A SPECIAL CASE OF SYLVESTER'S IDENTITY FOR QUANTUM MATRICES

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Abstract

Quantum matrices are matrices with entries that almost commute, e.g., they commute up to a constant. These structures arise naturally when studying Hopf algebras, representation theory, knot invariants, and they more recently have been found to be intimately related to totally nonnegative matrices (real matrices in which every minor is nonnegative). In the past two decades, several determinant identities from classical linear algebra have been shown to carry over to the quantum case.

In his 2014 work, Casteels developed a combinatorial method to embed the coordinate ring of quantum matrices into a subalgebra of the quantum torus, a similar, but simpler, algebra. Extending his work, we provide a new proof for a particular case of the quantum analog of Sylvester's Determinant Identity by calculating determinants using path systems on a lattice. We also demonstrate our attempts to generalize our proof to the full result. t

1. Introduction and Background

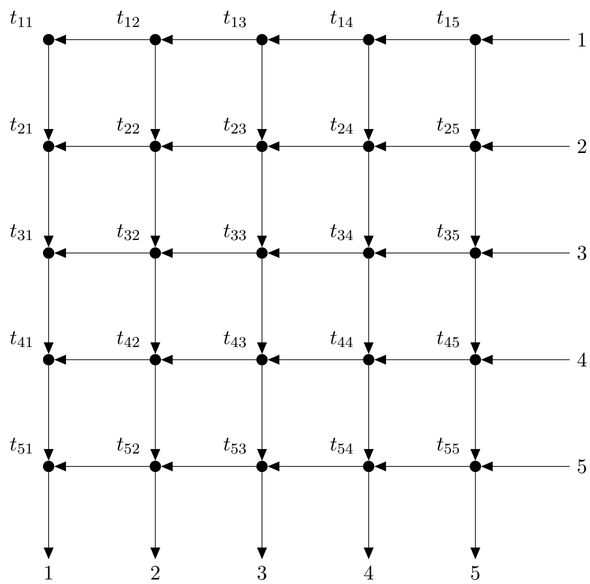
Quantum matrices can be thought of as the matrices for which every 2×2 submatrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies the following commutation relations:

$$ab = qba;\ ac = qca; bd = qdb;\ cd = qdc; bc = cb;\ ad = da + (q - q^{-1})bc.$$

When we discuss quantum "matrices," however, we really refer to the coordinate algebra $\mathcal{O}_q(\mathbb{M}_{m\times n})$, with invariants $x_{11}, x_{12}, \ldots, x_{mn}$ labeled as the entries to the $m \times n$ matrix $X = (x_{ij})$, that is, the ring $k\langle x_{11}, \ldots, x_{mn}\rangle$ for a field k, with these relations:

- $x_{ij}x_{ik} = qx_{ik}x_{ij}$ when $j < k$
- $x_{ij}x_{kj} = qx_{kj}x_{ij}$ when $i < k$
- $x_{ij}xk\ell = x_{k\ell}x_{ij} - (q - q^{-1})x_{i\ell}x_{kj}$ when $i < k$ and $j < \ell$
- $x_{ij}x_{k\ell} = x_{k\ell}x_{ij}$ when $i < k$ and $j < \ell$

Now when we speak of quantum matrices, we actually refer to the quantized coordinate ring $\mathcal{O}_q(M_{m\times n})$, whose generators $\{x_{ij}\}$ satisfy the above properties. Now we would like to embed this into a simpler algebra, one that does not have, in particular, the ugly relation $x_{ij}xk\ell = x_{k\ell}x_{ij} - (q - q^{-1})x_{i\ell}x_{kj}$. Now the quantum affine space, whose generators we will denote $\{t_{ij}\}_{1\leq i\leq m, 1\leq j\leq n}$, satisfy exactly the above relations, except that any two t_{ij} on a diagonal simply commute with each other. The quantum torus is the algebra generated by these elements of the quantum affine space and their inverses, and it is into this space that we can embed quantum matrices. We do this by considering an $m \times n$ path model for $m \times n$ quantum matrices. This is a grid of vertices labeled with the generators of the quantum affine space, with directed edges going left and down. From the right we also have the vertices labels $1, \ldots, m$ and on the bottom vertices labeled $1, \ldots, n$ (an example of 5×5 quantum matrices path model is shown below).



Now define the weight of a path from i to j as the product of the weights of the turns in the path. The weight of a turn is the label of the vertex it turns on if it goes from horizontal to vertical (a Γ turn), or the inverse of the label of the vertex if the turn is from vertical to horizontal (a ⅃ turn).

The result is that, the map from x_{ij} to the sum of the weights of all the paths from i on the right to j on the bottom in the path diagram defines the embedding of quantum matrices into the quantum torus.

This is more comprehensively spelled out in Casteel's 2004 paper, Matrices by Paths.

1.1. **Quantum Determinant.** There are several linear-algebraic concepts with analogues for quantum matrices, one of the most fundamental being the determinant (as well as the permanent, and in general the immanant). The classical determinant for an $n \times n$ matrix $A = (a_{ij})$ can be defined as

$$\det(A) = \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where $\ell(\sigma)$ is the number of inversions in σ.[1] Now we define the quantum determinant of an element $B = (b_{ij})$ of $n \times n$ quantum matrices with a similar equation:

$$\det_q(B) = \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} \prod_{i=1}^{n} b_{i\sigma(i)}.$$

With the quantum determinant defined, such identities as the Cayley-Hamilton Theorem, Muir's Theorem, Sylvester's Identity, and others from classical linear algebra have analogues for quantum matrices.

Now the quantum determinant can be calculated in these path models as well. It has been verified that on these path models, the equivalent definition of the determinant of a square $k \times k$ submatrix on $m \times n$ quantum matrices indexed by rows $\{i_1, \ldots, i_k\}$ and columns $\{j_1, \ldots, j_k\}$

2. Sylvester's Identity

Let T_{PQ} be a $k \times k$ submatrix of the $n \times n$ matrix A, where P, Q are the respective indices of the rows and the columns of A that make up T_{PQ}. For $1 \leq i, j \leq n$, let T_{ij} denote the $k+1 \times k+1$ submatrix of A from the rows indexed with the elements of $P \cup \{i\}$ and the columns indexed by the elements of $Q \cup \{j\}$. Let $U = (\det(T_{ij}))_{1 \leq i,j \leq n, i \notin P, j \notin Q}$. Then Sylvester's Identity states that $\det(U) = \det(A)\det(T_{PQ})^{n-k-1})$.

More casually, Sylvester's Identity states that there is a nice relationship between a matrix and its $k \times k$ minor given by the determinant of another matrix, the entries of which are all the $k+1 \times k+1$ minors having the $k \times k$ minor as a submatrix.

The quantum analogue of Sylvester's identity is exactly the same, with simply the replacement of "determinant" with "quantum determinant" (or det with $\det_q$). This was considered by, and proven. We attempt to prove this again, but with these path models, in order to demonstrate this new proof method, and with hopes that this could me more revealing and perhaps sleeker. Now we consider a particular case of quantum Sylvester's identity that itself is not too difficultly verified: In the original paper on the quantum Sylvesters identity, it is the first case considered; our goal past this was to generalize more and more from this base.

Given an $n \times n$ matrix

$$A = \begin{bmatrix} c & A_{12} & d \\ A_{21} & A_{22} & A_{23} \\ e & A_{32} & f \end{bmatrix},$$

[1] Probably the more common form of this equation uses the function $\mathrm{sgn}(\sigma)$ rather than $\ell(\sigma)$, both of which when used as the exponent of -1 produce the same results; the difference is as an exponent for $-q$; since there is no difference in the first case, we use this less standard function to emphasize the similarity between these two determinants.

where c, d, e, f are single entries and A_{22} is an $(n-2)\times(n-2)$ block, the minor in consider. We claim that $\det_q(A_{22})\det_q(A) = \det_q(C)\det_q(F) - q\det_q(D)\det_q(E)$, where

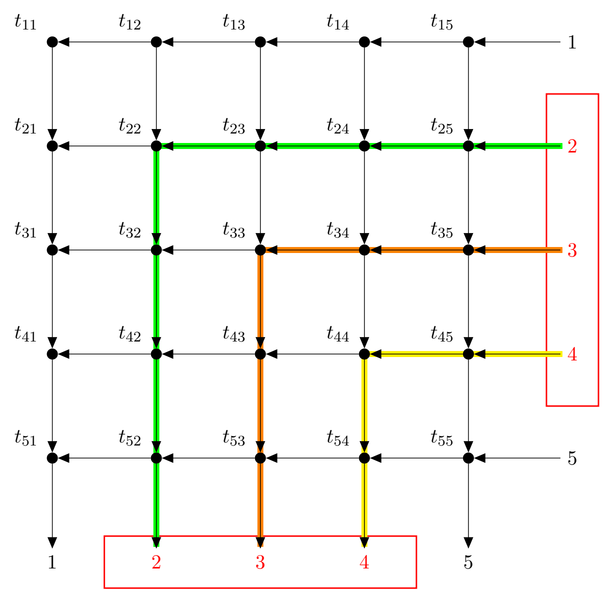
$$C = \begin{bmatrix} c & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

and we define D, E, F similarly, with the three nearest entries from A as written above included.Note this identity satisfies the general quantum Sylvester Identity statement, that is, $U = \begin{bmatrix} \det_q(C) & \det_q(D) \\ \det_q(E) & \det_q(F) \end{bmatrix}$, and $\det_q(U) = \det_q(A)\det_q(A_{22})^{n-(n-2)-1} = \det_q(A)\det_q(A_{22})$.

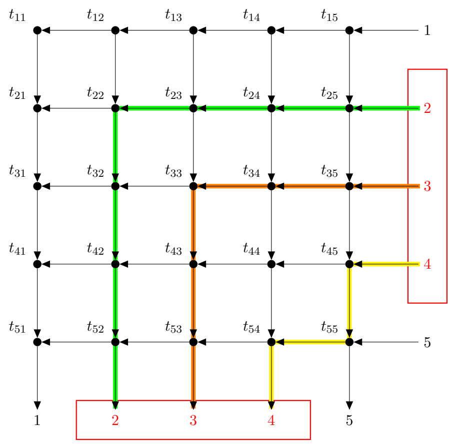
We demonstrate the proof using 5×5 quantum matrices path models as illustrative, but the proof itself speaks generally of any $n \times n$ quantum matrices.

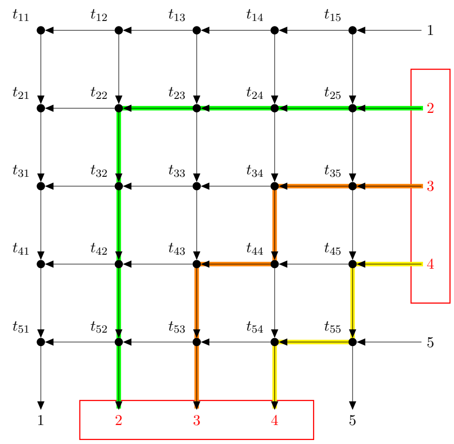
First we calculate $\det_q(A_{22})$, taking the sum of the weights of all vertex-disjoint path systems. We remark here that this calculation demonstrates the relative ease of proving the case chosen: the submatrix A_{22} makes up the majority of the matrix, so the possible ways of changing paths is limited; furthermore, that A_{22} is in one block in A ever more constrains the variability of paths. We now consider the three types of paths we can have.

The first type of path consists of the "most efficient" path, in the sense that it makes the fewest turns. There is only one path associate with this, the one that from i to i turns at $t_{i,i}$. The weight of this path system is $\prod_{i=2}^{n-1} t_{i,i}$.



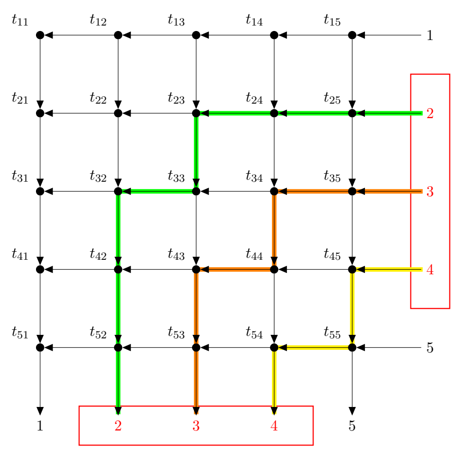
Now the second type of path systems made up partly of the straight paths from i to i turning only at t_{ii}, for $2 \le i < m$, and paths that turn at $t_{i,i+1}$, $t_{i+1,i+1}$, and $t_{i+1,i}$, noting that this, if it happens $i = m$, must happen for all i with $m \le i \le n-1$ because otherwise we could not have vertex-disjointness. But this is the only degree of freedom we have.We demonstrate both instances of this case for 5×5 quantum matrices:





The sum of the weights of these path models comes to $\sum_{j=2}^{n-1}(\prod_{i=2}^{j-1} t_{i,i} \cdot \prod_{i=j-1}^{n-1} t_{i,i+1}t_{i+1,i+1}^{-1}t_{i+1,i})$.

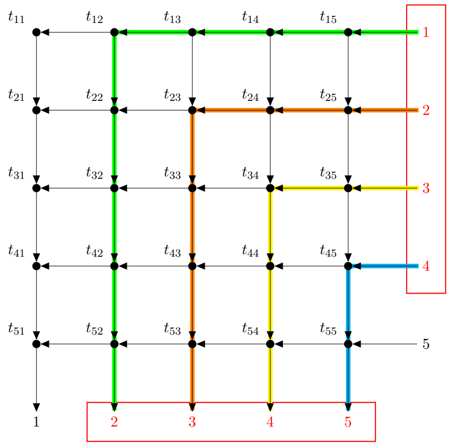
Now in the final case, we have that all the paths take this "detour." The calculation yields $\prod_{i=2}^{n-1} t_{i,i+1}t_{i+1,i+1}^{-1}t_{i+1,i}$.

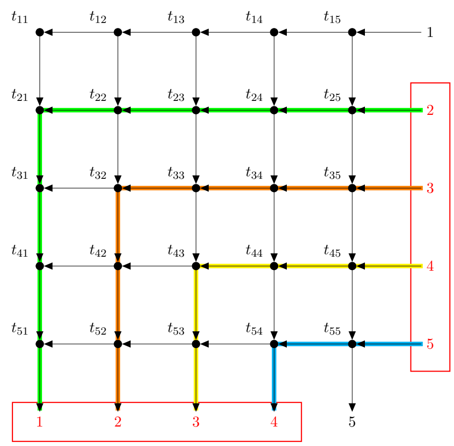


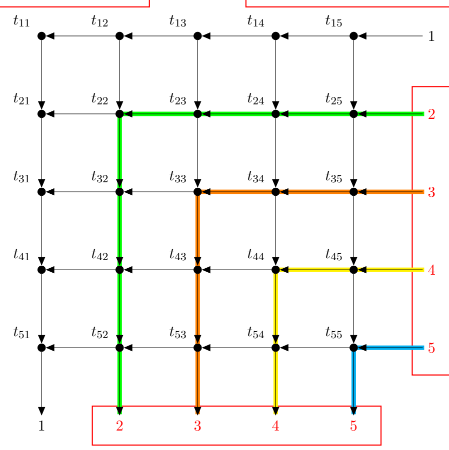
Thus our calculation comes to $\det_q(A_{22}) = \prod_{i=2}^{n-1} t_{i,i} + \sum_{j=2}^{n-1}(\prod_{i=2}^{j-1} t_{i,i} \cdot \prod_{i=j-1}^{n-1} t_{i,i+1}t_{i+1,i+1}^{-1}t_{i+1,i}) + \prod_{i=2}^{n-1} t_{i,i+1}t_{i+1,i+1}^{-1}t_{i+1,i}$

Note that there is only one vertex-disjoint path system when calculating $\det_q(A)$, the paths that turn only at $t_{i,i}$. We get $\det_q(A) = \prod_{i=1}^{n} t_{i,i}$.

Now we calculate the determinants of the $n-1 \times n-1$ minors. Being larger minor than even A_{22}, we have even fewer degrees of freedom for changing paths. In these first three cases, that is, for D, E, F, we in fact have only one vertex-disjoint path system for each:

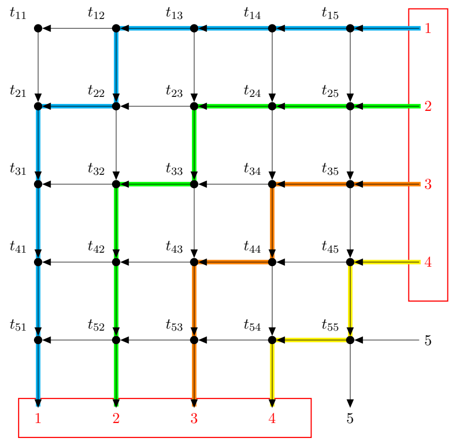






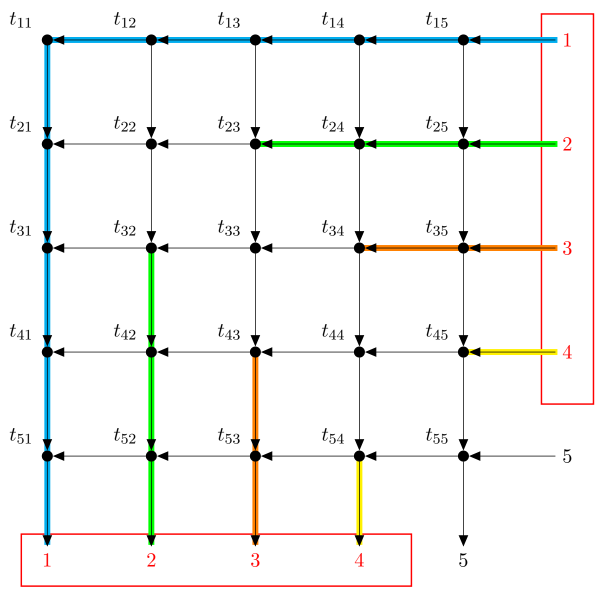
Then from the respective above illustrations, we calculate the determinants:
$\det_q(D) = \prod_{i=1}^{n-1} t_{i,i+1}$
$\det_q(E) = \prod_{i=1}^{n-1} t_{i+1,i}$
$\det_q(F) = \prod_{i=2}^{n} t_{i,i}$

Now for $\det_q(C)$, we again consider two cases, based on the path from 1 to 1. If the path has multiple turns, the path system must necessarily be this:



And the weight of this path system is $\prod_{i=1}^{n-1} t_{i,i+1}x_{i+1,i+1}^{-1}t_{i+1,i}$.

Now if the path from 1 to 1 turns only at t_{11}, notice that we have all the rest of the grid besides the vertices of the left and top for the remaining paths from $\{2, 3, \ldots, n-1\}$ to $\{2, 3, \ldots, n-1\}$. But notice, the paths could never have gone to the top vertices or the left, since the directions of the edges would not allow them to "get back." Thus we have the full grid available for all vertex-disjoint path systems from $\{2, 3, \ldots, n-1\}$ to $\{2, 3, \ldots, n-1\}$, which is exactly $\det_q(A_{22})$.



So the sum of these path models comes to $t_{11} \cdot \det_q(A_{22})$, and thus we have $\det_q(C) = t_{11}\det_q(A_{22}) + \prod_{i=1}^{n-1} t_{i,i+1}x_{i+1,i+1}^{-1}t_{i+1,i}$.

We have then

$$\det_q(A_{22}) \prod_{i=1}^{n} x_{i,i}$$

and

$$x_{11}\det_q(A_{22}) + \prod_{i=1}^{n-1} x_{i,i+1}x_{i+1,i+1}^{-1}x_{i+1,i} - \prod_{i=1}^{n-1} x_{i,i+1}\prod_{i=1}^{n-1} x_{i+1,i}$$

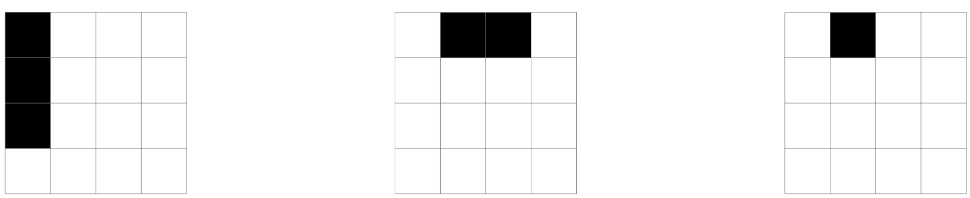
Careful algebraic manipulation yields the desired equality.

An Algebraic Approach

Every time the number of dimensions of the submatrix are reduced we have that the degrees of freedom our paths enjoy increases. This results in a combinatorial explosion of sorts in the expected number of path systems we must sum over. We can see

Finding a way to reduce the information, and for us, the number of path systems would be deterimental in making sense of the actual identity. One natural contender for reducing paths is to not allow turns on certain vertices, reducing the dimension of freedom each free column and row brings with it. We see below the expected simplification if we block 1, 2, 3, $n-2$, and $n-1$ vertices. Note that this blocking may result in a smaller number of paths in a system, or it may trivialize whole systems

At this point it is useful to define a systematic blocking (or coloring) scheme. We first associate to our matrix a natural closed diagram with $n \times n$ squares. In this case we let blocking of a vertex be denoted by coloring the associated square black. In addition we impose the seemingly artificial condition that is a square is colored black, we must color every square above or to the left of it black as well. In particular, we have defined a *Cauchon Diagram*.



The first two diagrams are examples of Cauchon Diagrams while the third is not.

To get a good understanding of the rigidity this additional coloring restriction results in consider a random coloring in the following fashion: with uniform probability color one square and with probability $\frac{1}{2}$ choose above or left of that square, and color all those squares. We have a significantly lower number of path systems path systems for subdiagrams of size $(n-2)\times(n-2)$. The drawback of this coloring and any coloring at all, is we do no longer get the full picture. Our determinant goes from a global statement on our submatrix over the entirety of the whole matrix to (possibly) a subset of our submatrix over (certainly) a subset of our whole matrix.

Two natural questions also arise:

- What advantage does a Cauchon coloring have over a random coloring?
- Does there exist any bridge from the local to global?

To answer the first question we address a technicality about the notion of prime ideal for non-communative rings. A prime ideal, P in a communative ring, P is a proper ideal such that if $ab \in P$ then either $a \in P$ or $b \in P$. Over a non-communative ring, a stronger definition is used. A proper ideal, p of a non-communative ring R is completely prime if, given two ideals A, B of R when $AB \subset P$ then either $A \subset P$ or $B \subset P$. In general every completely prime ideal is prime but the reverse may not be true however Goodearl proved in the quantized coordinate ring every prime ideal is completely prime.

For this reason we refer to the two definations interchancgably.

Now, back to question 1 we have that Cauchon Diagram's are so rich with structure they are in bijection with the set of automorphism invariant prime ideals of our coordinate quantum ring of $n \times n$ matrices. If we denote these automorphism invariant prime ideals as $\mathcal{H}$-primes, we see just how revealing this correspondence theorem is. That is right off the cuff we know there are only a finite number of such $\mathcal{H}$-primes, when the intersection of two $\mathcal{H}$-primes is $\mathcal{H}$-prime. Most importantly for us is that this theorem gives us an algebraic structure to piece together our local statements into a global one and attempt to use the correspondence to drawback our result.

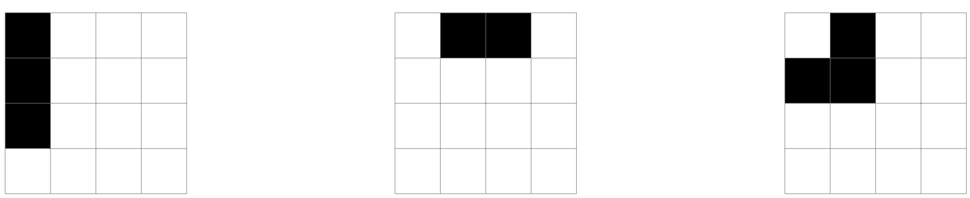
Our above discussion is summarized by the following theorems:

Theorem 2.1. *The set of $n \times n$ Cauchon diagrams is bijective to $\mathcal{H} - Spec(\mathcal{O}_q(M_{n\times n})$.*

Theorem 2.2. *Path model over cauchon diagram is equivlent to modding our ring by a prime ideal*

In this light, if we let $S = \det_q(U) - \det_q(A)\det_q(B)^{n-k-1}$ then our questions becomes: If we can show S is zero in a m diagrams can we reconstruct it and say S is zero always?

To try to gain some intuition, we take one of the most basic examples we can: a 4×4 matrix A, and choose a 2×2 submatrix B, say, the one made up of rows 1 and 3 and columns 3 and 4. Karel Casteels calculated S over a few Cauchon Diagrams and came to show, for this case 3 diagrams were sufficient to proving Sylvesters.



The question asked earlier becomes:

$$\text{If } S = 0 \text{ in } R/P_i \text{ for } i \in \{1, 2, ..., j\} \text{ is } S = 0 \text{ in } R?$$

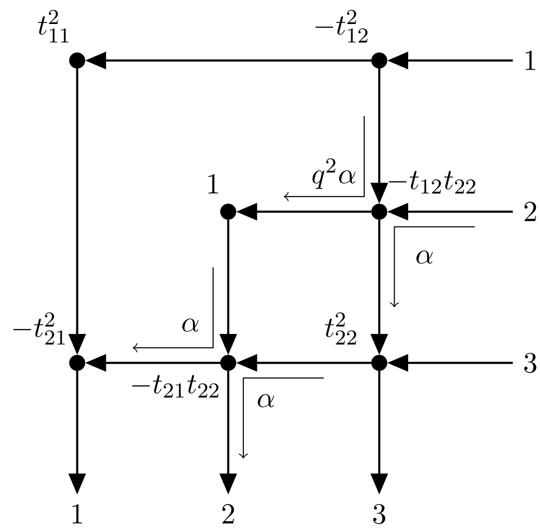
Unfortunately we could not solve this question.

2.1. **Moving forward.** We would like to be able to provide a similar proof for the more general quantum Sylvester's Identity; however, further attempts to do so required more complicated proofs than those already existing without the path diagrams. It was also unclear that these, in their complexity, would provide more information than the known proof does.

3. What else?

Special cases of quantum matrices have been defined, by adding more relations to the matrices. The quantum special orthogonal group, $\mathrm{SO}_q(n)$, is defined by relations on entries in a $n \times n$ martix of indeterminates. We calculated the relations for $n = 3, 4, 5$ by calculating an $n^2 \times n^2$ matrix of relations from an explicit formula given in Takeuchi's paper. We have been attempting to create path models for these.

The below image demonstrates such a path model for $\mathrm{SO}_q(3)$, where the non-edge arrows indicate that a turn in that direction multiplies the labeling constant to the path weight, where $\alpha = (q - q^{-1})$.



Now note that in this case we are not embedding $\mathrm{SO}_q(3)$ into the quantum torus; rather, it seems the natural algebra to embed it into is $\mathrm{SL}_{q^2}(2)$, which has the notable additional relation $t_{11} = t_{22}^{-1}$. This is because we have an isomorphism between a quotient of the quantized coordinate ring $\mathcal{O}(\mathrm{SL}_q(2))$ and $\mathcal{O}((SO)_q(3))$, given in Klimyk and Schmudgen.

Attempts for creating a path model for $\mathrm{SO}_q(4)$ have been able to satisfy some of the relations, but not all of them yet, and we have not yet tried with $\mathrm{SO}_q(5)$, though if we could, the hope for the future would be to generalize to any $\mathrm{SO}_q(n)$. Part of the goal in creating these diagrams would also be to find a less complicated algebra in which to embed each of these, as there is not yet given any isomorphism like the above for $\mathrm{SO}_q(3)$.

4. Future Work

While we only considered the Sylvester Determinant identity, we believe there is room in the field to generalize a number of classical linear algebraic identities. This idea is based off of the fact, that in addition to the identity we considered in this paper, a generalized

Cayley-Hamilton Theorem also exists. An interesting topic to consider would be trying to define an eigenvalue of a quantum matrix and seeing which spectral results survive q-communativity. This is interesting since Zhang provides 2 characteristic equations for our determinant.

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