In addition, from now on, for simplicity we assume that $p\nmid h_K$, where h_K is the class number of K.

Let $g\in S_l(N_g,\chi_g)$ be an ordinary newform not of CM-type. Let $\mathfrak{c}$ be an ideal of $\mathcal{O}_K$ coprime to p, and fix a Hecke character ψ_0 of infinity type $(-1,0)$ and conductor $\mathfrak{c}\mathfrak{p}^e$ with $e\in\{0,1\}$. We assume that the central character ε_{ψ_0} of ψ_0 is of the form

(H0) $$\varepsilon_{\psi_0}=\varepsilon_K\omega^{r_1}\text{ for some even integer } r_1,$$

where ω is the Teichmüller character.

4.1. **Lifting of automorphic representations.** Let π be the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ attached to g. The central character of π is the adelic character ω_g defined by the condition that for any prime $q\nmid N_g$ and any uniformizer ϖ_q we have $\omega_{g,p}(\varpi_q)=\chi_g(q)$. Since $p\nmid N_g$, the local component π_p is a spherical representation, and it follows from [Bum97, Thm. 4.6.4] and its proof that π_p is isomorphic to the principal series $\pi(\chi,\chi^{-1}\omega_g)$, where χ is the unramified character of $\mathbb{Q}_p^\times$ defined by $\chi(p)=\alpha_p(g)p^{(1-l)/2}$.

Since we are assuming that g is not of CM-type, and in particular it does not have CM by K, it follows from [GJ78, Prop. 2.3.3] that π admits, adopting the terminology of [*op. cit.*], a base change lifting to a cuspidal automorphic representation π_K of $\mathrm{GL}_2(\mathbb{A}_K)$. We fix such a lifting. Observe that if $\mathfrak{p},\overline{\mathfrak{p}}$ are the places of K above p, then $\pi_{K,\mathfrak{p}}\simeq\pi_{K,\overline{\mathfrak{p}}}\simeq\pi_p$.

From the assumption that g is not of CM-type we deduce that there is no non-trivial character η of $K^\times\backslash\mathbb{A}_K^\times$ such that $\pi_K\simeq\pi_K\otimes\eta$. Indeed, the existence of such a character would imply that there exists a quadratic extension L of K such that, for all prime ℓ, the restriction to G_K of the ℓ-adic Galois representation attached to g is induced from a character of G_L, which is not possible by [Rib85, Thm. 2.1]. Now, it follows from [GJ78, Thm. 9.3] that π_K admits an adjoint lifting to a cuspidal automorphic representation $\Pi_{\mathrm{Ad}^0(g)}$ of $\mathrm{GL}_3(\mathbb{A}_K)$. Fix such a lifting and define

$$\Pi:=\Pi_{\mathrm{Ad}^0(g)}\otimes\psi_0|\cdot|^{1/2}.$$

Observe that $\Pi_\mathfrak{p}\simeq\Pi_{\overline{\mathfrak{p}}}\simeq\pi(\chi^2\omega_{g,p}^{-1},1,\chi^{-2}\omega_{g,p})\otimes\psi_0|\cdot|^{1/2}$ and it follows from the definition of χ that $\chi^2\omega_{g,p}^{-1}\neq|\cdot|^{\pm1/2}$ and therefore that $\pi(\chi^2\omega_{g,p}^{-1},1,\chi^{-2}\omega_{g,p})=\mathrm{Ind}_B^{\mathrm{GL}_3}(\chi^2\omega_{g,p}^{-1},1,\chi^{-2}\omega_{g,p})$, where B denotes the Borel subgroup of $\mathrm{GL}_3(\mathbb{Q}_p)$.

4.2. **Descent to unitary groups.** Let $U(2,1)$ be the quasi-split unitary group corresponding to the quadratic extension $K/\mathbb{Q}$. Let $\Phi\in\mathrm{GL}_3(K)$ be the matrix whose entries are $\Phi_{ij}=(-1)^{i-1}\delta_{i,4-j}$. Then we can describe $U(2,1)$ by specifying its functor of points:

$$U(2,1)(R)=\left\{g\in\mathrm{GL}_3(R\otimes_\mathbb{Q}K)\ :\ g\Phi\,{}^t\overline{g}=\Phi\right\}$$

for any $\mathbb{Q}$-algebra R.

Let $U(3)$ be the definite unitary group whose functor of points is given by

$$U(3)(R)=\left\{g\in\mathrm{GL}_3(R\otimes_\mathbb{Q}K)\ :\ g\,{}^t\overline{g}=I_3\right\}.$$

Given a representation ρ of $\mathrm{GL}_3(\mathbb{A}_K)$, let $\tilde\rho$ be the representation defined on the same space by $\tilde\rho(x)=\rho({}^t\overline{x}^{-1})$. Then, the representation Π defined above satisfies $\Pi\simeq\tilde\Pi$, and so it follows from [Rog90, Thm. 13.3.3] that there exists a cuspidal automorphic representation σ' of $U(2,1)(\mathbb{A}_\mathbb{Q})$ whose base change to K is isomorphic to Π. Fix such a representation σ'. Observe that $\sigma'_p\simeq\pi(\chi^2\omega_g^{-1},1,\chi^{-2}\omega_g)\otimes\psi_0|\cdot|^{1/2}$ under the identification $U(2,1)(\mathbb{Q}_p)=\mathrm{GL}_3(\mathbb{Q}_p)$. Also, from [*op. cit.*, Prop. 13.2.2], the local representation σ'_∞ is square-integrable, so, applying [*op. cit.*, Prop. 14.6.2], we can transfer σ' to a representation σ of $U(3)$. The local components of σ at finite primes agree with those of σ', so in particular we have that $\sigma_p\simeq\sigma'_p$.

Remark 4.1. Let $GU(3)$ be the definite unitary similitude group whose functor of points is given by

$$GU(3)(R) = \left\{ g \in \mathrm{GL}_3(R \otimes_{\mathbb{Q}} K) \;:\; g\,{}^t\bar{g} = \nu(g) I_3 \text{ for some } \nu(g) \in R^\times \right\}.$$

As explained in [BR93, §1.8], one can extend σ to an irreducible automorphic representation of $GU(3)$ by choosing an extension of the central character of σ to the center of $GU(3)$.

4.3. **p-adic L-functions for unitary groups.** A construction of p-adic L-functions for unitary groups is given in [EW16], and, in great generality in [EHLS20]. Here we deduce from these works the existence of an anticyclotomic p-adic L-function for the conjugate self-dual representation V in §2.2.

Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ containing the values of ψ_0, and write

$$\Lambda^{\mathrm{ac}} = \mathcal{O}[\![\Gamma^{\mathrm{ac}}]\!]$$

for the anticyclotomic Iwasawa algebra, where Γ^{ac} is the Galois group of the anticyclotomic $\mathbb{Z}_p$-extension of K.

We will need to consider the following CM periods, as they are introduced in [BDP12]:

- $\Omega_\infty \in \mathbb{C}^\times$ is the complex period attached to K defined in [*op. cit.*, eq. (2-15)];
- $\Omega_p \in \mathbb{C}_p^\times$ is the p-adic period attached to K defined in [*op. cit.*, eq. (2-17)].

Theorem 4.2. *There exists an element*

$$L_p(\mathrm{ad}^0(g_K) \otimes \psi_0) \in \mathrm{Frac}\,\Lambda^{\mathrm{ac}}$$

such that for all characters ξ of Γ^{ac} crystalline at both $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ and corresponding to a Hecke character of infinity type $(-n, n)$ with $n \equiv r_1/2 \pmod{p-1}$ and $n \geq l-1$, we have

$$L_p(\mathrm{ad}^0(g_K) \otimes \psi_0)(\xi) = \left(\frac{\Omega_p}{\Omega_\infty}\right)^{6n+3} \cdot \pi^{3n} \cdot \Gamma(n,l) \cdot \mathcal{E}_p(\mathrm{ad}^0(g), \psi_0\xi)^2 \cdot L(\mathrm{ad}^0(g_K) \otimes \psi_0^{-1}\xi^{-1}\omega^n, 0),$$

where:

- $\Gamma(n,l) = (n+l-1)! \cdot n! \cdot (n-l+1)!$,
- $\mathcal{E}_p(\mathrm{ad}^0(g), \psi_0\xi) = \left(1 - \frac{\alpha_g(\psi_0\xi\omega^{-n})(\mathfrak{p})}{\beta_g p}\right) \cdot \left(1 - \frac{(\psi_0\xi\omega^{-n})(\mathfrak{p})}{p}\right) \cdot \left(1 - \frac{\beta_g(\psi_0\xi\omega^{-n})(\mathfrak{p})}{\alpha_g p}\right)$.

Proof. Let σ be the irreducible automorphic representation of $U(3)$ introduced in the previous subsection. Let Σ be the set of places of $\mathbb{Q}$ consisting of p, infinity, the primes dividing D_K, and the primes at which σ ramifies. On account of Remark 4.1, the main result of [EW16] yields an element $\mathscr{L}_p^\Sigma \in \Lambda^{\mathrm{ac}}[1/p]$ such that, for all ξ as in the statement, satisfies

$$\mathscr{L}_p^\Sigma(\xi) = \left(\frac{\pi\Omega_p}{\Omega_\infty}\right)^{6n+3} \cdot \mathcal{E}_p(\xi) \cdot \mathcal{E}_\infty(\xi) \cdot L^\Sigma(\tilde{\sigma}, \xi^{-1}\omega^n, 0),$$

where $\tilde{\sigma}$ is the contragredient of σ, and $\mathcal{E}_p(\xi)$ and $\mathcal{E}_\infty(\xi)$ are certain modified Euler factors at p and ∞, respectively. Since Σ contains p, infinity, and all the ramified primes, we have that

$$L^\Sigma(\tilde{\sigma}, \xi^{-1}\omega^n, 0) = L^\Sigma(\mathrm{ad}^0(g_K) \otimes \psi_0^{-1}\xi^{-1}\omega^n, 0).$$

Since we are assuming that p splits in K, the form of the modified Euler factor at p can be extracted from [EHLS20, eq. (86)]. Up to a nonzero rational factor independent of ξ, it is given by

$$\mathcal{E}_p(\xi) = \frac{\mathcal{E}_p(\mathrm{ad}^0(g), \psi_0\xi)^2}{L_p(\mathrm{ad}^0(g_K) \otimes \psi_0^{-1}\xi^{-1}\omega^n, 0)}.$$

The form of the modified Euler factor at infinity can be extracted from [EL20, eq. (2.3.1)]. This formula, with $a = 3$, $b = 0$, $\underline{\tau} = (l, 0, -l)$, $r = 2n + 2$ and $s = 0$, yields, up to a nonzero rational factor independent of ξ,

$$\mathcal{E}_\infty(\xi) = (2\pi i)^{-3n-3} \cdot \Gamma(n, l).$$

Finally, the Euler factors at primes $\ell \in \Sigma \setminus \{p, \infty\}$ can be p-adically interpolated by certain elements $\mathcal{P}_\ell \in \Lambda^{\mathrm{ac}}$, and, multiplying by their inverses, we obtain the p-adic L-function in the statement. □

4.4. **CM Hida family.** Let Γ_K be the Galois group of the $\mathbb{Z}_p^2$-extension K_∞/K and put

$$\Gamma_{\mathfrak{p}} = \mathrm{Gal}(K_{\mathfrak{p}^\infty}/K) \simeq \mathbb{Z}_p,$$

where $K_{\mathfrak{p}^\infty}$ is the maximal subfield of K_∞ unramified outside of $\mathfrak{p}$, so that $K_{\mathfrak{p}^\infty}$ is the $\mathbb{Z}_p$-extension of K inside the ray class field $K(\mathfrak{p}^\infty)$. Since we are assuming that $p \nmid h_K$, viewing $1 + p\mathbb{Z}_p$ as a subgroup of $\mathcal{O}_{K,\mathfrak{p}}^\times$, the restriction of the (geometrically normalized) Artin map to $K_{\mathfrak{p}}^\times$ induces an isomorphism $\mathrm{art}_{\mathfrak{p}} : 1 + p\mathbb{Z}_p \simeq \Gamma_{\mathfrak{p}}$. Let $\gamma_{\mathfrak{p}} \in \Gamma_{\mathfrak{p}}$ be the topological generator corresponding to $1+p$ under this isomorphism and, for the variable S, let $\Psi_S : \Gamma_K \to \mathbb{Z}_p[\![S]\!]^\times$ be the character given by

$$\Psi_S(\sigma) = (1 + S)^{l(\sigma)},$$

where $l(\sigma) \in \mathbb{Z}_p$ is defined by $\sigma|_{K_{\mathfrak{p}^\infty}} = \gamma_{\mathfrak{p}}^{l(\sigma)}$. Consider the formal q-expansion

$$\boldsymbol{\theta}_{\psi_0}(S)(q) := \sum_{(\mathfrak{a}, p\mathfrak{c})=1} \psi_0(\sigma_{\mathfrak{a}}) \Psi_S^{-1}(\sigma_{\mathfrak{a}}) q^{\mathbf{N}(\mathfrak{a})} \in \mathcal{O}[\![S]\!][\![q]\!], \tag{4.1}$$

where $\sigma_{\mathfrak{a}} \in \mathrm{Gal}(K(\mathfrak{c}\mathfrak{p}^\infty)/K)$ is the Artin symbol of $\mathfrak{a}$. Then, for every $k \geq 2$, the specialization of $\boldsymbol{\theta}_{\psi_0}$ at $S = (1 + p)^{k-2} - 1$ is given by the theta series

$$\mathbf{f}_k = \sum_{(\mathfrak{a}, p\mathfrak{c})=1} \psi_0(\mathfrak{a}) \lambda^{k-2}(\mathfrak{a}) q^{\mathbf{N}(\mathfrak{a})} \in S_k(D_K \mathbf{N}(\mathfrak{c}) p, \omega^{2+r_1-k}),$$

where λ is the unique (since $p \nmid h_K$) Hecke character of infinity type $(-1, 0)$ and conductor $\mathfrak{p}$ whose p-adic avatar factors through $\Gamma_{\mathfrak{p}}$. In particular, $\mathbf{f}_2$ is the ordinary p-stabilization of θ_{ψ_0}.

Remark 4.3. If ψ is a Hecke character of infinity type $(1 - k, 0)$ as in §2.2, then $\psi_0 := \psi\lambda^{2-k}$ is a Hecke character as above (in particular, satisfying (H0) with e.g. $r_1 = k - 2$), and so the resulting $\mathbf{f}_k$ recovers the p-stabilization of θ_ψ. From now on we shall always assume that ψ and ψ_0 are related in this manner, and refer to $\mathbf{f} = \boldsymbol{\theta}_{\psi_0}$ as the CM Hida family attached to ψ (or ψ_0).

4.5. **A factorization formula.** In this section we prove a factorization formula relating the p-adic L-function attached to V in Theorem 4.2 to anticyclotomic p-adic L-functions attached to the other two representations in the decomposition (2.1).

Put $N = \mathrm{lcm}(N_g, N_\psi)$, where $N_\psi := D_K \mathbf{N}(\mathfrak{c})$. In addition to the previous hypotheses, from now on we shall also assume that:

(a) $\varepsilon_\ell(V_{fgg^*}) = +1$ for all primes $\ell \mid N$,
(b) $\gcd(N_g, N_\psi)$ is squarefree.

With notations as in Remark 2.3, here $\varepsilon_\ell(V_{fgg^*})$ denotes the epsilon-factor of the Weil–Deligne representation attached to the restriction of V_{fgg^*} to $G_{\mathbb{Q}_\ell}$.

Note that it follows from (H0) that the Galois representation of the Hida family $\mathbf{f} = \boldsymbol{\theta}_{\psi_0}$ attached to ψ is residually irreducible and p-distinguished (see also [LLZ15, Rem. 5.1.3]). For the following result, we adopt the definition of congruence ideal in [Hsi21, §3.3].

Theorem 4.4. *Under the above hypotheses, there exists an element*

$$\mathscr{L}_p(\mathrm{ad}(g_K)\otimes\psi_0)\in\mathrm{Frac}\,\Lambda^{\mathrm{ac}}$$

such that for all characters ξ of Γ^{ac} crystalline at both $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ and corresponding to a Hecke character of infinity type $(-n,n)$ with $n\equiv r_1/2 \pmod{p-1}$ and $n\geq l-1$, we have

$$\mathscr{L}_p(\mathrm{ad}(g_K)\otimes\psi_0)(\xi)^2=\Gamma(n,l,l)\cdot\frac{\mathcal{E}_p(\mathrm{ad}(g),\psi_0\xi)^2}{\mathcal{E}_0(\psi_0\xi)^2\cdot\mathcal{E}_1(\psi_0\xi)^2}\cdot\prod_{\ell\mid N}\tau_\ell\cdot\frac{L(\mathrm{ad}(g_K)\otimes\psi_0^{-1}\xi^{-1}\omega^n,0)}{(2\pi i)^{4n+4}\cdot\langle\theta_{\psi_0\xi_n},\theta_{\psi_0\xi_n}\rangle^2},$$

where:

- $\Gamma(n,l,l)=(n+l-1)!\cdot(n!)^2\cdot(n-l+1)!$,
- $\mathcal{E}_p(\mathrm{ad}(g),\psi_0\xi)=\Bigl(1-\frac{\alpha_g(\psi_0\xi\omega^{-n})(\mathfrak{p})}{\beta_g p}\Bigr)\cdot\Bigl(1-\frac{(\psi_0\xi\omega^{-n})(\mathfrak{p})}{p}\Bigr)^2\cdot\Bigl(1-\frac{\beta_g(\psi_0\xi\omega^{-n})(\mathfrak{p})}{\alpha_g p}\Bigr)$,
- $\mathcal{E}_0(\psi_0\xi)=\Bigl(1-\frac{(\psi_0\xi\omega^{-n})(\mathfrak{p})}{(\psi_0\xi\omega^{-n})(\bar{\mathfrak{p}})}\Bigr)$, $\mathcal{E}_1(\psi_0\xi)=\Bigl(1-\frac{(\psi_0\xi\omega^{-n})(\mathfrak{p})}{p(\psi_0\xi\omega^{-n})(\bar{\mathfrak{p}})}\Bigr)$,
- *τ_ℓ is an explicit nonzero rational number independent of n,*
- *$\theta_{\psi_0\xi_n}$ is the theta series of weight $2n+2$ attached to $\psi_0\xi_n:=\psi_0\xi\omega^{-n}|\cdot|^{-n}$.*

Moreover, if $\mathcal{H}$ is any generator of the congruence ideal of $\boldsymbol{\theta}_{\psi_0}$, then $\mathcal{H}\cdot\mathscr{L}_p(\mathrm{ad}(g_K)\otimes\psi_0)$ belongs to Λ^{ac}.

Proof. This is essentially a reformulation of [Hsi21, Thm. A] specialized to our setting. Let $\mathbf{f}=\boldsymbol{\theta}_{\psi_0}$ be the Hida family attached to the Hecke character ψ_0 as in (4.1), with associated big Galois representation $\mathbb{V}_{\mathbf{f}}$, and denote by $\mathbb{V}(\mathbf{f},g,g^*)$ the Kummer self-dual twist of the triple tensor product $\mathbb{V}_{\mathbf{f}}\hat{\otimes}_{\mathcal{O}}T_g\otimes_{\mathcal{O}}T_{g^*}$ introduced in [ACR21, §7.1] (and recalled in §3 above). Since $\mathbb{V}_{\mathbf{f}}\simeq\mathrm{Ind}_K^{\mathbb{Q}}(\psi_0^{-1}\Psi_S)$, we immediately find that

$$\mathbb{V}(\mathbf{f},g,g^*)\simeq\mathrm{ad}(T_g)\otimes\mathrm{Ind}_K^{\mathbb{Q}}(\psi_0^{-1}\omega^{r_1/2}\Psi_S^{(1-\tau)/2}),$$

where for a character χ of G_K we denote by χ^τ the composition of χ with the action of the non-trivial automorphism τ of $K/\mathbb{Q}$, and put $\chi^{1-\tau}:=\chi(\chi^\tau)^{-1}$.

By [Hsi21, Thm. A], attached to $(\mathbf{f},g,g^*)$ (and a specific choice of level-N test vectors for this triple), there is an "unbalanced" triple product p-adic L-function $\mathscr{L}_p(\mathbf{f},g,g^*)\in\mathrm{Frac}\,\mathcal{O}[\![\Gamma_{\mathfrak{p}}]\!]$ interpolating, for all $k'\equiv r_1+2 \pmod{2(p-1)}$ with $k'\geq 2l$, the (central) values at $s=0$ of the triple product L-function

$$L(\mathbb{V}(\mathbf{f}_k,g,g^*),s)=L(\mathrm{ad}(g_K)\otimes\psi_0^{-1}\xi^{-1}\omega^{r_1/2},s).$$

where we put ξ to denote the specialization of $\Psi_S^{(\tau-1)/2}$ at $S=(1+p)^{k'-2}-1$, so ξ^{-1} is a character of Γ^{ac} crystalline at both $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ corresponding to a Hecke character of infinity type $(-(k'/2-1),k'/2-1)$. Taking $\mathscr{L}_p(\mathrm{ad}(g_K)\otimes\psi_0)$ to be the image of $\mathscr{L}_p(\mathbf{f},g,g^*)$ under the map $\mathrm{Frac}\,\mathcal{O}[\![\Gamma_{\mathfrak{p}}]\!]\to\mathrm{Frac}\,\Lambda^{\mathrm{ac}}$ determined by $\gamma_{\mathfrak{p}}\mapsto\gamma_{\mathfrak{p}}^{\tau-1}$, we thus see that the result follows from [Hsi21, Thm. A]. □

We next discuss an anticyclotomic p-adic L-function associated with V', arising from a suitable restriction of Katz's p-adic L-function.

Denote by Σ the set of algebraic Hecke characters ξ of K for which $s=0$ is a critical point for $L(\xi,s)$ in the sense of Deligne. This set can be written as the disjoint union $\Sigma=\Sigma_{\mathfrak{p}}\cup\Sigma_{\bar{\mathfrak{p}}}$, where

$$\Sigma_{\mathfrak{p}}=\{\xi\in\Sigma\text{ of infinity type }(a,b),\text{ with }a\geq 1,\,b\leq 0\},$$
$$\Sigma_{\bar{\mathfrak{p}}}=\{\xi\in\Sigma\text{ of infinity type }(a,b),\text{ with }a\leq 0,\,b\geq 1\}.$$

Note that the involution $\xi\mapsto\xi^\tau$ takes characters in $\Sigma_{\mathfrak{p}}$ to characters in $\Sigma_{\bar{\mathfrak{p}}}$, and vice versa.

Let $G_{\mathfrak{c}}=\mathrm{Gal}(K(\mathfrak{c}p^\infty)/K)$ be the Galois group of the ray class field of K of conductor $\mathfrak{c}p^\infty$, and denote by $\mathbb{Z}_p^{\mathrm{ur}}$ the completion of the ring of integers of the maximal unramified extension of $\mathbb{Q}_p$. The following result is originally due to Katz.

Theorem 4.5. *There exists an element $\mathscr{L}_{\mathfrak{p},\mathfrak{c}}^{\mathrm{Katz}}(K) \in \mathbb{Z}_p^{\mathrm{ur}}[\![G_{\mathfrak{c}}]\!]$ uniquely characterized by the property that for every character of $\Gamma_{\mathfrak{c}}$ corresponding to a Hecke character $\xi \in \Sigma_{\mathfrak{p}}$ of infinity type (k_1, k_2) and conductor dividing $\mathfrak{c}$ we have*

$$\mathscr{L}_{\mathfrak{p},\mathfrak{c}}^{\mathrm{Katz}}(K)(\xi) = \left(\frac{\Omega_p}{\Omega_\infty}\right)^{k_1-k_2} (k_1-1)! \cdot \left(\frac{\sqrt{D_K}}{2\pi}\right)^{k_2} \cdot (1-p^{-1}\xi^{-1}(\mathfrak{p})p^{-1})(1-\xi(\bar{\mathfrak{p}})) \cdot L_{\mathfrak{c}}(\xi, 0),$$

where $L_{\mathfrak{c}}(\xi, s)$ is the L-function of ξ with the Euler factors at the primes $\mathfrak{l}|\mathfrak{c}$ removed. Moreover, we have the functional equation

$$\mathscr{L}_{\mathfrak{p},\mathfrak{c}}^{\mathrm{Katz}}(K)(\xi) = \mathscr{L}_{\mathfrak{p},\bar{\mathfrak{c}}}^{\mathrm{Katz}}(K)(\xi^{-\tau}\mathbf{N}^{-1}),$$

where the equality is up to a p-adic unit.

Proof. See [dS87, Thm. II.4.14] for a construction of $\mathscr{L}_{\mathfrak{p},\mathfrak{c}}^{\mathrm{Katz}}(K)$ (corresponding to the measure on $G_{\mathfrak{c}}$ denoted by $\mu(\mathfrak{c}\bar{\mathfrak{p}}^\infty)$ in [*loc. cit.*]), and [dS87, Thm. II.6.4] for the functional equation. □

Assume that $\mathfrak{c}$ is fixed under complex conjugation, i.e., $\bar{\mathfrak{c}} = \mathfrak{c}$. Denote by $\Delta_{\mathfrak{c}}$ the torsion subgroup of $G_{\mathfrak{c}}$, and put $\Gamma_K := G_{\mathfrak{c}}/\Delta_{\mathfrak{c}} \simeq \mathbb{Z}_p^2$, which is identified with the Galois group of the unique $\mathbb{Z}_p^2$-extension K_∞/K. We fix a decomposition

$$G_{\mathfrak{c}} \simeq \Delta_{\mathfrak{c}} \times \Gamma_K. \tag{4.2}$$

Put $\bar{\psi}_0 = \psi_0|_{\Delta_{\mathfrak{c}}}$ and $\bar{\psi}_0^- = \bar{\psi}_0^{\tau-1}$, and note that the latter defines a finite order anticyclotomic Hecke character of conductor dividing $\mathfrak{c}p^s$ for some $s \geq 0$. Denote by $\mathscr{L}_{\mathfrak{p},\psi_0}^{\mathrm{Katz}}(K)^-$ the image of $\mathscr{L}_{\mathfrak{p},\mathfrak{c}}^{\mathrm{Katz}}(K)$ under the composite map

$$\mathbb{Z}_p^{\mathrm{ur}}[\![G_{\mathfrak{c}}]\!] \to \mathbb{Z}_p^{\mathrm{ur}}[\![\Gamma_K]\!] \to \mathbb{Z}_p^{\mathrm{ur}}[\![\Gamma^{\mathrm{ac}}]\!],$$

where the first arrow is the projection defined by $\bar{\psi}_0^-$ and the second arrow is given by $\gamma \mapsto \gamma^{\tau-1}$ for $\gamma \in \Gamma_K$.

From now on we shall assume that the above $\mathfrak{c}$ and ψ_0 satisfy the conditions (H1)–(H4) in the following result.

Proposition 4.6. *In addition to* (H0)*, assume that:*

- (H1) $\mathfrak{c}$ *is only divisible by primes that are split in K;*
- (H2) $\bar{\psi}_0^-$ *has order prime-to-p and the prime-to-p part of its conductor it exactly* $\mathfrak{c}$*;*
- (H3) $\bar{\psi}_0^-|_{G_{K_v}} \neq 1$ *for all primes $v|p$ in K;*
- (H4) $\bar{\psi}_0^-$ *has order at least* 3.

Then, as an ideal of $\mathbb{Z}_p^{\mathrm{ur}}[\![\Gamma^{\mathrm{ac}}]\!]$, the congruence ideal $C(\boldsymbol{\theta}_{\psi_0})$ is generated by

$$h_K \cdot \mathscr{L}_{\mathfrak{p},\psi_0}^{\mathrm{Katz}}(K)^-$$

where h_K is the class number of K.

Proof. A generator of $C(\boldsymbol{\theta}_{\psi_0})$ is given by a congruence power series $H(\boldsymbol{\theta}_{\psi_0})$ attached to $\boldsymbol{\theta}_{\psi_0}$ as in [Hid06]. By our assumptions, this $H(\boldsymbol{\theta}_{\psi_0})$ corresponds to a branch character satisfying the hypotheses (1)–(4) in [Hid06, p. 466], so as noted in p. 469 of [*op. cit.*], the result follows from the proof of the anticyclotomic Iwasawa main conjecture by Hida–Tilouine [HT93a, HT94] and Hida [Hid06]. □

Definition 4.7. Put

$$L_p(\mathrm{ad}(g_K) \otimes \psi_0) := \left(\mathscr{L}_p(\mathrm{ad}(g_K) \otimes \psi_0) \cdot h_K \cdot \mathscr{L}_{\mathfrak{p},\psi_0}^{\mathrm{Katz}}(K)^-\right)^2,$$

which by Theorem 4.4 and Proposition 4.6 defines an element in $\mathbb{Z}_p^{\mathrm{ur}}[\![\Gamma^{\mathrm{ac}}]\!]$.

We can now derive an anticyclotomic analogue of Dasgupta's factorization [Das16, Thm. 1], relating the p-adic L-function of Theorem 4.4 to the product of the p-adic L-functions in Theorem 4.2 and Theorem 4.5. Similarly as in [*loc. cit.*], our result is a p-adic analogue of the factorization of complex L-functions

$$L(\mathrm{ad}(g_K)\otimes\chi,s)=L(\mathrm{ad}^0(g_K)\otimes\chi,s)\cdot L(\chi,s)$$

arising from the decomposition of G_K-representations

$$\mathrm{ad}(V_g)\otimes\chi\simeq(\mathrm{ad}^0(V_g)\otimes\chi)\oplus\chi.$$

However, our proof is largely simplified by the fact that the three p-adic L-functions involved have a Zariski dense overlapping set of characters in the range of interpolation.

Our factorization formula will in fact involve the following anticyclotomic projection of the Katz p-adic L-function.

Definition 4.8. Viewing ψ_0 as a character of $G_{\mathfrak{c}}$, write $\psi_0=\bar{\psi}_0\cdot\psi_\Gamma$ according to the factorization (4.2), with $\bar{\psi}_0$ (resp. ψ_Γ) a character of $\Delta_{\mathfrak{c}}$ (resp. Γ_K). We denote by $\mathscr{L}_{\mathfrak{p}}^{\mathrm{Katz}}(\psi_0)^{-,\iota}\in\mathbb{Z}_p^{\mathrm{ur}}[\![\Gamma^{\mathrm{ac}}]\!]$ the image of $\mathscr{L}_{\mathfrak{p},\mathfrak{c}}^{\mathrm{Katz}}(K)$ under the composite map

$$\mathbb{Z}_p^{\mathrm{ur}}[\![G_{\mathfrak{c}}]\!]\to\mathbb{Z}_p^{\mathrm{ur}}[\![\Gamma_K]\!]\to\mathbb{Z}_p^{\mathrm{ur}}[\![\Gamma_K]\!]\to\mathbb{Z}_p^{\mathrm{ur}}[\![\Gamma^{\mathrm{ac}}]\!]\to\mathbb{Z}_p^{\mathrm{ur}}[\![\Gamma^{\mathrm{ac}}]\!],$$

where the first arrow is given by the projection defined by $\bar{\psi}_0^{-1}\omega^{r_1/2}$, the second by twisting by ψ_Γ^{-1}, the third is the natural projection, and the last arrow is the involution given by $\gamma\mapsto\gamma^{-1}$ for $\gamma\in\Gamma^{\mathrm{ac}}$. In other words, $\mathscr{L}_{\mathfrak{p}}^{\mathrm{Katz}}(\psi_0)^{-,\iota}$ is the element of $\mathbb{Z}_p^{\mathrm{ur}}[\![\Gamma^{\mathrm{ac}}]\!]$ defined by

$$\mathscr{L}_{\mathfrak{p}}^{\mathrm{Katz}}(\psi_0)^{-,\iota}(\xi)=\mathscr{L}_{\mathfrak{p},\mathfrak{c}}^{\mathrm{Katz}}(K)(\psi_0^{-1}\xi^{-1}\omega^{r_1/2})$$

for all characters ξ of Γ^{ac}.

Denote by τ_N the product of constants $\prod_{\ell|N}\tau_\ell$ appearing in Theorem 4.4.

Theorem 4.9. *The following equality holds*

$$L_p(\mathrm{ad}(g_K)\otimes\psi_0)=u\cdot L_p(\mathrm{ad}^0(g_K)\otimes\psi_0)\cdot\mathscr{L}_{\mathfrak{p}}^{\mathrm{Katz}}(\psi_0)^{-,\iota}\cdot\tau_N$$

where u is a unit in $(\Lambda^{\mathrm{ac}})^\times$.

Proof. Let ξ be a character of Γ^{ac} as in the statement of Theorem 4.2 and Theorem 4.4, hence in particular corresponding to a Hecke character, still denoted by ξ, of infinity type $(-n,n)$ with $n\geq l-1$. Noting that $\theta_{\psi_0\xi_n}$ has weight $2n+2$, from Hida's formula for the adjoint L-value (see [HT93b, Thm .7.1]) and Dirichlet's class number formula we obtain (cf. [JSW17, p. 414])

$$\langle\theta_{\psi_0\xi_n},\theta_{\psi_0\xi_n}\rangle=(2n+1)!\cdot D_K^2\cdot\frac{1}{2^{4n+4}\pi^{2n+3}}\cdot\frac{2\pi h_K}{w_K\sqrt{D_K}}\cdot L(\psi_0^{1-\tau}\xi^{1-\tau},1),\tag{4.3}$$

where w_K is the number of units in $\mathcal{O}_K$. Since $L(\psi_0^{1-\tau}\xi^{1-\tau},1)$ corresponds to the value at $s=0$ of the L-function for the Hecke character $\psi_0^{\tau-1}\xi^{\tau-1}\mathbf{N}^{-1}$ of infinity type $(2n+2,-2n)$, using (4.3) the interpolation formula in Theorem 4.5 can be rewritten as

$$\begin{aligned}\mathscr{L}_{\mathfrak{p},\mathfrak{c}}^{\mathrm{Katz}}(K)(\psi_0^{\tau-1}\xi^{\tau-1}\mathbf{N}^{-1})=\left(\frac{\Omega_p}{\Omega_\infty}\right)^{4n+2}&\cdot\frac{2^{6n+4}\pi^{4n+2}}{\sqrt{D_K}^{2n+1}}\cdot\frac{w_K}{D_K^2h_K}\\&\times\left(1-\frac{(\psi_0\xi\omega^{-n})(\mathfrak{p})}{(\psi_0\xi\omega^{-n})(\bar{\mathfrak{p}})}\right)\left(1-\frac{(\psi_0\xi\omega^{-n})(\mathfrak{p})}{p(\psi_0\xi\omega^{-n})(\bar{\mathfrak{p}})}\right)\cdot\langle\theta_{\psi_0\xi_n},\theta_{\psi_0\xi_n}\rangle.\end{aligned}$$

Thus together with Theorem 4.4 we find that

(4.4)
$$\begin{aligned}&\mathscr{L}_p(\mathrm{ad}(g_K)\otimes\psi_0)(\xi)^2\cdot\mathscr{L}_{\mathfrak{p},\mathfrak{c}}^{\mathrm{Katz}}(K)(\psi_0^{\tau-1}\xi^{\tau-1}\mathbf{N}^{-1})^2\cdot h_K^2\\&=\left(\frac{\Omega_p}{\Omega_\infty}\right)^{8n+4}\cdot\frac{2^{8n+4}\pi^{4n}}{\sqrt{D_K}^{4n}}\cdot\Gamma(n,l,l)\cdot\mathcal{E}(\mathrm{ad}(g),\psi_0\xi)^2\cdot\frac{w_K^2}{D_K^4}\cdot\tau_N\cdot L(\mathrm{ad}(g_K)\otimes\psi_0^{-1}\xi^{-1}\omega^n,0).\end{aligned}$$

On the other hand, we have the factorization

$$L(\mathrm{ad}(g_K)\otimes\psi_0^{-1}\xi^{-1}\omega^n,0)=L(\mathrm{ad}^0(g_K)\otimes\psi_0^{-1}\xi^{-1}\omega^n,0)\cdot L(\psi_0^{-1}\xi^{-1}\omega^n,0). \tag{4.5}$$

The character $\psi_0^{-1}\xi^{-1}\omega^n$ has infinity type $(n+1,-n)$, and so is in the range of interpolation for $\mathscr{L}_{\mathfrak{p},\mathfrak{c}}^{\mathrm{Katz}}(K)$. Thus combining Theorem 4.2 and Theorem 4.5 and using (4.5) we find

(4.6)
$$\begin{aligned}L_p(\mathrm{ad}^0(g_K)\otimes\psi_0)(\xi)\cdot\mathscr{L}_{\mathfrak{p},\mathfrak{c}}^{\mathrm{Katz}}(K)(\psi_0^{-1}\xi^{-1}\omega^n)&=\left(\frac{\Omega_p}{\Omega_\infty}\right)^{6n+3}\cdot\pi^{3n}\cdot\Gamma(n,l)\cdot\mathcal{E}(\mathrm{ad}^0(g),\psi_0\xi)^2\\&\times\left(\frac{\Omega_p}{\Omega_\infty}\right)^{2n+1}\cdot n!\cdot\left(\frac{2\pi}{\sqrt{D_K}}\right)^n\cdot(1-p^{-1}\psi_0\xi(\mathfrak{p}))^2\cdot L(\mathrm{ad}(g_K)\otimes\psi_0^{-1}\xi^{-1}\omega^n,0).\end{aligned}$$

Comparing (4.4) and (4.6) we see that their ratio is given by $2^{7n+4}\cdot\sqrt{D_K}^{-3n-8}\cdot\tau_N$; since for varying n the first two factors are interpolated by a unit in $(\Lambda^{\mathrm{ac}})^\times$, applying the functional equation of Theorem 4.5 this gives the result. □

5. The Euler system

Let $g\in S_l(N_g,\chi_g)$ be a newform as in §2.1, and let ψ be a Hecke character of K of infinity type $(1-k,0)$ for some even integer $k\geq 2$, conductor $\mathfrak{c}$ prime to p, and central character ε_K. Recall from §4 that we assume that $(p)=\mathfrak{p}\overline{\mathfrak{p}}$ splits in K and (for simplicity) that p does not divide the class number of K.

5.1. **Modified diagonal cycles.** Recall that, for a positive integer m, we write $K[m]$ for the maximal p-extension inside the ring class field of K of conductor m. Recall further that $H^1_{\mathrm{Iw}}(K[mp^\infty],T)=\varprojlim_r H^1(K[mp^r],T)$. Then, for a positive integer m, let

$$\kappa_{\psi,\mathrm{ad}(g),m,\infty}\in H^1_{\mathrm{Iw}}(K[mp^\infty],T^\psi_{\mathrm{ad}(g)}) \tag{5.1}$$

be the class $\kappa_{\psi,g,g^*,m,\infty}$ constructed in [ACR21, Thm. 6.5]. (For $m=1$, this is essentially the class $\kappa^{(3)}(\mathbf{f},g,g^*)$ defined in §3, after an application of Shapiro's lemma and twisting by the inverse of the anticyclotomic character ξ in (5.4) below.) Since we have a direct sum decomposition

$$H^1_{\mathrm{Iw}}(K[mp^\infty],T^\psi_{\mathrm{ad}(g)})=H^1_{\mathrm{Iw}}(K[mp^\infty],T)\oplus H^1_{\mathrm{Iw}}(K[mp^\infty],T'),$$

we can project the class $\kappa_{\psi,\mathrm{ad}(g),m,\infty}$ to each of the summands. We denote its projection to the first summand as $\kappa_{\psi,\mathrm{ad}^0(g),m,\infty}$. For the following results, we keep the notations for Selmer groups introduced in Section 2.3.

Theorem 5.1. *Let $\mathcal{S}$ be the set of all squarefree products of primes q split in K and coprime to pN_gN_ψ, and assume that $H^1(K[mp^s],T)$ is torsion-free for every $m\in\mathcal{S}$ and for every $s\geq 0$. There exists a collection of classes*

$$\left\{\kappa_{\psi,\mathrm{ad}^0(g),m,\infty}\in\mathrm{Sel}_{\mathrm{bal}}(K[mp^\infty],T)\,:\,m\in\mathcal{S}\right\}$$

such that whenever $m,mq\in\mathcal{S}$ with q a prime, we have

$$\mathrm{cor}_{K[mq]/K[m]}(\kappa_{\psi,\mathrm{ad}^0(g),mq,\infty})=P_{\mathfrak{q}}(V^\psi_{\mathrm{ad}(g)};\mathrm{Fr}_{\mathfrak{q}}^{-1})\,\kappa_{\psi,\mathrm{ad}^0(g),m,\infty}.$$

Proof. This is an immediate consequence of [ACR21, Thm. 6.5] and [*op. cit.*, Prop. 6.6]. □

The Euler factors appearing in the previous theorem are not the ones that we want. Indeed, let $q = \mathfrak{q}\bar{\mathfrak{q}}$ be a prime which splits in K. Then

$$P_{\mathfrak{q}}(V^{\psi}_{\mathrm{ad}(g)}; X) = \left(1 - \frac{\psi(\mathfrak{q})X}{q^{k/2}}\right) P_{\mathfrak{q}}(V; X),$$

so there is an unwanted extra factor. We now deal with this problem.

Definition 5.2. Let $\mathcal{P}'$ be the set of primes $q \nmid pN_gN_\psi$ split in K such that

- $q = 1$ modulo p,
- $T/(\mathrm{Fr}_q - 1)T$ is a cyclic $\mathbb{Z}_p$-module,
- $\mathrm{Fr}_q - 1$ is bijective on T'.

Here Fr_q denotes any arithmetic Frobenius element for q. Since $T^\vee(1) \simeq T^c$ and $(T')^\vee(1) \simeq (T')^c$, the definition does not depend on this choice.

Remark 5.3. Under certain conditions, we will show in Proposition 6.3 below that there exists $\sigma \in G_K$ such that if q is a prime such that Fr_q is conjugate to σ in $\mathrm{Gal}\,(K(\mu_p, \overline{T}, \overline{T'})/K)$, then it belongs to $\mathcal{P}'$.

Theorem 5.4. *Let $\mathcal{S}'$ be the set of squarefree products of primes in $\mathcal{P}'$, and assume that $H^1(K[mp^s], T)$ is torsion-free for every $n \in \mathcal{S}'$ and for every $s \geq 0$. There exists a collection of classes*

$$\left\{\kappa_m \in \mathrm{Sel}_{\mathrm{bal}}(K[mp^\infty], T) \;:\; m \in \mathcal{S}'\right\}$$

such that $\kappa_1 = \kappa_{\psi,\mathrm{ad}^0(g),1,\infty}$ and, whenever $m, mq \in \mathcal{S}'$ with q a prime, we have

$$\mathrm{cor}_{K[mq]/K[m]}(\kappa_{mq}) = P_{\mathfrak{q}}(V; \mathrm{Fr}_{\mathfrak{q}}^{-1})\, \kappa_m,$$

where $\mathfrak{q}$ is any of the primes of K above q.

Proof. We construct the classes κ_m by modifying the classes $\kappa_{\psi,\mathrm{ad}^0(g),m,\infty}$ in Theorem 5.1 appropriately as done in the proof of [LZ19, Thm. 5.3.3].

For each $m \in \mathcal{S}'$, let $\Gamma_m = \mathrm{Gal}\,(K[mp^\infty]/K)$. For each prime $q \mid m$, let $F_q \in \mathcal{S}'$ denote the unique element of Γ_m which acts trivially on $K[q]$ and maps to Fr_q in $\Gamma_{m/q}$. Then, the factor $1 - q^{-k/2}\psi(\mathfrak{q})F_{\mathfrak{q}}^{-1}$ is invertible in $\mathbb{Z}_p[\![\Gamma_m]\!]$. We now take

$$\kappa = \prod_{q \mid m} \left(1 - \frac{\psi(\mathfrak{q})}{q^{k/2}} F_{\mathfrak{q}}^{-1}\right)^{-1} \kappa_{\psi,\mathrm{ad}^0(g),m,\infty}.$$

These classes clearly satisfy the required properties. □

5.2. **The explicit reciprocity law.** Let K_∞ denote the anticyclotomic $\mathbb{Z}_p$-extension of K and let

$$\kappa_{\psi,\mathrm{ad}(g),\infty} \in H^1_{\mathrm{Iw}}(K_\infty, T^{\psi}_{\mathrm{ad}(g)})$$

be the image of the class $\kappa_{\psi,\mathrm{ad}(g),1,\infty}$ in (5.1) under the corestriction map for $K[p^\infty]/K_\infty$. By [ACR21, Prop. 6.6] we have $\kappa_{\psi,\mathrm{ad}(g),\infty} \in \mathrm{Sel}_{\mathrm{bal}}(K_\infty, T^{\psi}_{\mathrm{ad}(g)})$; in particular, the restriction $\mathrm{res}_{\bar{\mathfrak{p}}}(\kappa_{\psi,\mathrm{ad}(g),\infty})$ lands in the image of the natural map

$$H^1_{\mathrm{Iw}}(K_{\infty,\bar{\mathfrak{p}}}, \mathscr{F}^{\mathrm{bal}}_{\bar{\mathfrak{p}}}(T^{\psi}_{\mathrm{ad}(g)})) \longrightarrow H^1_{\mathrm{Iw}}(K_{\infty,\bar{\mathfrak{p}}}, T^{\psi}_{\mathrm{ad}(g)})$$

(see (2.3)). Note that this map is an injection under our hypotheses. On the other hand, let

$$\kappa_{\psi,\mathrm{ad}^0(g),\infty} \in \mathrm{Sel}_{\mathrm{bal}}(K_\infty, T) \tag{5.2}$$

be the image of the class $\kappa_1 = \kappa_{\psi,\mathrm{ad}^0(g),1,\infty}$ of Theorem 5.4 under the corestriction map. Thus $\kappa_{\psi,\mathrm{ad}^0(g),\infty}$ is the projection of $\kappa_{\psi,\mathrm{ad}(g),\infty}$ onto the first direct summand in the decomposition

$$\mathrm{Sel}_{\mathrm{bal}}(K_\infty, T^{\psi}_{\mathrm{ad}(g)}) = \mathrm{Sel}_{\mathrm{bal}}(K_\infty, T) \oplus \mathrm{Sel}_{\mathrm{bal}}(K_\infty, T'),$$

and since $\mathscr{F}^{\rm bal}_{\bar{\mathfrak{p}}}(T^{\psi}_{{\rm ad}(g)})$ is contained in T, it is clear that

$$\mathrm{res}_{\bar{\mathfrak{p}}}(\kappa_{\psi,{\rm ad}(g),\infty}) = \mathrm{res}_{\bar{\mathfrak{p}}}(\kappa_{\psi,{\rm ad}^0(g),\infty}). \tag{5.3}$$

Definition 5.5. Put $\psi_0 = \psi\lambda^{2-k}$ as in Remark 4.3, and define

$$L_p({\rm ad}^0(g_K)\otimes\psi) := \mathrm{Tw}_\xi\big(L_p({\rm ad}^0(g_K)\otimes\psi_0)\big), \tag{5.4}$$

where $\mathrm{Tw}_\xi : \Lambda^{\rm ac} \to \Lambda^{\rm ac}$ is the twisting homomorphism for the character $\xi := (\lambda^{1-\tau})^{k/2-1}$. Similarly, define $\mathscr{L}_p({\rm ad}(g_K)\otimes\psi)$, $\mathscr{L}^{\rm Katz}_{\mathfrak{p},\psi}(K)^-$, $L_p({\rm ad}(g_K)\otimes\psi)$, and $\mathscr{L}^{\rm Katz}_{\mathfrak{p}}(\psi)^{-,\iota}$ by twisting the corresponding p-adic L-functions defined for ψ_0 in §4.5.

For the statement of the next result, note that ψ_0 has the same restriction to $\Delta_{\mathfrak{c}}$ as ψ.

Theorem 5.6. *There exists an injective $\Lambda^{\rm ac}$-module map with pseudo-null cokernel*

$$\mathfrak{Log} : H^1_{\rm Iw}(K_{\infty,\bar{\mathfrak{p}}}, \mathscr{F}^{\rm bal}_{\bar{\mathfrak{p}}}(T^{\psi}_{{\rm ad}(g)})) \longrightarrow \mathbb{Z}_p^{\rm ur}[\![\Gamma^{\rm ac}]\!]$$

such that

$$\mathfrak{Log}(\mathrm{res}_{\bar{\mathfrak{p}}}(\kappa_{\psi,{\rm ad}^0(g),\infty})) = h_K\cdot\mathscr{L}^{\rm Katz}_{\mathfrak{p},\psi}(K)^-\cdot\mathscr{L}_p({\rm ad}(g_K)\otimes\psi).$$

Proof. Let $\mathbb{V}(\mathbf{f},g,g^*)$ and $\mathbb{V}(\mathbf{f},g,g^*)_f$ be as in §3 (corresponding to $\mathbb{V}^\dagger_{\mathbf{f}gg^*}$ and $\mathbb{V}^{gg^*}_{\mathbf{f}}$, respectively, in the notation of [ACR21, §8.2]). Then, identifying $G_{\mathbb{Q}_p}$ with $G_{K_{\bar{\mathfrak{p}}}}$ via the composition of the embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ fixed in the introduction with complex conjugation, we get an isomorphism of $\Lambda_{\mathbf{f}}[G_{K_{\bar{\mathfrak{p}}}}]$-modules

$$\mathbb{V}(\mathbf{f},g,g^*)_f = \mathbb{V}^-_{\mathbf{f}}\hat{\otimes}_{\mathcal{O}}T^+_g\otimes T^+_{g^*}(\epsilon_{\rm cyc}^{1-l}\boldsymbol{\kappa}_f^{-1/2}) \simeq \mathscr{F}^{\rm bal}_{\bar{\mathfrak{p}}}(T^{\psi}_{{\rm ad}^0(g)})\otimes\xi\Psi_S^{(\tau-1)/2}. \tag{5.5}$$

By [ACR21, Thm. 7.4], after extending scalars to $\mathbb{Z}_p^{\rm ur}$, the composition of the $\Lambda_{\mathbf{f}}$-linear map

$$H^1(\mathbb{Q}_p, \mathbb{V}(\mathbf{f},g,g^*)_f) \longrightarrow \Lambda_{\mathbf{f}}$$

of [*op. cit.*, Prop. 7.3] with the isomorphism $\mathbb{Z}_p^{\rm ur}\hat{\otimes}\Lambda_{\mathbf{f}} \simeq \mathbb{Z}_p^{\rm ur}[\![\Gamma^{\rm ac}]\!]$ given by $\gamma\mapsto\gamma^{\tau-1}$ sends the class $\kappa^{(3)}(\mathbf{f},g,g^*)$ recalled in §3 to the product

$$h_K\cdot\mathscr{L}^{\rm Katz}_{\mathfrak{p},\psi_0}(K)^-\cdot\mathscr{L}_p({\rm ad}(g_K)\otimes\psi_0),$$

noting that by Proposition 4.6 the first two factors generate the congruence ideal of $\mathbf{f}$. Taking twists by ξ and using the isomorphism

$$H^1_{\rm Iw}(K_{\infty,\bar{\mathfrak{p}}}, \mathscr{F}^{\rm bal}_{\bar{\mathfrak{p}}}(T^{\psi}_{{\rm ad}^0(g)})\otimes\xi) \simeq H^1(\mathbb{Q}_p, \mathbb{V}(\mathbf{f},g,g^*)_f)$$

induced by (5.5) and using (5.3), the result follows. □

Corollary 5.7. *The map $\mathfrak{Log}$ of Theorem 5.6 satisfies*

$$\big(\mathfrak{Log}(\mathrm{res}_{\bar{\mathfrak{p}}}(\kappa_{\psi,{\rm ad}^0(g),\infty}))^2\big) = \big(L_p({\rm ad}^0(g_K)\otimes\psi)\cdot\mathscr{L}^{\rm Katz}_{\mathfrak{p}}(\psi)^{-,\iota}\big)$$

as ideals in $\mathbb{Z}_p^{\rm ur}[\![\Gamma^{\rm ac}]\!]\otimes\mathbb{Q}_p$.

Proof. This is clear from Theorem 5.6 and the factorization in Theorem 4.9. □

6. Verifying the hypotheses

Let g and ψ be as introduced in §2.1 and §2.2, respectively. Recall that, given a rational prime p and a sufficiently large finite extension $E/\mathbb{Q}_p$, we define the G_K-representation

$$V = \mathrm{ad}^0(V_g)(\psi_{\mathfrak{P}}^{-1})(1-k/2),$$

where $\rho_g : G_{\mathbb{Q}} \to \mathrm{Aut}_E(V_g) \simeq \mathrm{GL}_2(E)$ is the p-adic Galois representation attached to g. The aim of this section is to give conditions under which the hypotheses in the general results of [JNS] are satisfied for V. Let $K(p^\infty)^\circ$ denote the maximal abelian extension of K unramified at primes not dividing p. Then, the crucial condition that we need to verify is the existence of an element $\sigma \in \mathrm{Gal}\,(\bar{K}/K(p^\infty)^\circ)$ such that $T/(\sigma-1)T$ is a free $\mathcal{O}$-module of rank one, where $\mathcal{O}$ is the ring of integers of E.

As in [Loe17, §3.1] we define an open subgroup $H_g \subseteq G_{\mathbb{Q}}$, a quaternion algebra B_g and an algebraic group G_g. Let $H = H_g \cap G_{K(\mathfrak{c})^\circ}$. Then we have an adelic representation

$$\tilde{\rho}_g : H \longrightarrow G_g(\hat{\mathbb{Q}})$$

and representations

$$\tilde{\rho}_{g,p} : H \longrightarrow G_g(\mathbb{Q}_p)$$

for every rational prime p, and, according to [Loe17, Thm. 2.2.2], for all but finitely many p we can conjugate $\tilde{\rho}_{g,p}$ so that $\tilde{\rho}_{g,p}(H) = G_g(\mathbb{Z}_p)$.

Let L be a finite extension of K containing the Fourier coefficients of g and the image of the Hecke character ψ. Let $\mathfrak{P}$ be a prime of L above some rational prime p, and let $E = L_{\mathfrak{P}}$.

Definition 6.1. We say that the prime $\mathfrak{P}$ is *good* if the following conditions hold:

- $p \geq 3$;
- p is unramified in B_g;
- p is coprime to $\mathfrak{c}$ and N_g;
- $\tilde{\rho}_{g,p}(H) = G_g(\mathbb{Z}_p)$;
- $E = \mathbb{Q}_p$.

Lemma 6.2. *Assume that there is at least one prime which divides D_K but not N_g. Then, if $\mathfrak{P}$ is a good prime,*

$$\rho_{g,\mathfrak{P}}(H \cap G_{K(p^\infty)^\circ}) = \mathrm{SL}_2(\mathbb{Z}_p).$$

Proof. The proof of this result is very similar to the proof of [ACR21, Lem. 8.9]. We include it here for the convenience of the reader.

Let $\mathbb{Q}(\rho_g)$ be the Galois extension of $\mathbb{Q}$ cut out by the representations ρ_g. It is unramified outside pN_g. Therefore, the condition on D_K implies that $K \cap \mathbb{Q}(\rho_g) = \mathbb{Q}$. Moreover, since any Galois extension of $\mathbb{Q}$ contained in the anticyclotomic $\mathbb{Z}_p$-extension K_∞ of K must itself contain K, we also have $K_\infty \cap \mathbb{Q}(\rho_g) = \mathbb{Q}$.

The conditions on $\mathfrak{P}$ imply that

$$\rho_{g,\mathfrak{P}}(H \cap G_{\mathbb{Q}(\mu_{p^\infty})}) = \mathrm{SL}_2(\mathbb{Z}_p),$$

and, from the remarks in the previous paragraph, it follows that

$$\rho_{g,\mathfrak{P}}(H \cap G_{K_\infty(\mu_{p^\infty})}) = \mathrm{SL}_2(\mathbb{Z}_p).$$

Finally, since $H \cap G_{K(p^\infty)^\circ}$ is a normal subgroup of $H \cap G_{K_\infty(\mu_{p^\infty})}$ of index dividing $p-1$ and there are no such subgroups in $\mathrm{SL}_2(\mathbb{Z}_p)$, the lemma follows. □

Now fix a good prime $\mathfrak{P}$ and define $\mathbb{Z}_p[G_K]$-modules $T = \mathrm{ad}^0\,T_g(\psi_{\mathfrak{P}}^{-1})(1-k/2)$ and $T' = \mathbb{Z}_p(\psi_{\mathfrak{P}}^{-1})(1-k/2)$. Let $V = T \otimes \mathbb{Q}_p$ and $V' = T' \otimes \mathbb{Q}_p$.

Proposition 6.3. *Assume that there is at least one prime which divides D_K but not N_g. Suppose that there exists $\eta \in G_{K(p^\infty)^\circ}$ such that $\chi_g(\eta)\psi_{\mathfrak{P}}(\eta)$ is a square modulo p and $\psi_{\mathfrak{P}}(\eta)^2 \neq 1$ modulo p. Then there exists $\sigma \in G_{K(p^\infty)^\circ}$ such that*

- *$T/(\sigma-1)T$ is free of rank 1 over $\mathbb{Z}_p$,*
- *$\sigma - 1$ acts invertibly on T'.*

Proof. We closely follow the proof of [LZ19, Prop. 5.2.1] (see also the proof of [ACR21, Lem 5.10] and [Loe17, Prop. 4.2.1]). By the previous lemma the image of $\eta H \cap G_{K(p^\infty)^\circ}$ under $\rho_{g,\mathfrak{P}}$ contains all the elements of the form

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1}\chi_g(\eta) \end{pmatrix}, \quad x \in \mathbb{Z}_p^\times.$$

Choose x such that $x^2 = \chi_g(\eta)\psi_{\mathfrak{P}}(\eta)$. Choose $\sigma \in \eta H \cap G_{K(p^\infty)^\circ}$ whose image under $\rho_{g,\mathfrak{P}}$ is given by the element above, with the choice of x which we have just specified. Then, the eigenvalues of σ acting on T are 1, $\psi_{\mathfrak{P}}^{-1}(\eta)$ and $\psi_{\mathfrak{P}}^{-2}(\eta)$ and the eigenvalue of σ acting on T' is $\psi_{\mathfrak{P}}^{-1}(\eta)$. The result follows from the assumptions on η. □

7. Applications

Let $g \in S_l(N_g, \chi_g)$ and ψ be a Hecke character of K of infinity type $(1-k, 0)$ for some even integer $k \geq 2$ as introduced in §2, and recall that we consider the E-valued G_K-representation V in Definition 2.1. We begin by collecting a set of hypotheses for our later reference.

Hypotheses 7.1.

- (h1) p splits in K,
- (h2) $p \nmid h_K$,
- (h3) the conditions in Proposition 4.6 hold,
- (h4) g is ordinary at p and non-Eisenstein mod p,
- (h5) g is not of CM type,
- (h6) $\mathfrak{P}$ is a good prime in the sense of Definition 6.1,
- (h7) the conditions in Proposition 6.3 hold.

7.1. The Bloch–Kato conjecture.

We begin with a standard lemma, whose proof follows from the same argument as in [ACR21, Lem. 9.1].

Lemma 7.2. *The Bloch–Kato Selmer group of V is given by*

$$\mathrm{Sel}(K,V) \simeq \begin{cases} \mathrm{Sel}_{\mathrm{bal}}(K,V) & \text{if } 2 \leq k < 2l, \\ \mathrm{Sel}_{\mathrm{unb}}(K,V) & \text{if } k \geq 2l. \end{cases}$$

Let $\kappa_{\psi,\mathrm{ad}^0(g),\infty}$ be as in (5.2), and denote by

$$\kappa_{\psi,\mathrm{ad}^0(g)} \in \mathrm{Sel}_{\mathrm{bal}}(K,T)$$

the image of $\kappa_{\psi,\mathrm{ad}^0(g),\infty}$ under the corestriction $H^1_{\mathrm{Iw}}(K_\infty, T) \to H^1(K,T)$.

Theorem 7.3. *Assume hypotheses (h1)–(h7). Then the following implication holds:*

$$\kappa_{\psi,\mathrm{ad}^0(g)} \neq 0 \quad \Longrightarrow \quad \dim_E \mathrm{Sel}_{\mathrm{bal}}(K,V) = 1.$$

In particular, if $2 \leq k < 2l$ and $\kappa_{\psi,\mathrm{ad}^0(g)} \neq 0$ then the Bloch–Kato Selmer group $\mathrm{Sel}(K,V)$ is one-dimensional.

Proof. This follows from the general theory of anticyclotomic Euler systems developed in [JNS] (see [ACR21, §8] for a summary) applied to the Euler system constructed in Theorem 5.4. By Proposition 6.3, Hypotheses 7.1 give sufficient conditions for the general results of [JNS] to apply in our case. Note also that for the application of these results it suffices to have an anticyclotomic Euler system consisting of classes indexed by squarefree products of primes q in a positive density set $\mathcal{P}'$ of primes split in K, as is the case for the anticyclotomic Euler system of Theorem 5.4 (see Remark 5.3). □

Theorem 7.3 can be viewed as a result towards the Bloch–Kato conjecture for V in rank 1. The next result establishes cases of the same conjecture in rank 0.

Theorem 7.4. *Assume hypotheses (h1)–(h7), and in addition that:*

- $\varepsilon_\ell(V_{fgg^*}) = +1$ *for all primes* $\ell \mid N$,
- $\gcd(N_g, N_\psi)$ *is squarefree,*
- $L(\theta_\psi, k/2) \neq 0$.

If $k \geq 2l$ *then the following implication holds:*

$$L(V, 0) \neq 0 \quad \Longrightarrow \quad \mathrm{Sel}(K, V) = 0.$$

Proof. By Theorem 4.2 and Definition 5.5 we see that

$$L(V, 0) \neq 0 \quad \Longrightarrow \quad L_p(\mathrm{ad}^0(g_K) \otimes \psi)(\xi_{\mathrm{triv}}) \neq 0,$$

where ξ_{triv} is the trivial character of Γ^{ac}. Similarly, from Theorem 4.5 and Definition 5.5 we see that

$$L(\theta_\psi, k/2) \neq 0 \quad \Longrightarrow \quad \mathscr{L}_{\mathfrak{p}}^{\mathrm{Katz}}(\psi)^{-,\iota}(\xi_{\mathrm{triv}}) \neq 0.$$

Therefore by the factorization in Theorem 4.9 we thus see that $L_p(\mathrm{ad}(g_K) \otimes \psi)(\xi_{\mathrm{triv}}) \neq 0$, and so $\kappa_{\psi,\mathrm{ad}^0(g)} \neq 0$ by the explicit reciprocity law of Corollary 5.7. The result now follows from Theorem 7.3 and global duality by the same argument as in [ACR21, Thm. 9.5]. □

Remark 7.5. The hypotheses in Theorem 7.4 and the decomposition (2.1) imply that the sign of the functional equation for $L(V, s)$ is $+1$, and so the nonvanishing of $L(V, 0)$ is expected to hold generically.

7.2. **The Iwasawa main conjecture.** Here we deduce our main result towards the anticyclotomic Iwasawa main conjecture for V.

Since ψ has central character ε_K by assumption, its associated theta series θ_ψ has trivial nebentypus. In the following we denote by $\varepsilon(\theta_\psi)$ its global root number.

Theorem 7.6. *Assume hypotheses (h1)–(h7), and in addition that:*

- $\varepsilon_\ell(V_{fgg^*}) = +1$ *for all primes* $\ell \mid N$,
- $\varepsilon(\theta_\psi) = +1$,
- $\gcd(N_g, N_\psi)$ *is squarefree.*

If the p*-adic* L*-function* $L_p(\mathrm{ad}^0(g_K)\otimes\psi)$ *is nonzero, then the Pontryagin dual of* $\mathrm{Sel}_{\mathrm{unb}}(K_\infty, A)$ *is* Λ^{ac}*-torsion, with*

$$\mathrm{Char}_{\Lambda^{\mathrm{ac}}}\big(\mathrm{Sel}_{\mathrm{unb}}(K_\infty, A)^\vee\big) \supset \big(L_p(\mathrm{ad}^0(g_K) \otimes \psi) \cdot \mathscr{L}_{\mathfrak{p}}^{\mathrm{Katz}}(\psi)^{-,\iota}\big)$$

in $\mathbb{Z}_p^{\mathrm{ur}}[\![\Gamma^{\mathrm{ac}}]\!] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Proof. The assumption that $\varepsilon(\theta_\psi) = +1$ implies that the anticyclotomic projection $\mathscr{L}_{\mathfrak{p}}^{\mathrm{Katz}}(\psi)^{-,\iota}$ is nonzero by Greenberg's nonvanishing results [Gre83]. Since $L_p(\mathrm{ad}^0(g_K) \otimes \psi) \neq 0$ by hypothesis, together with the factorization in Theorem 4.9 it follows that

$$L_p(\mathrm{ad}(g_K) \otimes \psi) \neq 0.$$

By Corollary 5.7, this shows that the class $\kappa_{\psi,\mathrm{ad}^0(g),\infty}$ is non-torsion. By the general results of [JNS] (see also [ACR21, Thm. 8.5]), we thus conclude that $X_{\mathrm{bal}}(K_\infty, A)$ and $\mathrm{Sel}_{\mathrm{bal}}(K_\infty, T)$ both have Λ^{ac}-rank one, with

$$\mathrm{Char}_{\Lambda^{\mathrm{ac}}}\big(X_{\mathrm{bal}}\big(K_\infty, A\big)_{\mathrm{tors}}\big) \supset \mathrm{Char}_{\Lambda^{\mathrm{ac}}}\left(\frac{\mathrm{Sel}_{\mathrm{bal}}(K_\infty, T)}{\Lambda^{\mathrm{ac}} \cdot \kappa_{\psi,\mathrm{ad}^0(g),\infty}}\right)^2.$$

The result now follows from this by the same argument as in the proof of [ACR21, Thm. 7.15] based on Poitou–Tate duality and the explicit reciprocity law of Corollary 5.7. □

References

[ACR21] Raúl Alonso, Francesc Castella, and Óscar Rivero, *Iwasawa theory for* $\mathrm{GL}_2 \times \mathrm{GL}_2$ *and diagonal cycles*, preprint, arXiv:2106.05322.

[AH06] Adebisi Agboola and Benjamin Howard, *Anticyclotomic Iwasawa theory of CM elliptic curves*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 4, 1001–1048. MR 2266884

[Arn07] Trevor Arnold, *Anticyclotomic main conjectures for CM modular forms*, J. Reine Angew. Math. **606** (2007), 41–78. MR 2337641

[BDP12] Massimo Bertolini, Henri Darmon, and Kartik Prasanna, *p-adic Rankin L-series and rational points on CM elliptic curves*, Pacific J. Math. **260** (2012), no. 2, 261–303. MR 3001796

[BK90] Spencer Bloch and Kazuya Kato, *L-functions and Tamagawa numbers of motives*, The Grothendieck Festschrift, Vol. I, Progr. Math., vol. 86, Birkhäuser Boston, Boston, MA, 1990, pp. 333–400. MR 1086888

[BR93] Don Blasius and Jonathan D. Rogawski, *Motives for Hilbert modular forms*, Invent. Math. **114** (1993), no. 1, 55–87. MR 1235020

[BSV21] Massimo Bertolini, Marco Adamo Seveso, and Rodolfo Venerucci, *Reciprocity laws for balanced diagonal cycles*, Astérisque, to appear (2021), available at the website https://www.esaga.uni-due.de/massimo.bertolini/publications/.

[Bum97] Daniel Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics, vol. 55, Cambridge University Press, Cambridge, 1997. MR 1431508

[Das16] Samit Dasgupta, *Factorization of p-adic Rankin L-series*, Invent. Math. **205** (2016), no. 1, 221–268. MR 3514962

[Del79] P. Deligne, *Valeurs de fonctions L et périodes d'intégrales*, Automorphic forms, representations and *L*-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, With an appendix by N. Koblitz and A. Ogus, pp. 313–346. MR 546622

[dS87] Ehud de Shalit, *Iwasawa theory of elliptic curves with complex multiplication*, Perspectives in Mathematics, vol. 3, Academic Press, Inc., Boston, MA, 1987, *p*-adic *L* functions. MR 917944

[EHLS20] Ellen Eischen, Michael Harris, Jianshu Li, and Christopher Skinner, *p-adic L-functions for unitary groups*, Forum Math. Pi **8** (2020), e9, 160. MR 4096618

[EL20] Ellen Eischen and Zheng Liu, *Archimedean zeta integrals for unitary groups*, preprint, `arXiv:2006.04302` (2020).

[EW16] Ellen Eischen and Xin Wan, *p-adic Eisenstein series and L-functions of certain cusp forms on definite unitary groups*, J. Inst. Math. Jussieu **15** (2016), no. 3, 471–510. MR 3505656

[GJ78] Stephen Gelbart and Hervé Jacquet, *A relation between automorphic representations of* GL(2) *and* GL(3), Ann. Sci. École Norm. Sup. (4) **11** (1978), no. 4, 471–542. MR 533066

[Gre83] Ralph Greenberg, *On the Birch and Swinnerton-Dyer conjecture*, Invent. Math. **72** (1983), no. 2, 241–265. MR 700770

[Gre94] ———, *Iwasawa theory and p-adic deformations of motives*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 193–223.

[GS93] Ralph Greenberg and Glenn Stevens, *p-adic L-functions and p-adic periods of modular forms*, Invent. Math. **111** (1993), no. 2, 407–447.

[Hid06] Haruzo Hida, *Anticyclotomic main conjectures*, Doc. Math. (2006), no. Extra Vol., 465–532.

[How07] Benjamin Howard, *Variation of Heegner points in Hida families*, Invent. Math. **167** (2007), no. 1, 91–128.

[Hsi21] Ming-Lun Hsieh, *Hida families and p-adic triple product L-functions*, American Journal of Mathematics **143** (2021), no. 2, 411–532.

[HT93a] H. Hida and J. Tilouine, *Anti-cyclotomic Katz p-adic L-functions and congruence modules*, Ann. Sci. École Norm. Sup. (4) **26** (1993), no. 2, 189–259.

[HT93b] ———, *Anti-cyclotomic Katz p-adic L-functions and congruence modules*, Ann. Sci. École Norm. Sup. (4) **26** (1993), no. 2, 189–259. MR 1209708

[HT94] ———, *On the anticyclotomic main conjecture for CM fields*, Invent. Math. **117** (1994), no. 1, 89–147.

[JNS] Dimitar Jetchev, Jan Nekovář, and Christopher Skinner, preprint.

[JSW17] Dimitar Jetchev, Christopher Skinner, and Xin Wan, *The Birch and Swinnerton-Dyer formula for elliptic curves of analytic rank one*, Camb. J. Math. **5** (2017), no. 3, 369–434.

[LLZ15] Antonio Lei, David Loeffler, and Sarah Livia Zerbes, *Euler systems for modular forms over imaginary quadratic fields*, Compos. Math. **151** (2015), no. 9, 1585–1625.

[Loe17] David Loeffler, *Images of adelic Galois representations for modular forms*, Glasg. Math. J. **59** (2017), no. 1, 11–25.

[LZ19] David Loeffler and Sarah Livia Zerbes, *Iwasawa theory for the symmetric square of a modular form*, J. Reine Angew. Math. **752** (2019), 179–210. MR 3975641

[Rib85] Kenneth A. Ribet, *On l-adic representations attached to modular forms. ii.*, Glasgow Math. J. **27** (1985), 185–194.

[Rog90] Jonathan D. Rogawski, *Automorphic representations of unitary groups in three variables*, Annals of Mathematics Studies, vol. 123, Princeton University Press, Princeton, NJ, 1990. MR 1081540

R. A.: Department of Mathematics, Princeton University, Fine Hall, Princeton, NJ 08544-1000, USA
Email address: raular@math.princeton.edu

F. C.: Department of Mathematics, University of California, Santa Barbara, CA 93106, USA
Email address: castella@ucsb.edu

O. R.: Simons Laufer Mathematical Sciences Institute, 17 Gauss Way, Berkeley, CA 94720, United States of America
Email address: riverosalgado@gmail.com