COMBINATORIAL MODELS OF QUANTUM ALGEBRAS

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Abstract. A combinatorial model of the algebra of quantum matrices was established by Casteels in 2014. During our time in the REU program, we obtained two results by applying or modifying this model. The first application is to to the quantum Grassmannian $\mathcal{G}_q^{m,n}(\Bbbk)$, for which we are able to compute generating sets for $\mathcal{H}$-primes in the $m = 2$ case. The second application is to the algebra of quantum skew-symmetric matrices, for which we construct a new combinatorial model in the spirit of the algebra of quantum matrices.

Contents

1. Introduction

Quantizations of traditional structures like the general and special linear groups have been studied since the 1980's, but the more general algebra of quantum matrices $\mathcal{O}_q(\mathcal{M}_{m,n}(\Bbbk))$ became a central object of study much more recently. While $\mathcal{O}_q(\mathcal{M}_{m,n}(\Bbbk))$ has nice properties as far as noncommutative algebras go, such as being expressible as an iterated Ore extension, many fundamental questions about its structure require heavy machinery from noncommutative algebra and representation theory. The prime spectrum of $\mathcal{O}_q(\mathcal{M}_{m,n}(\Bbbk))$ remained a mystery, even with additional hypotheses about the base field $\Bbbk$ and the quantum parameter q. It became more tractable upon the development of the $\mathcal{H}$-stratification theory by Goodearl and Letzter (see [5]), but even then we did not have a complete understanding, lacking even generating sets for even the most important prime ideals.

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In [3] the author presents a model for the algebra of quantum matrices in terms of paths in a directed graph. The development of this model made it easier to tackle questions about quantum matrices by embedding $\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{k}))$ and some of its prime quotients in an algebra $\mathcal{S}_q^{m,n}$ that has much simpler structure in terms of commutation relations, among other nice properties. This model allowed for the solution of the problem of finding generating sets for the so-called $\mathcal{H}$-prime ideals. For this reason, studying combinatorial models of quantum algebras is a promising way to investigate these structures.

Section 3 of the paper applies the combinatorial model of $\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{k}))$ to one of its subalgebras $\mathcal{G}_q^{m,n}$, called the *quantum Grassmannian.* There is a general interest in studying the quantum Grassmannian because of its connections to questions about total non-negativity/total positivity, as developed by Postnikov in [11]. The prime spectrum of $\mathcal{G}_q^{m,n}$ has special significance because the $\mathcal{H}$-primes of $\mathcal{G}_q^{m,n}$ are in bijection with the cells of the totally non-negative Grassmannian, and so we strove to understand these primes. Some results about the $\mathcal{H}$-prime spectrum were obtained in [8], but obtaining generating sets for these primes remained an open question. Our work is based on the following conjecture.

Conjecture 1.1. An $\mathcal{H}$-prime P in $\mathcal{G}_q^{m,n}$ is generated by the maximal minors it contains.

Developing an outline for such a proof involves adapting Sagbi and Gröbner basis theory to ideals of subalgebras, work that was first done by Miller in [10]. Repeating Miller's analysis for our noncommutative setting requires understanding all of the relations in $\mathcal{G}_q^{m,n}$, and these are quite complicated in general. We were able make some progress, with the following result.

Theorem 1.2. *An $\mathcal{H}$-prime P in $\mathcal{G}_q^{2,n}$ is generated by the maximal minors it contains.*

We had success in the $m = 2$ case because we were able to understand these relations. Obtaining results using the same methods for higher m would require a more complete understanding of these relations.

In Section 4, we develop an analog of the path model for quantum matrices to the algebra of quantum skew-symmetric matrices. After going over some preliminary definitions and constructing the graph model, we present a non-inductive argument for an initial result that focuses on manipulations of paths in the graph. This is the style of proof used for proving the validity of the model in the case of quantum matrices, and we had hoped that this style of proof would be easily adapted to proving the validity of the skew-symmetric model, but this does not seem to be the case. We prove that the quantized coordinate ring of $n \times n$ skew-symmetric matrices is isomorphic to our graph model, and the proof of this is the main content of Sections 5. The proof performs induction on the size of the matrix of generators, and shows that the commutation relations are preserved in the paths model by enumerating all possible configurations of paths as they embed from the $(n-1)\times(n-1)$ graph into paths in the $n \times n$ graph. This first step is essential for making progress towards understanding the quantum skew-symmetric matrix algebra as a whole.

2. Preliminaries

We introduce several objects and definitions necessary to talk about the problems we investigated. Throughout the paper, we fix positive integers $m, n \geq 2$ with $m \leq n$, an infinite field $\mathbb{k}$, and $q \in \mathbb{k}^\times$ that is not a root of unity. However note that our work concerning quantum skew-symmetric matrices only requires $q \neq 0$. For positive integers $k < \ell$, let $[k, \ell] = \{k, k+1, \dots, \ell\}$. When $k = 1$, we simply write $[\ell] = [1, \ell]$. The set of all $m \times n$ matrices over $\mathbb{k}$ is denoted $\mathcal{M}_{m,n}(\mathbb{k})$.

2.1. Quantum Matrices.

Definition 2.1. Let $\mathcal{O}_q(\mathcal{M}_{m,n}(k))$ be the $\mathbb{k}$-algebra generated by an $m \times n$ matrix of indeterminates $X = [x_{i,j}]$ with the following relations: For any 2×2 submatrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

of X,

(1) $ab = qba, cd = qdc$,
(2) $ac = qca, bd = qdb$,
(3) $bc = cb$,
(4) $ad = da + (q - q^{-1})bc$.

We call this algebra the *quantized coordinate ring of $m \times n$ matrices*, or simply *$m \times n$ quantum matrices.*

Definition 2.2. Changing the fourth relation in the above definition to $ad = da$ defines the *quantum affine space* $T_q^{m,n}$, or T_q^n if $m = n$. We write the generators of this algebra as indeterminates $t_{i,j}$. The localization of $\mathcal{T}_q^{m,n}$ at the multiplicative set generated by the $t_{i,j}$ is called the $m \times n$ *quantum torus* $\mathcal{S}_q^{m,n}$.

Definition 2.3. For subsets $I = \{i_1 < \cdots < i_k\} \subset [m], J = \{j_1 < \cdots < j_k\} \subset [n]$ of the same size, the *quantum minor*, or simply *minor*, associated to I and J is the element of $\mathcal{O}_q(\mathcal{M}_{m,n}(\Bbbk))$ defined by

$$[I|J] = \sum_{\sigma \in S_k} (-q)^{\ell(\sigma)} x_{i_1,j_{\sigma(1)}} \cdots x_{i_k,j_{\sigma(k)}},$$

where S_k is the symmetric group on k elements and $\ell(\sigma)$ is the number of inversions in σ, that is, the number of pairs (i,j) with $i < j$ and $\sigma(i) > \sigma(j)$.

Definition 2.4. A *maximal* quantum minor is a minor of the form $[[m]|J]$. In this case, we simply denote this minor by $[J]$. The set of all index sets J of maximal minors $\mathcal{O}_q(\mathcal{M}_{m,n}(\Bbbk))$ is denoted $\Pi_{m,n}$.

2.2. **Quantum Matrices by Paths.** In [3], the author develops a combinatorial method to view quantum matrices. A particularly nice consequence is that quantum minors have a useful interpretation here.

Let $M \in \mathcal{M}_{m,n}(\Bbbk)$. Then we set

$$\mathbf{t}^M = t_{1,1}^{(M)_{1,1}} t_{1,2}^{(M)_{1,2}} \cdots t_{m,n}^{(M)_{m,n}},$$

where the indices are written from left to right in lexicographic order from smallest to largest. We call these the *lexicographic monomials* of $\mathcal{S}_q^{m,n}$. It is well-known that the set of lexicographic monomials forms a basis for $\mathcal{S}_q^{m,n}$ as a $\Bbbk$-vector space.

Notation 2.5. For $N \in \mathcal{M}_{m,n}(\Bbbk)$, set $M \prec N$ if $(M)_{i,j} < (N)_{i,j}$, where (i,j) is the smallest coordinate in which the entries of M and N differ. This is a total order on $\mathcal{M}_{m,n}(\Bbbk)$. We also set $\mathbf{t}^M \prec \mathbf{t}^N$ if $M \prec N$. Call $\prec$ the *lexicographic order*.

Since $q \neq 0$, in the expression of any $a \in \mathcal{S}_q^{m,n}$ as a linear combination of lexicographic monomials, there is a largest lexicographic monomial appearing with a non-zero coefficient. Define $\mathrm{in}(a)$ to be this monomial and $\ell c(a)$ to be the coefficient of $\mathrm{in}(a)$.

Note that if $a, b \in \mathcal{G}_q^{m,n}$, then $\mathrm{in}(ab) = q^c \mathrm{in}(a)\mathrm{in}(b)$ for some $c \in \mathbb{Z}$.

Definition 2.6. Let $H_{m,n}$ be the directed graph defined as follows. The vertex set of $H_{m,n}$ consists of *white* vertices labelled $[m] \times [n]$, *row* vertices labelled by $[m]$ and *column* vertices labelled by $[n]$. The potential ambiguity between row and column vertex labels will be resolved by always explicitly stating which type of vertex we refer to.

The set of directed edges is as follows.

(1) For each row vertex i, an edge directed from i to the white vertex (i,n).
(2) For each column vertex j, an edge directed from the white vertex (m,j) to j.
(3) For each white vertex (i,j) with $i \neq 1$, an edge directed from (i,j) to $(i-1,j)$.
(4) For each white vertex (i,j) with $j \neq m$, an edge directed from (i,j) to $(i,j+1)$.

There is a natural planar embedding of $H_{m,n}$. This is the obvious generalisation of the embedding of $H_{3,4}$ in Figure 2.1. We here always assume $H_{m,n}$ is equipped with this embedding and so can use common directional terms (horizontal, vertical etc.) without confusion.

Let P be a directed path starting at a row vertex i and ending at a column vertex j. This path will be uniquely identified with the sequence of white vertices in which P changes direction, i.e., where P changes from horizontal to vertical (a Γ-*turn*) or where P changes from vertical to horizontal (a $\lrcorner$-*turn*). So suppose $P = ((i_1,j_1),(i_2,j_2),\ldots,(i_{2k+1},j_{2k+1}))$, which we note tells us that (i_ℓ,j_ℓ) is a Γ-turn if ℓ is odd, and a $\lrcorner$-turn if ℓ is even. We assign to P the element $w(P) \in \mathcal{S}_q^{m,n}$ with

$$w(P) = t_{i_1,j_1} t_{i_2,j_2}^{-1} t_{i_3,j_3} \cdots t_{i_{2k},j_{2k}}^{-1} t_{i_{2k+1},j_{2k+1}}.$$

For a given $(i,j) \in [m] \times [n]$, let $\Gamma(i \mid j)$ be the set of all paths starting at row vertex i and ending at column vertex j.

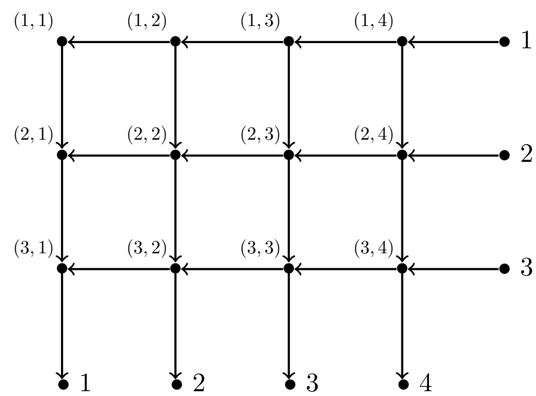

FIGURE 2.1. The graph $H^{3\times 4}$. The row vertices are those labelled $1, 2, 3$ on the right side of the graph, while the column vertices are those labelled $1, 2, 3, 4$ along the bottom of the graph.

Theorem 2.7 (Casteels [3])**.** *For $(i, j) \in [m] \times [n]$, let $\hat{x}_{i,j} \in \mathcal{S}_q^{m,n}$ be the element*

$$\hat{x}_{i,j} = \sum_{P \in \Gamma(i|j)} w(P).$$

Then the subalgebra of $\mathcal{S}_q^{m,n}$ generated by the $\hat{x}_{i,j}$ for all $(i, j) \in [m] \times [n]$ is isomorphic to $\mathcal{O}_q(\mathcal{M}_{m,n}(\Bbbk))$ under the map $x_{i,j} \mapsto \hat{x}_{i,j}$.

From now on, we lose the ˆ notation, and identify elements of $\mathcal{O}_q(\mathcal{M}_{m,n}(\Bbbk))$ with their image under the above map.

Quantum minors, considered as elements of $\mathcal{S}_q^{m,n}$ have the following interpretation. First, a *path system* is a tuple $\mathcal{P} = (P_1, \ldots, P_k)$ of paths in $H_{m,n}$, each starting at a row vertex and ending at a column vertex. The path system is *vertex-disjoint* if no two paths have a common vertex. The weight of $\mathcal{P}$ is $w(\mathcal{P}) = w(P_1)w(P_2)\cdots w(P_k)$.

Theorem 2.8 (Casteels [2])**.** *For subsets $I = \{i_1 < \cdots < i_k\} \subset [m], J = \{j_1 < \cdots < j_k\} \subset [n]$, we have*

$$[I|J] = \sum_{\mathcal{P}} w(\mathcal{P}),$$

where the sum is over all vertex-disjoint path systems $\mathcal{P} = (P_1, \ldots, P_k)$ where for each $s \in [k]$, P_s starts at i_s and ends at j_s.

2.3. **Skew-symmetric Quantum Matrices.**

Definition 2.9. Let $\mathcal{O}_q(Sk_n(\Bbbk))$ be the *the quantized coordinate ring of skew-symmetric matrices*. It is the $\Bbbk$-algebra generated by indeterminates $x_{i,j}$ for $1 \leq i \leq j < n$ with relations:

$$\begin{aligned}
x_{i,j}x_{i,l} &= qx_{i,l}x_{i,j} && \text{for ffr} i < j < l \\
x_{i,j}x_{j,l} &= qx_{j,l}x_{i,j} && \text{for } i < j < l \\
x_{i,j}x_{k,j} &= qx_{k,j}x_{i,j} && \text{for } i < k < j \\
x_{i,j}x_{k,l} &= qx_{k,l}x_{i,j} && \text{for } i < k < l < j \\
x_{i,j}x_{k,l} &= qx_{k,l}x_{i,j} + (q - q^{-1})x_{i,l}x_{k,j} && \text{for } i < k < j < l \\
x_{i,j}x_{k,l} &= qx_{k,l}x_{i,j} + (q - q^{-1})x_{i,k}x_{j,l} - q(q - q^{-1})x_{i,l}x_{j,k} && \text{for } i < j < k < l.
\end{aligned}$$

We also take $x_{i,i} = 0$ for all $i \in [n]$ and $x_{j,i} = -qx_{i,j}$ for $j > i$.

We move towards a combinatorial interpretation for this algebra in the following way. Let

$$U = \{(i, j) \in [n] \times [n] : i < j\},$$

and order the elements of U lexicographically. Furthermore, let $(i,j)^T = (j,i)$. Lastly, set

$$L = U^T = \{(i,j) \in [n] \times [n] : i > j\},$$

and

$$D = \{(i,i) \in [n] \times [n] : i \in [n]\}.$$

Ultimately, we will view $\mathcal{O}_q(Sk_n(\mathbb{k}))$ as a subalgebra of $\mathcal{S}_q^n$ that involves what we call the *skew Postnikov-* or skP_n-graph on $n^2 + n$ vertices. This graph consists of *white* vertices labeled by the set

$$W = \{([n] \times [n]) \setminus (i,i) | i \in [n]\},$$

as well as *row* and *column* vertices, both labeled by $R = [n]$ and $C = [n]$, respectively. The white vertices of the graph sit nicely on an $n \times n$ diagram with the diagonal removed. We put an additional column of vertices to the right of the diagram for the row vertices, and additional row below the diagram for the column vertices. An example of this is shown in Figure 2.2. Furthermore, we identify each white vertex using coordinates oriented as we would for a matrix. So the top row of white vertices would have coordinates $(1,2),(1,3),(1,4),(1,5),(1,6)$, and similarly for the other rows. Row and column vertices are identified as labeled in Figure 2.2.

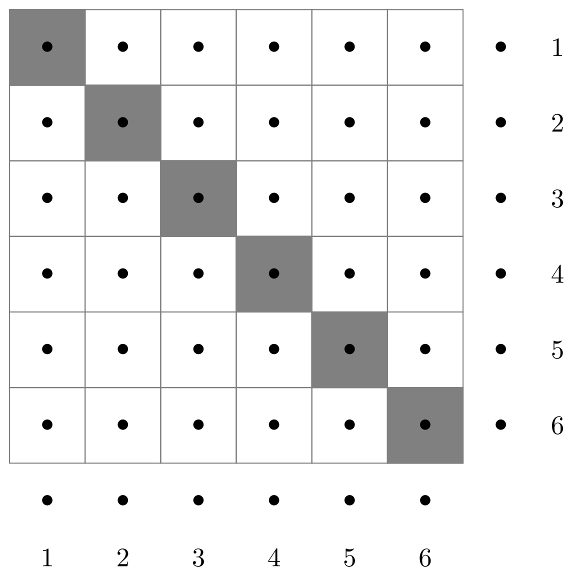

FIGURE 2.2. The orientation of the vertices for $n = 6$ on top of the 6×6 diagram.

Directed edges are drawn on the graph according to coordinates according to the following scheme:

(1) For each $i \in R$, draw an edge from i to (i,j) such that j is the largest integer with $(i,j) \in W$.
(2) For each $j \in C$, draw an edge to j from (i,j) such that i is the largest integer with $(i,j) \in W$.
(3) For each pair $(i,j),(i,j') \in W$ with $j > j'$ and such that there is no j'' with $j > j'' > j'$ and $(i,j'') \in W$, draw an edge from (i,j) to (i,j').
(4) For each pair $(i,j),(i',j) \in W$ with $i < i'$ and such that there is no i'' with $i' > i'' > i$ and $(i'',j) \in W$, draw and edge from (i,j) to (i',j).

Figure 2.3 shows the full graph construction for $n = 6$.

We can identify elements of $\mathcal{S}_q^n$ with paths in skP_n by designing a weighting scheme based on the path's expression as a series of edges.

Definition 2.10. For the set of edges E in an skP_n graph, define the edge-weighting function $w : E \to \mathcal{S}_q^n$ as follows:

(1) For an edge e starting at row vertex i going to the first white vertex (i,j) to its left,

$$w(e) = \begin{cases} t_{i,j} & \text{if } (i,j) \in U \\ -qt_{i,j} & \text{if } (i,j) \in L \end{cases}.$$

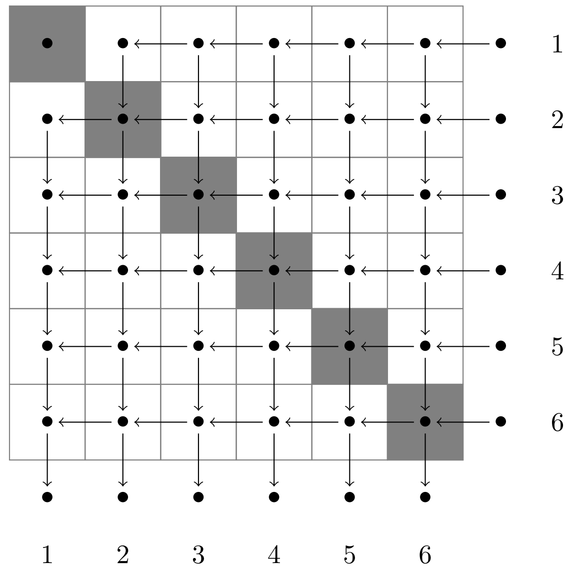

FIGURE 2.3. The full graph for $n = 6$ on top of the 6×6 diagram.

(2) For an edge e from white vertex (i, j) to white vertex (i, j') with $j > j'$,

$$w(e) = \begin{cases} t_{i,j}^{-1} t_{i,j'} & \text{if } (i,j), (i,j') \in U \\ -q t_{i,j}^{-1} t_{i,j'} & \text{if } (i,j) \in U \text{ and } (i,j') \in L \\ t_{j,i}^{-1} t_{j',i} & \text{if } (i,j), (i,j') \in L \end{cases}$$

(3) If e is a vertical edge, $w(e) = 1$.

For a positive integer t, consider a path $P = (v_1, \dots, v_t)$ where $v_s \in W$ for $s \in [t]$. The sequence P can be turned into a sequence of edges $(e_1, \dots, e_{t-1})$, and we define the weight of a path $w(P)$ to be the product of all of the edge weights $w(P) = w(e_1)w(e_2)\cdots w(e_{t-1})$.

For a path P that goes from a row vertex r to a column vertex c, there is a quicker way to compute the weight of P by identifying the sequence of turns in P. A turn in P can be identified as subsequence of consecutive vertices (v_{s-1}, v_s, v_{s+1}) such that if $v_{s-1}, v_s, v_{s+1} \in W$, the ordered pairs $(i_{s-1}, j_{s-1}), (i_{s+1}, j_{s+1})$ corresponding to the coordinates of v_{s-1} and v_{s+1} satisfy $i_{s-1} \neq i_{s+1}$ and $j_{s-1} \neq j_{s+1}$, if two of $\{v_{s-1}, v_s, v_{s+1}\}$ are in W, say v_s, v_{s+1}, then $|\{i_s, j_s\} \cap \{i_{s+1}, j_{s+1}\}| = 1$, or if only one of $\{v_{s-1}, v_s, v_{s+1}\}$ is in W.

We can further identify each of these turns as either Γ-turns or ⅃-turns. A Γ-turn corresponds to a turn (v_{s-1}, v_s, v_{s+1}) where (v_{s-1}, v_s) is a horizontal edge and (v_s, v_{s+1}) is a vertical edge. A ⅃-turn corresponds to a turn where (v_{s-1}, v_s) is a vertical edge, and (v_s, v_{s+1}) is a horizontal edge. We define the subsequence $r(P)$ of P to be the subsequence consisting of the middle vertices from the triples corresponding to turns. That is, the sequence of vertices where the path pivots. This subsequence has odd length, and by assigning weights to these vertices by $w_v : W \to T_n^\times$:

$$w_v(i,j) = \begin{cases} t_{i,j} & \text{if } (i,j) \in U \\ -t_{j,i} & \text{if } (i,j) \in L \end{cases},$$

then we can see that for a path P from row vertex r to column vertex c with subsequence of turns $r(P) = (v_{i_1}, \dots, v_{i_m})$,

$$w(P) = \begin{cases} w_v(v_{i_1}) w_v(v_{i_2})^{-1} \cdots w_v(v_{i_m}^T) & \text{if } P \text{ has a vertical cross-edge,} \\ q w_v(v_{i_1}) w_v(v_{i_2})^{-1} \cdots w_v(v_{i_m}^T) & \text{if } P \text{ has a horizontal cross-edge.} \end{cases}$$

Let

$$x_{ij} = \sum_{P: i \to j} w(P)$$

that is, the sum of weights of paths from i to j.

Definition 2.11. Let A_n be the $\mathbb{k}$-algebra generated by the elements x_{ij} with $i < j \leq n$.

3. The Quantum Grassmannian

3.1. **Basic definitions and combinatorial properties.** Recall that $\Pi_{m,n}$ denotes the set of maximal quantum minors of $\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{k}))$.

Definition 3.1. The subalgebra $\mathcal{G}_q^{m,n}(\mathbb{k})$ of $\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{k}))$ generated by $\Pi_{m,n}$ is referred to as the $m \times n$ *quantum Grassmannian.*

The generating set $\Pi_{m,n}$ of the quantum Grassmannian has a combinatorial structure that is essential to our work. To develop this structure, we introduce some orderings on $\Pi_{m,n}$.

Definition 3.2. For each $s \in [n]$, define the total order $<_s$ on $[n]$ by

$$s <_s s+1 <_s \cdots <_s n <_s 1 <_s 2 <_s \cdots <_s s-1.$$

Next, for $[J], [K] \in \Pi_{m,n}$ where $[J] = [j_1 <_s \cdots <_s j_m]$ and $[K] = [k_1 <_s \cdots <_s k_m]$, we define the partial order $\leq_s$ on $\Pi_{m,n}$ by

$$[J] \leq_s [K] \iff j_\ell \leq_s k_\ell, \forall \ell \in [m].$$

When dealing the order $<_1$, we will often suppress the subscript since it is the "standard" order.

Definition 3.3. For the partial order $<_s$ on $\Pi_{m,n}$, a monomial $[I_1][I_2]\cdots[I_k] \in \mathcal{G}_q^{m,n}(\mathbb{k})$ is called a *standard monomial* if $[I_i] \leq [I_{i+1}]$ for all $i \in [k-1]$.

The poset $(\Pi_{m,n}, \leq_s)$ is a distributive lattice with meet and join determined as follows:

(a) The join of $[J], [K] \in \Pi_{m,n}$ is given by

$$[J \vee K] = [\max(j_1, k_1) <_s \max(j_2, k_2) <_s \cdots <_s \max(j_m, k_m)]. \quad (1)$$

(b) The meet of $[J], [K] \in \Pi_{m,n}$ is given by

$$[J \wedge K] = [\min(j_1, k_1) <_s \min(j_2, k_2) <_s \cdots <_s \min(j_m, k_m)]. \quad (2)$$

Note that the maxima and minima above are taken with respect to $<_s$.

Example 3.4. We look at the case of $\mathcal{G}_{2,4}(\mathbb{k})$, presenting in Figure 3.1 the Hasse diagrams that arise from each of the different orderings.

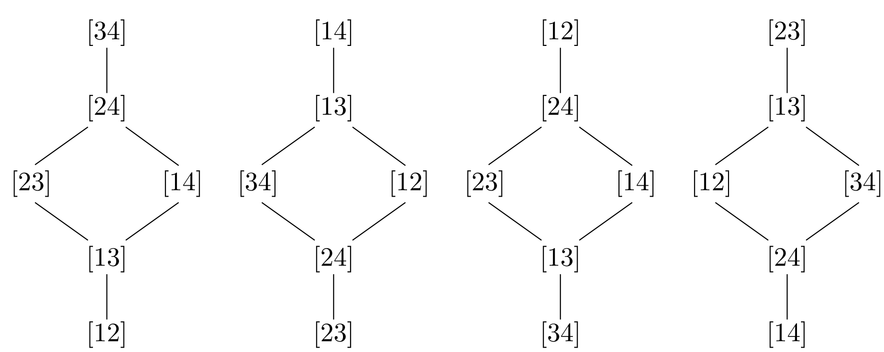

Figure 3.1. From left to right, we have $(\Pi_{2,4}, \leq_s)$ for $s = 1, 2, 3, 4$.

Definition 3.5. For an $\mathbb{N}$-graded $\mathbb{k}$-algebra $\mathcal{A}$ and Π a finite subset of $\mathcal{A}$ partially ordered by $<$, we say that $\mathcal{A}$ is a *quantum graded algebra with a straightening law* (QGASL) if the following hold:

1. The elements of Π are homogeneous of positive degree.
2. The set Π generates $\mathcal{A}$ as a $\mathbb{k}$-algebra.
3. The standard monomials in the elements of Π are linearly independent.
4. If α, β are incomparable with respect to $<$, then $\alpha\beta$ is a linear combination of $\lambda, \mu \in \Pi$ such that $\lambda < \mu$ and $\lambda < \alpha, \lambda < \beta$.

(5) For all $\alpha, \beta \in \Pi$, there exists $c_{\alpha\beta} \in \Bbbk^\times$ such that $\alpha\beta - c_{\alpha\beta}\beta\alpha$ is a linear combination of λ or $\lambda\mu$ for $\lambda, \mu \in \Pi$, $\lambda < \mu, \lambda < \alpha, \lambda < \beta$.

Theorem 3.6 (Lenagan and Russell [9])**.** *For each $s \in [n]$, $\mathcal{G}_q^{m,n}(\Bbbk)$ is a QGASL on $(\Pi_{m,n}, \leq_s)$.*

3.2. **Relations in $\mathcal{G}_q^{m,n}(\Bbbk)$.** In this section, we describe completely the relations in $\mathcal{G}_q^{m,n}(\Bbbk)$ that we need to deal with in order to perform computations with ideals.

There are two kinds of relations among the generators of $\mathcal{G}_q^{m,n}(\Bbbk)$ that we need to consider: those that can be deduced from the commutation rules in quantum matrices, and the quantum Plücker relations, which are q-analogues of the classical Plücker relations.

A general formula for the commutation relations between maximal minors, that is, for relations of the form (5) in Definition 3.5, can be found in [4]. We will here be concerned only with the case $m = 2$, and these are computed explicitely as follows.

Lemma 3.7. *The following relations hold in $\mathcal{G}_q^{2,n}(\Bbbk)$.*

$$\begin{aligned}
&[ab][cd] = q[cd][ab], \text{ if } |\{a,b\} \cap \{c,d\}| = 1 \text{ and } a < b \text{ or } c < d,\\
&[ab][cd] = q^2[cd][ab], \text{ if } a < b < c < d,\\
&[ab][cd] = [cd][ab], \text{ if } a < c < d < b,\\
&[ab][cd] = [cd][ab] + (q - q^{-1})[cb][ad], \text{ if } a < c < b < d.
\end{aligned}$$

The quantum Plücker relations are computed from the following formula derived from [6].

Proposition 3.8 (Quantum Plücker Coordinates (see [7])**.** *Given positive integers m, n, let $J_1, J_2, K \subset [n]$ be such that $|J_1|, |J_2| \leq m$ and $|K| = 2m - |J_1| - |J_2| > m$. Then*

$$\sum_{K' \sqcup K''} (-q)^{\ell(J_1;K')+\ell(K';K'')+\ell(K'',J_2)}[J_1 \sqcup K'][K'' \sqcup J_2] = 0,$$

where $\ell(I;J) = |\{(i,j) \in I \times J : i > j\}|$.

It is important to note that the above sum is defined only for certain partitions of K, specifically those for which $J_1 \cap K' = J_2 \cap K'' = \emptyset$. In practice, quantum Plücker relations are difficult to compute. This difficulty formed one barrier to extending our results from $\mathcal{G}_q^{2,n}(\Bbbk)$ to $\mathcal{G}_q^{m,n}(\Bbbk)$.

One can infer based on the formula from Proposition 3.8 that the quantum Plücker relations in the $m = 2$ case all involve 4 distinct indices, so there are in general $\binom{n}{4}$ relations obtained from this formula. The formula may yield duplicates based on the ordering of the indices involved, and so we now show that for each 4-subset of $[n]$, we have the following Plücker relation.

Lemma 3.9. *For $0 < a < b < c < d < n$,*

$$[ab][cd] - q[ac][bd] + q^2[ad][bc] = 0.$$

Proof. This is a direct computation that uses the quantum matrix relations and definition of quantum minor. In doing this we may replace a, b, c, d by $1, 2, 3, 4$ since doing computations in $\mathcal{G}_q^{2,n}(\Bbbk)$ with only a 4-subset of possible indices can be identified with performing the same computation in $\mathcal{G}_q^{2,4}(\Bbbk)$. For this reason, we refer to [8], where it is established that

$$[12][34] - q[13][24] + q^2[14][23] = 0$$

is a quantum Plücker relation in $\mathcal{G}_q^{2,4}(\Bbbk)$. The result follows.

□

3.3. **$\mathcal{H}$-primes.** The most productive way to study the prime spectrum of $\mathcal{G}_q^{m,n}(\Bbbk)$ has been to focus on primes that are invariant under the action of an algebraic torus. This is the "stratification theory" of Goodearl and Letzter [5]. In this paper we continue this method. Letting $\mathcal{H} = (\Bbbk^\times)^{m+n}$ denote an algebraic torus, the $(m+n)$-tuples of $\Bbbk^\times$, we obtain an automorphism of $\mathcal{O}_q(\mathcal{M}_{m,n}(\Bbbk))$ for each $h = (\rho_1, \dots, \rho_m, \gamma_1, \dots, \gamma_n)$ in $\mathcal{H}$ defined on the generators by

$$(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n) \cdot x_{i,j} = \alpha_i \beta_j x_{i,j}.$$

This action induces an action on $\mathcal{G}_q^{m,n}(\Bbbk)$ as a subalgebra of quantum matrices, except that the multiplication by the α_i coordinates becomes irrelevant. So for $\mathcal{G}_q^{m,n}(\Bbbk)$, the induced action happens only from $\mathcal{H} = (\Bbbk^\times)^n$ and looks like

$$(\beta_1, \dots, \beta_n) \cdot \gamma = \left(\prod_{j \in \gamma} \beta_j \right) \gamma,$$

for a maximal minor γ. It is well-understood that the set of prime ideals of $\mathcal{G}_q^{m,n}(\Bbbk)$, denoted $\mathrm{Spec}(\mathcal{G}_q^{m,n}(\Bbbk))$ can be partitioned into strata based on which primes are invariant under the automorphisms defined by $\mathcal{H}$, that is, those primes P for which $h \cdot P = P \ \forall h \in \mathcal{H}$. These are the so-called *$\mathcal{H}$-primes*.The set of all such primes is denoted by $\mathcal{H}\text{-}\mathrm{Spec}(\mathcal{G}_q^{m,n}(\Bbbk))$.

The $\mathcal{H}$-primes of $\mathcal{G}_q^{m,n}(\Bbbk)$ have a nice characterization through the different poset orders in the following way:

Remark 3.10 (A characterization of ideal membership for $\mathcal{H}$-primes)**.** For a fixed $\mathcal{H}$-prime P, there is a unique $[J_s]$ for each of the s-orderings on $\Pi_{m,n}$ such that $[J_s] \notin P$, but $[I] \in P$ for each $I \ngeq_s J_s$.

This characterization is equivalent to understanding that for each $\mathcal{H}$-prime there is a corresponding combinatorial object called a *Grassmann necklace*.

Definition 3.11. A Grassmann necklace I is an n-tuple of m-subsets (or *beads*) $I = (I_1, \dots, I_m)$ with the properties that for each $s \in [n]$ we have

(1) $I_{s+1} = I_s$ if $s \notin I_s$
(2) $I_{s+1} = (I_s \setminus \{s\}) \cup \{s'\}$ for any $s' \in [n]$ if $s \in I_s$.

This correspondence comes from Postnikov in [11]. To obtain an $\mathcal{H}$-prime based on a given necklace I, we look at each bead I_s as an element of the poset $(\Pi_{m,n}, \leq_s)$ and let all elements that are incomparable to $[I_s]$ become part of the generating set for the ideal. The connection to Remark 3.10 should be clear. These characterizations are essential in order to use the result of Lemma 3.9. We also have the following result based on the poset structure of $\Pi_{2,n}$ that will help us determine ideal membership.

Lemma 3.12 (Minor inclusions in $\mathcal{G}_q^{2,n}(\Bbbk)$)**.** *For $a, b, c, d \in [n]$ such that*

$$a <_1 c <_1 b <_1 d$$

and there exists $s \in [n]$ such that $[ab] \ngeq_s [i_1 i_2]$ or $[cd] \ngeq_s [i_1 i_2]$, where $i_1, i_2 \in [n]$ with $i_1 <_s i_2$, then

(1) $[ad] \ngeq_s [i_1 i_2]$ or $[bc] \ngeq_s [i_1 i_2]$,
(2) $[ac] \ngeq_s [i_1 i_2]$ or $[bd] \ngeq_s [i_1 i_2]$.

Proof. This lemma relies on what we can determine based on the possible positions of s relative to a, b, c, d in the 1-order. Note that we arrange the indices of the minors to reflect the changing s-order.

If $s <_1 a$ or $d \leq_1 s$, we have $a <_s c <_s b <_s d$. Then if $[ab] \ngeq_s [i_1 i_2]$ we have $a <_s i_1$ or $b <_s i_2$. For $a <_s i_1$ then $[ad] \ngeq_s [i_1 i_2]$ and $[ac] \ngeq_s [i_1 i_2]$ clearly. For $b <_s i_2$ then $[cb] \ngeq_s [i_1 i_2]$ clearly and $[ac] \ngeq_s [i_1 i_2]$ since $c <_s b <_s i_2$. If $[cd] \ngeq_s [i_1 i_2]$ then $c <_s i_1$ or $d <_s i_2$. For $c <_s i_1$, then $[ad] \ngeq_s [i_1 i_2]$ since $a <_s c$ and $[ac] \ngeq_s [i_1 i_2]$ since $i_1 <_s i_2$.

We suppress the remaining three cases because they work out nearly identically to this first case. □

This lemma gives us that if we have a relation between minors of the form $[ab][cd] - q^\circ [cd][ab] = q^\bullet [cb][ad]$, where $q^\circ, q^\bullet$ are integer powers of q, then having either $[ab]$ or $[cd]$ in P gives at least two additional minors contained in P. Before moving on, we mention that with respect to a fixed $\mathcal{H}$-prime ideal P, we refer to a *q-Hibi* expression

$$[I][J] - q^\bullet [I \wedge J][I \vee J]$$

as being of *type 1* if at least one of $\{[I], [J]\} \in \mathrm{Pl}(P)$ and being of *type 2* otherwise.

3.4. **A Sagbi basis for $\mathcal{G}_q^{m,n}(\Bbbk)$.** We want to show that the set of minors $[\gamma]$ for all $\gamma \in \Pi_{m,n}$ form a Sagbi basis for $\mathcal{G}_q^{m,n}(\Bbbk)$. That is, we want

$$\mathrm{in}(\mathcal{G}_q^{m,n}(\Bbbk)) = \Bbbk[\mathrm{in}(\gamma) : \gamma \in \Pi_{m,n}],$$

where in denotes the so-called *initial algebra* of the quantum Grassmannian. This is the algebra generated by the leading terms of all elements of the algebra with respect to the lexicographic order. The theory of generating sets in algebraic structures (Gröbner bases, Sagbi bases, etc.) allows us to work primarily with these initial algebras, which often have simpler computational structure because their inherited relations aren't as complicated as the ones from the original algebra.

Since we may view $\mathcal{G}_q^{m,n}(\Bbbk)$ as a subalgebra of the quantum matrix algebra, we may use the paths model of [3], which gives the monomials that we obtain a nice structure.

Proposition 3.13. *If* $\gamma = (j_1 < \cdots < j_m) \in \Pi_{m,n}$, *then*

$$in(\gamma) = t_{1,j_1} t_{2,j_2} \cdots t_{m,j_m},$$

which corresponds to the weight of the path system $\mathcal{P} = (P_1, \ldots, P_m)$ *where for each* $i \in [m]$ *we have that* P_i *represents a path* $P_i = (i, (i, j_i), j_i)$.

Proof. Suppose that $\mathcal{Q} = (Q_1, \ldots, Q_m)$ is a vertex disjoint path system corresponding to a minor γ, where for each $i \in [m]$ we have Q_i starting at row vertex i and ending at column vertex j_i. Since each white vertex in the set is used by at most one path in the system $\mathcal{Q}$, rearranging the weight of the system $w(\mathcal{Q})$ into a standard monomial $\mathbf{t}^{M_\mathcal{Q}}$ gives the expression $\mathbf{t}^{M_\mathcal{Q}} = q^r w(\mathcal{Q})$ for some integer q.

If $\mathcal{Q} \neq \mathcal{P}$, then $\mathcal{Q}$ contains a path with a ⅃-turn, let k denote the minimal index so that Q_k contains such a turn. Since k is minimal, the coordinates of $M_\mathcal{P}$ and $M_\mathcal{Q}$ that are lexicographically less than (k, j_k) are equal, but at the coordinate (k, j_k), $M_\mathcal{P}$ has a 1 where $M_\mathcal{Q}$ has a 0. Hence $t^{M_\mathcal{Q}} \prec t^{M_\mathcal{P}}$, where $\prec$ denotes the ordering described in Notation 2.5. □

This proposition gives us that the leading term of any minor is represented by the path system containing only single Γ-turns.

Definition 3.14. An element $g \in \mathcal{G}_q^{m,n}(\Bbbk)$ has a *Sagbi expression* if it can be written in the form

$$g = \sum_i c_i \gamma_1^{i_1} \gamma_2^{i_2} \cdots \gamma_{k_i}^{i_{k_i}}$$

for nonzero scalars c_i and where the product of $\gamma_i \in (\Pi_{m,n}, \leq_1)$ terms is a standard monomial in the quantum Grassmannian. Furthermore, the above expression must also satisfy

$$\mathrm{in}(g) = q^\bullet \max_i(\mathrm{in}(\gamma_1^{i_1})\mathrm{in}(\gamma_2^{i_2}) \cdots \mathrm{in}(\gamma_k^{i_{k_i}}))$$

where the maximum refers to the lex order on standard monomials of the quantum affine space $\mathcal{T}_q^{m,n}$ of the $t_{i,j}$.

For each $\gamma \in \Pi_{m,n}$, we know that $\mathrm{in}(\gamma)$ can be viewed as a monomial in $\mathcal{T}_q^{m,n}$. Thus we have that for any pair of minors $\gamma, \delta \in \Pi_{m,n}$ there is an integer $c(\gamma, \delta)$ such that

$$\mathrm{in}(\gamma)\mathrm{in}(\delta) = q^{c(\gamma,\delta)}\mathrm{in}(\delta)\mathrm{in}(\gamma),$$

and also there is an integer $h(\gamma, \delta)$ such that

$$\mathrm{in}(\gamma)\mathrm{in}(\delta) = q^{h(\gamma,\delta)}\mathrm{in}(\gamma \wedge \delta)\mathrm{in}(\gamma \vee \delta).$$

Proposition 3.15. *The algebra* $\Bbbk[in(\gamma) : \gamma \in \Pi_{m,n}]$ *is a QGASL on* $(\Pi_{m,n}, \leq_1)$.

Proof. We check that the set of standard monomials on $(\Pi_{m,n}, \leq_1)$ forms a linearly independent set, since the other properties of a QGASL clearly hold. Since we are considering the standard monomials as elements of $\mathcal{T}_q^{m,n}$ we know that they admit the following expression:

$$\mathrm{in}(\gamma_1) \cdots \mathrm{in}(\gamma_k) = q^r \mathbf{t}^{M_{\gamma_1} + \cdots + M_{\gamma_k}},$$

where r is an integer and $\mathrm{in}(\gamma_i) = \mathbf{t}^{M_{\gamma_i}}$. We would like to verify that if $\mathrm{in}(\delta_1) \cdots \mathrm{in}(\delta_\ell)$ is another standard monomial, then $M_{\gamma_1} + \cdots + M_{\gamma_k} \neq M_{\delta_1} + \cdots + M_{\delta_k}$.

To this end, we claim that $\gamma_1 \cup \cdots \cup \gamma_k \neq \delta_1 \cup \cdots \cup \delta_k$ are not equal as multisets. So suppose that they are equal, and let i be the least index such that $\gamma_i \neq \delta_i$. Then writing $\gamma_i = (j_1, \ldots, j_m)$ and $\delta_i = (j'_1, \ldots, j'_m)$, suppose that s is the least index with $j_s \neq j'_s$. Without loss of generality we have $j_r < j'_r$, and then since the multisets are equal we have that there exists $i' > i$ such that $j_r \in \delta_{i'}$. This then contradicts the assumption that the sequence of δ terms is a standard monomial, i.e. $\delta_i \leq \delta_{i'}$ for all $i' \geq i$. Thus the multisets $\gamma_1 \cup \cdots \cup \gamma_k, \delta_1 \cup \cdots \cup \delta_k$ are not equal, so $\mathbf{t}^{M_{\gamma_1}+\cdots+M_{\gamma_k}} \neq \mathbf{t}^{M_{\delta_1}+\cdots+M_{\delta_k}}$. It is well known that the set of standard monomials in $\mathcal{T}_q^{m,n}$ are linearly independent, and it then follows that the standard monomials of $\mathbb{k}[\mathrm{in}(\gamma) : \gamma \in \Pi_{m,n}]$ are also linearly independent. ☐

Remark 3.16. Somewhat strangely, the Proposition 3.15 does not hold in general for $\leq_s$ with $s \neq 1$, even though it is true for all orders in the non-initial algebras. This is easy to see even in the relatively small case $(\Pi_{2,4}, \leq_2)$. Indeed, attempting to follow the method of proof in [9] leads somewhat surprisingly to the conclusion that the noncommutative dehomogenisations of the initial algebras at consecutive quantum minors have *multiple* isomorphism classes. This shows us that even though the initial algebras allow us to more easily obtain computational results pertaining to the whole algebra, we sometimes lose relevant data.

Corollary 3.17. *Let $g \in \mathcal{G}_q^{m,n}(\mathbb{k})$. If*

$$g = \sum_i c_i \boldsymbol{\gamma}_i \in \mathcal{G}_q^{m,n}(\mathbb{k})$$

is the expression of g as a linear combination of standard monomials taken with respect to $\leq_1$, then this is a Sagbi expression for g.

Proof. If the above expression is not a Sagbi expression for g, then let A be the subset of indices for which $\mathrm{in}(\boldsymbol{\gamma}_i)$ is maximum. Then there exist integers d_i such that

$$\sum_{i \in A} c_i q^{d_i} \mathrm{in}(\boldsymbol{\gamma}_i) = 0,$$

which contradicts the linear independence of standard monomials established by the previous proposition. ☐

Corollary 3.18. *The set of elements $\gamma \in \Pi_{m,n}$ form a Sagbi basis for $\mathcal{G}_q^{m,n}(\mathbb{k})$.*

3.5. **Computations with $\mathcal{H}$-primes.** In this section we fix an $\mathcal{H}$-prime P and let $\mathrm{Pl}(P)$ denote the set of quantum minors contained in P. Our goal with this section of the project is to demonstrate $\mathrm{Pl}(P)$ generate P as a Sagbi-Gröbner basis. That is,

$$\mathrm{in}(\langle \mathrm{Pl}(P) \rangle) = \langle \mathrm{in}(\gamma) : \gamma \in P \rangle \subset \mathrm{in}(\mathcal{G}_q^{m,n}(\mathbb{k})).$$

We immediately have one direction of inclusion,

$$P \supseteq \mathrm{Pl}(P),$$

but in order to show that $\mathrm{Pl}(P)$ forms the desired generating set, we also need

$$P \subseteq \mathrm{Pl}(P).$$

Working towards this goal, we adapt the work of [10] to the noncommutative setting.

Proposition 3.19. *The set $Pl(P)$ is a Sagbi-Gröbner basis for $\langle Pl(P) \rangle$ if and only if every $g \in \langle Pl(P) \rangle$ has a Sagbi expression of the form*

$$g = \sum_{i=1}^{\ell} \gamma_i A_i$$

where each $\gamma_i \in Pl(P)$ and A_i is a standard monomial in $\mathcal{G}_q^{m,n}(\mathbb{k})$.

Proof. If $\mathrm{Pl}(P)$ is a Sagbi-Gröbner basis for $\langle \mathrm{Pl}(P) \rangle$ then for $g \in \langle \mathrm{Pl}(P) \rangle$ we can write $\mathrm{in}(g) = q^r \mathrm{in}(\gamma) a$ for some $r \in \mathbb{Z}, \gamma \in \mathrm{Pl}(P), a \in \mathrm{in}(\mathcal{G}_q^{m,n}(\mathbb{k}))$. Then choosing $A \in \mathcal{G}_q^{m,n}(\mathbb{k})$ so that $\mathrm{in}(A) = \mathrm{in}(a)$ and letting $g' = g - \mathrm{lc}(g) q^r \mathrm{in}(\gamma) A$, we see $g' \in \langle \mathrm{Pl}(P) \rangle$ and $\mathrm{in}(g') < \mathrm{in}(g)$, where $\mathrm{lc}(g)$ is the coefficient of $\mathrm{in}(g)$. Repeating this process gives the desired Sagbi expression for g.

Conversely, we certainly have $\mathrm{in}(\langle \mathrm{Pl}(P) \rangle) \supseteq \langle \mathrm{in}(\gamma) : \gamma \in \mathrm{Pl}(P) \rangle$. Then for $g \in \mathrm{in}(\langle \mathrm{Pl}(P) \rangle)$ having the desired Sagbi expression, we certainly have $\mathrm{in}(g) \in \langle \mathrm{in}(\gamma) : \gamma \in \mathrm{Pl}(P) \rangle$. ☐

We now present a result analogous to Buchberger's criterion for Gröbner bases.

Label the elements of $\mathrm{Pl}(P)$ by $\{\gamma_1,\dots,\gamma_\ell\}$ such that $\gamma_i <_1 \gamma_j$ for all $i,j\in[\ell]$ with $i<j$, and let $S=\mathrm{Syz}(\mathrm{in}(\gamma_1),\dots,\mathrm{in}(\gamma_\ell))$ be the right $\mathrm{in}(\mathcal{G}_q^{m,n}(\Bbbk))$-module consisting of $(a_1,\dots,a_\ell)\in(\mathrm{in}(\mathcal{G}_q^{m,n}(\Bbbk)))^\ell$ with

$$\sum_{i=1}^{\ell}\mathrm{in}(\gamma_i)a_i=0.$$

An element $(a_1,\dots,a_\ell)\in S$ is called a *homogeneous syzygy* if $\mathrm{in}(\gamma_i a_i)=\mathrm{in}(\gamma_j a_j)$ for all $i,j\in[\ell]$ with $a_i,a_j\neq 0$. In this case we refer to the common term $\mathrm{in}(\gamma a_i)$ as the *degree* of the syzygy.

Lemma 3.20. *Suppose that M is a finite generating set for S consisting of the homogeneous syzygies in which every coordinate is an element of $in(\mathcal{G}_q^{m,n}(\Bbbk))$, and for each $(a_1,\dots,a_\ell)\in M$, fix $(A_1,\dots,A_\ell)\in(\mathcal{G}_q^{m,n}(\Bbbk))^\ell$ such that $in(A_i)=in(a_i)$ for all i. If for all $(a_1,\dots,a_\ell)\in M$ there is a Sagbi expression of the form $\sum_{i=1}^{\ell}\gamma_iA_i$, then $Pl(P)$ is a Gröbner basis for $\langle Pl(P)\rangle$.*

Proof. From Proposition 3.19, we see that we need to find a suitable expression for $g\in\langle\mathrm{Pl}(P)\rangle$. So fix $g\in\langle\mathrm{Pl}(P)\rangle$. We have an initial expression for g of the form

$$g=\sum_{i=1}^{\ell}\gamma_iB_i,$$

be we do not know if $\mathrm{in}(g)=\max_i\{\mathrm{in}(\gamma_iB_i)\}$. If this is the case, then we would be done, so suppose that

$$\mathrm{in}(g)<\max_i\{\mathrm{in}(\gamma_iB_i)\}:=\boldsymbol{t}^{M_0}.$$

In this case, we show that there is another expression

$$g=\sum_{i=1}^{\ell}\gamma_iD_i,$$

where $\max_i\{\mathrm{in}(\gamma_iD_i)\}<t^{M_0}$. Repeating this argument finitely many times leaves us with the desired expression for g.

Set $b_i=\mathrm{lc}(B_i)\mathrm{in}(B_i)$ for all $i\in[\ell]$. Since $\mathrm{in}(g)<\mathbf{t}^{M_0}$, there is a maximal nonempty subset $S\subseteq[\ell]$ such that for all $i\in S$, $\mathrm{in}(\gamma_iB_i)=\mathbf{t}^{M_0}$, and

$$\sum_{i\in S}\mathrm{in}(\gamma_i)b_i=0.$$

Without loss of generality, $S=[s]$ for some $s\in[\ell]$ and so

$$(b_1,\dots,b_s,0,\dots,0)\in\mathrm{Syz}(\mathrm{in}(\gamma_1),\dots,\mathrm{in}(\gamma_\ell)).$$

Moreover, $\mathrm{in}(\gamma_iB_i)=\mathbf{t}^{M_0}$ for $i\in[s]$ and $\mathrm{in}(\gamma_iB_i)<\mathbf{t}^{M_0}$ for $i\in[\ell]\setminus[s]$. Thus $(b_1,\dots,b_s,0,\dots,0)$ is a homogeneous syzygy of degree $\mathbf{t}^{M_0}$.

We can choose k elements $(a_{1,j},a_{2,j},\dots,a_{\ell,j})\in M$ (not necessarily distinct) and monomials $c_j\in\mathrm{in}(\mathcal{G}_q^{2,n}(\Bbbk))$ such that

$$(b_1,\dots,b_s,0,\dots,0)=\sum_{j=1}^{k}(a_{1,j},a_{2,j},\dots,a_{\ell,j})c_j, \tag{3}$$

where $(a_{1,j}c_j,\dots,a_{\ell,j}c_j)$ is a homogeneous syzygy of degree $\mathbf{t}^{M_0}$ for each $j\in[k]$.

For every $i\in[\ell]$ and $j\in[k]$, choose $C_j,A_{i,j}\in\mathcal{G}_q^{m,n}(\Bbbk)$ so that $\mathrm{lc}(C_j)\mathrm{in}(C_j)=c_j$ and $\mathrm{lc}(A_{i,j})\mathrm{in}(A_{i,j})=a_{i,j}$. These choices imply the following. For $i\in[s]$, we have

$$\mathrm{in}\left(B_i-\sum_{j=1}^{k}A_{i,j}C_j\right)<\mathrm{in}(B_i).$$

Secondly, for $i\in[\ell]\setminus[s]$, we have that $\mathrm{in}(\gamma_iB_i)<\mathbf{t}^{M_0}$ and $\sum_{j=1}^{k}a_{i,j}c_j=0$. Hence for all $i\in[\ell,$

$$\mathrm{in}\left(\gamma_i\left(B_i-\sum_{j=1}^{k}A_{i,j}C_j\right)\right)<\mathbf{t}^{M_0}. \tag{4}$$

Next, since $(a_{1,j},\dots,a_{\ell,j}) \in M$, it follows that $h_j = \sum_{i=1}^{\ell} \gamma_i A_{i,j}$ is *not* a good expression for itself. By assumption however, there does exist good expression for itself, say $\sum_{i=1}^{\ell} \gamma_i \hat{A}_{i,j}$. Thus for all $j \in [k]$, we must have that $\max_i\{\mathrm{in}(\gamma_i \hat{A}_{i,j})\} < \max_i\{\mathrm{in}(\gamma_i A_{i,j})\}$. Hence,

$$\max_{i,j} \mathrm{in}(\gamma_i \hat{A}_{i,j} C_j) < \max_{i,j} \mathrm{in}(\gamma_i A_{i,j} C_j) = \mathbf{t}^{M_0}. \tag{5}$$

Finally,

$$\begin{aligned} g &= \sum_{i=1}^{\ell} \gamma_i B_i - \sum_{i=1}^{\ell}\sum_{j=1}^{k} \gamma_i A_{i,j} C_j + \sum_{j=1}^{k}\sum_{i=1}^{\ell} \gamma_i A_{i,j} C_j \\ &= \sum_{i=1}^{\ell} \gamma_i \left(B_i - \sum_{j=1}^{k} A_i^j C_j \right) + \sum_{j=1}^{k}\sum_{i=1}^{\ell} \gamma_i \hat{A}_{i,j} C_j. \end{aligned}$$

Because of the inequalities (4) and (5), we see that every coefficient D_i of γ_i in the above expression is such that $\mathrm{in}(\gamma_i D_i) < \mathbf{t}^{M_0}$. In particular, we have found the desired better expression for g.

□

In order to apply this lemma, we need a finite generating set M, and also need to determine what S is in our case for the quantum Grassmannian. To this end, define a surjective map

$$\pi : \Bbbk[y_{J_1},\dots,y_{J_{\binom{n}{2}}}] \to \mathrm{in}(\mathcal{G}_q^{2,n}(\Bbbk))$$

where the y_{J_i} indeterminates correspond to the $[J_i]$ minors in the standard partial ordering induced by the 1-order. This is not a commutative polynomial ring, as the relations between the y_{J_i} are induced by the relations on the generators of $\mathrm{in}(\mathcal{G}_q^{2,n})$. We still want to apply Gröbner basis theory to this algebra, and luckily the noncommutative versions of S-polynomials and Buchberger's criterion are almost identical to the commutative versions. The details of these are found in [1], as Definition 6.1 and Theorem 6.5, respectively. The map π acts as follows on the generators of $\Bbbk[y_{J_1},\dots,y_{J_{\binom{n}{2}}}]$:

$$\pi : y_J \mapsto \mathrm{in}[J].$$

We now present a result that follows directly from Proposition 4.10 of [10].

Proposition 3.21. *Let $G = \{\gamma_1,\dots,\gamma_\ell\}$ be any subset of $\Pi_{m,n}$. Let*

$$\{\mathbf{P_i} = (P_{i,1}, P_{i,2},\dots,P_{i,\ell}) : i = 1,\dots,K\}$$

be a generating set for Syz$(y_{\gamma_1},\dots,y_{\gamma_\ell})$. Let

$$\{\sum_{i=1}^{\ell} \gamma_j P_{i,j} : i = K+1,\dots,M\}$$

be a generating set for $\ker(\pi) \cap \langle G\rangle$ *and set*

$$\mathbf{P_i} = (P_{i,1}, P_{i,2},\dots,P_{i,\ell})$$

for each $i = K+1,\dots,M$. Then $\{\pi(\mathbf{P_i}) : i = 1,\dots,M)\}$ generates Syz$(in(\gamma_1),\dots,in(\gamma_\ell))$ where $\pi(\mathbf{P_i}) = (\pi(P_{i,1}),\dots,\pi(P_{i,\ell}))$.

With this map and proposition in hand, we would like to compute a Gröbner basis for $\ker(\pi)\cap\langle G\rangle$, in the special case of $m = 2$ and where G denotes the set of y_J such that $[J] \in \mathrm{Pl}(P)$. Note the use of Gröbner basis here as opposed to Sagbi-Gröbner basis. Indeed, the theory of Gröbner bases applies to this noncommutative setting nicely, although it will not later. The details regarding S-polynomials and Buchberger's criterion adapted to the noncommutative setting can be found in [1].

The ideal $\ker(\pi)$ is generated by all of the polynomials in the y_I indeterminates obtained from the q-Hibi relations in the target algebra. So a type 1 q-Hibi relation $[I][J] - q^{\bullet}[I \wedge J][I \vee J]$ in $\mathrm{in}(\mathcal{G}_q^{2,n})$ corresponds to the ring element $y_I y_J - q^{\bullet} q_{I\wedge J} q_{I\vee J}$. We refer to these polynomials as "Hibi-like relations". We hope that the generating set obtained for this intersection then reduces to 0 under division by the set of minors contained in P.

We want our Gröbner basis to consist of the type 1 Hibi-like relations, along with elements of the form γy_J, where γ is a type 2 Hibi-like relation, and y_J corresponds to a $[J] \in \mathrm{Pl}(P)$. So we follow the standard process for computing a generating set for the intersection of ideals, and then add in these additional basis elements. That is: we look at the generators for

$$t\ker(\pi) + (1-t)\langle G\rangle,$$

where t is a new indeterminate that commutes with all of the y_I, and add to this set all of the type 1 Hibi-like relations (without the factor of t) and the products of type 2 relations with indeterminates corresponding to minors contained in the prime. This set still generates the intersection ideal. To this larger set we apply Buchberger's algorithm with respect to the standard elimination ordering, and then take the output set's intersection with the algebra of the indeterminates y_I. We then show that the resulting set reduces to 0 under division by the set of y_J corresponding to $[J] \in \mathrm{Pl}(P)$, which will give the desired result.

So let

$$\begin{aligned} M = \{&\{t(y_Ay_B - q^{\bullet}y_{A\wedge B}y_{A\vee B}) : [A][B] - q^{\bullet}[A\wedge B][A\vee B] \text{ is a } q\text{-Hibi relation}\}\cup \\ &\{(1-t)y_J : J \text{ corresponds to a } [J] \in\mathrm{Pl}(P)\}\cup \\ &\{y_Cy_D - q^{\bullet}y_{C\wedge D}y_{C\vee D} : [C][D] - q^{\bullet}[C\wedge D][C\vee D] \text{ is a type 1 } q\text{-Hibi relation}\}\cup \\ &\{(y_Ey_F - q^{\bullet}y_{E\wedge F}y_{E\vee F})y_K : \\ &y_K \in \mathrm{Pl}(P), [E][F] - q^{\bullet}[E\wedge F][E\vee F] \text{ is a type 2 } q\text{-Hibi relation}\}\}. \end{aligned}$$

Applying Buchberger's algorithm to this set means computing $\binom{4}{2} + 4 = 10$ types of S-polynomial. Once each of these is written out, it is clear that they are all zero since we are considering the whole ideal $\langle M\rangle$, which means that we can replace any leading term of a member of the ideal with the lower elements. This leads to very simple manipulations since $\langle M\rangle$ is a binomial ideal. We thus have that

$$\begin{aligned} M \cap \mathbb{k}[y_{J_1}, \ldots, y_{J_{\binom{n}{2}}}] =& \{y_Cy_D - q^{\bullet}y_{C\wedge D}y_{C\vee D} : \\ & [C][D] - q^{\bullet}[C\wedge D][C\vee D] \text{ is a type 1 } q\text{-Hibi relation}\}\cup \\ & \{(y_Ey_F - q^{\bullet}y_{E\wedge F}y_{E\vee F})y_K : \\ & y_K \in \mathrm{Pl}(P), [E][F] - q^{\bullet}[E\wedge F][E\vee F] \text{ is a type 2 } q\text{-Hibi relation}\}\}, \end{aligned}$$

which shows that we have the desired Gröbner basis for $\ker(\pi) \cap \langle G\rangle$.

By the proposition, the existence of this Gröbner basis gives a generating set for the syzygy S corresponding to relations between the maximal minors of P. These relations form the finite generating set M of Lemma 3.20 with most of the desired properties. It remains though to check whether or not the Sagbi-basis property is satisfied by the set of relations, and in order to do that we need to verfiy that the set goes to zero under division by $\mathrm{Pl}(P)$.

Lemma 3.22. *Given the set*

$$H_0 = \{[I][J] - q^{\bullet}[J][I] : [I], [J] \in Pl(P)\}$$

of commutation relations for the minors contained in $Pl(P)$, along with the sets

$$H_1 = \{[I][J] - q[I\wedge J][I\vee J] : \textit{one of } [I][J] - q[I\wedge J][I\vee J] \textit{ is a type-1 q-Hibi relation}\},$$

$$H_2 = \{([I][J] - q[I\wedge J][I\vee J])[K] : [K] \in Pl(P), [I][J] - q[I\wedge J][I\vee J] \textit{ is a type 2 q-Hibi relation.}\},$$

the set $H = H_0 \cup H_1 \cup H_2$ reduces to 0 under division by $Pl(P)$.

Proof. We show that the claim holds by proving it for an element from each of H_0, H_1, H_2. For H_0, the only non-trivial case is when we have minors that are related by a $\hat{q}$ term, and in this case we have

$$[ab][cd] - [cd][ab] = \hat{q}[cb][ad],$$

where $a < c < b < d$. We then want at least one of $[cb], [ad] \in \mathrm{Pl}(P)$, which we have by Lemma 3.12. Then from Definition 3.7 we deduce that $[cb][ad] = [ad][bc]$, so $\hat{q}[cb][ad]$ is a $\mathcal{G}_q^{2,n}(\mathbb{k})$ multiple of a minor in $\mathrm{Pl}(P)$.

For H_1, given a type 1 q-Hibi relation $[ab][cd] - q[ac][bd]$, with $a < b < c < d$, we see that by Lemma 3.9, we have

$$[ab][cd] - q[ac][bd] = -q^2[ad][bc].$$

Then see that we may apply Lemma 3.12 again after relabeling slightly, and obtain that one of $[ad], [bc] \in \mathrm{Pl}(P)$, and as in the previous case, see that these commute, so that we again have a $\mathcal{G}_q^{2,n}(\Bbbk)$ multiple of a minor in $\mathrm{Pl}(P)$.

Lastly, for H_2, given a type 2 q-Hibi relation multiplied by something in $\mathrm{Pl}(P)$: $([ab][cd] - q[ac][bd])[K]$, with $a < b < c < d$ and $[K] \in \mathrm{Pl}(P)$, we apply Lemma 3.9 to rewrite this to $-q^2[ad][bc][K]$ and are done. □

Theorem 3.23. *The set $Pl(P)$ of minors contained in an $\mathcal{H}$-prime P of $\mathcal{G}_q^{2,n}(\Bbbk)$ forms a Gröbner basis for $\langle Pl(P)\rangle$.*

Proof. This follows from Lemmas 3.20, 3.22, Proposition 3.21, and the preceding computation of a generating set for $\ker(\pi) \cap \langle G\rangle$. □

Remark 3.24. The shift from the abstract $[I], [J]$ notation to specific indices a, b, c, d in the above proof is possible because of how much we know about the relations between the specific minors. The statements about specific orderings of the indices arise from the structure of the poset, this is Lemma 3.9. This is what makes the $m = 2$ case relatively straightforward: the relations between minors become much more complicated even for $m = 3$.

Now we return to the original problem of finding a generating set for P, in the case $m = 2$. We introduce the notation $a =_q b$ if there exists an integer α such that $a = q^\alpha b$. For integers $a \le b$, we write $[a, b] = \{a, a+1, \ldots, b\}$.

Theorem 3.25. *For an $\mathcal{H}$-prime P of $\mathcal{G}_q^{2,n}$, let $Pl(P)$ be the set of maximal minors contained in P. Then*

$$P = \langle Pl(P)\rangle.$$

Proof. Suppose that P has associated Grassmann necklace $(I_1, \ldots, I_n)$. It follows from the definition of Grassmann necklace $i \in I_1$ and $i \ge s$, then $i \in I_2, I_3, \ldots, I_s$. Similarly, if $i \in I_s$ and $i < s$, then $i \in I_{s+1}, \ldots, I_n, I_1$.

Suppose that $\langle \mathrm{Pl}(P)\rangle \subsetneq P$ and take an $a \in P \setminus \langle \mathrm{Pl}(P)\rangle$ with the property that $\mathrm{in}(a)$ is minimal amongst all elements of $P \setminus \langle \mathrm{Pl}(P)\rangle$ with respect to the lex order. From the results of [8], there is a nonnegative integer k so that $a[I_1]^k \in \langle \mathrm{Pl}(P)\rangle$. Furthermore, since Corollary 3.18 gives us a Sagbi basis for $\mathcal{G}_q^{m,n}(\Bbbk)$, we may write $\mathrm{in}(a) = \mathrm{in}([J_1]) \cdots \mathrm{in}([J_r])$, where $J_1, \ldots, J_r \in \Pi_{2,n}$. So then

$$\mathrm{in}(a[I_1]^k) = \mathrm{in}([J_1]) \cdots \mathrm{in}([J_r])\mathrm{in}([I_1])^k.$$

By Theorem 3.23, we can then write

$$\mathrm{in}([J_1]) \cdots \mathrm{in}([J_r])\mathrm{in}([I_1])^k =_q \mathrm{in}([L_1]) \cdots \mathrm{in}([L_{r+k}]),$$

where $[L_1] \in \langle \mathrm{Pl}(P)\rangle$. In order to proceed, we would like to know that we are able to manipulate these $[L_i]$ indeterminates to recover k factors of $[I_1]$.

So let us write

$$\mathrm{in}\left([J_1]\right) \cdots \mathrm{in}\left([J_r]\right) \mathrm{in}\left([I_1]\right)^k =_q \mathrm{in}\left([L_1]\right) \cdots \mathrm{in}([L_{r+k}]), \tag{6}$$

where, without loss of generality, $J_p \ge_1 I_1$ for all $p \in [1, n]$ and $L_1 \in \langle \mathrm{Pl}(P)\rangle$. Again, our goal is to show that the Product (6) either contains at least k $\mathrm{in}([I_1])$'s or can be transformed into such a product with at least one factor in $\langle \mathrm{Pl}(P)\rangle$.

Note that if any $J_p \in \langle \mathrm{Pl}(P)\rangle$, then our desired result is trivially achieved, so we may as well further assume that $J_p \notin \langle \mathrm{Pl}(P)\rangle$, i.e., $J_p \ge_s I_s$ for all $s \in [1, n]$.

We now make some notational conventions. For all $p \in [1, r]$, we always write $[J_p] = [j_1^{(p)} j_2^{(p)}]$ where $j_1^{(p)} <_1 j_2^{(p)}$. Similarly, when we write $L_i = [\ell_1 \ell_2]$, we tacitly take $\ell_1 <_1 \ell_2$. However, for $s \in [1, n]$ we write $I_s = [i_1^{(s)} i_2^{(s)}]$ where now $i_1^{(s)} <_s i_2^{(s)}$. Now, there are three cases to consider.

Case 1: $L_1 = [j_1^{(p)} j_2^{(q)}]$ **for some** $p, q \in [1, r]$**.** If at least k of the L_a's in Product (6) are equal to I_1, then there is nothing to do. Otherwise, there is an L_i with $L_i = [i_1^{(1)} j_2^{(m)}]$ for some m, and so there must also be an $L_{m'} = [j_1^{(m')} i_2^{(1)}]$. Now, $j_2^{(m)} \ge_1 i_2^{(m)}$ but $j_1^{(m')} <_1 i_2^{(1)}$ in order for $L_{m'}$ to even be defined. Now replace $\mathrm{in}([L_m])\mathrm{in}([L_{m'}])$ in Product (6) with $\mathrm{in}([j_1^{(m')} j_2^{(m)}])\mathrm{in}([I_1])$, thus increasing the number of $\mathrm{in}([I_1])$'s while keeping in($[L_1]$). Repeat this process until we obtain k in($[I_1]$)'s.

Case 2: $L_1 = [i_1^{(1)} j_2^{(p)}]$ **for some** $p \in [1, r]$**.** We can and do take an $s \in [1, n]$ with the property that $L_1 \not\geq_s I_s$.

First, suppose that $i_1^{(1)} <_s j_2^{(p)}$. Then either $i_1^{(1)} <_s i_1^{(2)}$ or $j_2^{(p)} <_s i_2^{(s)}$. But the former case implies the falsity $I_1 \not\geq_s I_s$ (regardless of the s-order of $i_1^{(1)}$ and $i_2^{(2)}$). On the other hand, if $i_1^{(1)} \geq_s i_2^{(s)}$ and $j_2^{(p)} <_s i_2^{(s)}$, then

$$j_2^{(p)} <_s i_2^{(s)} \leq_s i_1^{(1)} <_s j_2^{(p)},$$

a contradiction.

Now suppose $i_1^{(1)} >_s j_2^{(p)}$. Since $i_1^{(1)} <_1 j_2^{(p)}$, we have in fact

$$i_1^{(1)} <_1 s \leq_1 j_2^{(p)}.$$

Now either $j_2^{(p)} <_s i_1^{(s)}$ or $i_1^{(1)} <_s i_2^{(s)}$. The former case implies that $J_p \not\geq_s I_s$, a contradiction. So we must have $j_2^{(p)} \geq_s i_1^{(s)}$ and $i_1^{(1)} <_s i_2^{(s)}$. The latter condition implies that $i_1^{(1)} <_s i_1^{(2)}$ as otherwise $I_1 \not\geq_s I_s$. Since also $i_1^{(1)} <_1 i_1^{(2)}$, we must have $\{i_1^{(1)} i_2^{(2)}\}$ is contained in either $[1, s-1]$ or $[s, n]$. Since we already know that $i_1^{(1)} <_1 s$, it must be the former.

So $i_1^{(1)}, i_2^{(1)} \in [1, s-1]$. Since $i_1^{(1)} <_s i_2^{(s)}$, we have $i_2^{(s)} \in [1, s-1]$. As noted above, This in turn implies $i_2^{(s)} \in I_1$. As $i_1^{(1)} <_s i_2^{(s)}$, we can only have $i_2^{(s)} = i_2^{(1)}$.

Now since $L_1 = [i_1^{(1)} j_2^{(p)}]$, there must be some a, r such that $L_a = [j_1^{(r)} i_1^{(2)}]$. So $i_1^{(1)} \leq_1 j_1^{(r)} <_1 i_1^{(2)}$. Summarizing the above we have the chain of inequalities

$$s \leq_s i_1^{(s)} \leq_s j_2^{(p)} \leq_s n <_s 1 \leq_s i_1^{(1)} \leq_s j_1^{(r)} <_s i_2^{(1)} = i_2^{(s)} <_s j_1^{(p)}.$$

Thus we see that $[j_1^{(r)} j_2^{(p)}] \not\geq_s I_s$. So in Expression 6, we replace in($[L_1]$)in($[L_a]$) with in($[j_1^{(r)} j_2^{(p)}]$)in($[I_1]$), bringing us back to Case 1.

Case 3: $L_1 = [j_1^{(p)} i_2^{(1)}]$ **for some** $p \in [1, r]$**.** We show that this can never occur under the above assumptions. Let s be such that $L_1 \not\geq_s I_s$.

First suppose $j_1^{(p)} <_s i_2^{(1)}$. Then either $j_1^{(p)} <_s i_1^{(s)}$ or $i_2^{(1)} <_s i_2^{(s)}$. The former case implies the falsity $J_p \not\geq_s I_s$ so we may assume that $j_1^{(p)} \geq_s i_1^{(s)}$ and $i_2^{(1)} <_s i_2^{(s)}$.

Since $I_1 \geq_s I_s$, we also have $i_2^{(1)} <_s i_1^{(1)}$. Thus $i_1^{(1)} <_1 s \leq_1 i_2^{(1)}$. From $s \leq_1 i_2^{(1)}$, it follows that $i_2^{(1)} \in I_s$. As $i_2^{(1)} <_s i_2^{(s)}$, we in fact have $i_2^{(1)} = i_1^{(s)}$. But we now have deduced the absurd chain of inequalities

$$j_1^{(p)} <_s i_2^{(1)} = i_1^{(s)} \leq_s j_1^{(p)}.$$

Finally, suppose $i_2^{(1)} <_s j_1^{(p)}$. Since $j_1^{(p)} <_1 i_2^{(1)}$ we must have

$$j_1^{(p)} <_1 s \leq_1 i_2^{(1)}.$$

Now, as before we may assume that $i_2^{(1)} \geq_s i_1^{(s)}$ and $j_1^{(p)} <_s i_2^{(s)}$. But now we have the following chain of inequalities

$$s \leq_s i_1^{(s)} \leq_s i_2^{(1)} \leq_s n <_s 1 \leq_s i_1^{(1)} \leq_s j_1^{(p)} <_s i_2^{(s)}.$$

But now we see that $I_1 \not\geq_s I_s$, a contradiction.

So we may write

$$\begin{aligned} \text{in}(a[I_1]^k) &=_q \text{in}([J_1]) \cdots \text{in}([J_r])\text{in}([I_1])^k \\ &=_q \text{in}([L_1]) \cdots \text{in}([L_r])\text{in}([I_1])^k \end{aligned}$$

with $L_1 \in \mathrm{Pl}(P)$. Then there exists $\alpha \in \Bbbk$ so that the element

$$a - \alpha \mathrm{in}([L_1]) \cdots \mathrm{in}([L_r]) \in \mathrm{Pl}(P).$$

This element will have leading term strictly smaller than a, and since $\mathrm{in}([L_1]) \cdots \mathrm{in}([L_r]) \in \langle \mathrm{Pl}(P) \rangle$, we conclude $a \in \mathrm{Pl}(P)$, which contradicts our choice of $a \notin \langle \mathrm{Pl}(P) \rangle$. Thus we have $P \subseteq \langle \mathrm{Pl}(P) \rangle$. □

4. Skew-symmetric Matrices

We walk through various results we collected from the different problems we explored during the program.

4.1. **Verification of $x_{i,i} = 0$ via paths.** For $i \in [n]$, let $\Gamma_D(i,j)$ be the set of paths from i to j relative to diagram D. For a path P let $w(P)$ denote the weight of P, and let P^T denote the transpose of P. Furthermore, let P_U be the set of vertices of P contained in U, and respectively P_L be the set of vertices of P contained in L.

Given a path $P \in \Gamma_D(i,j)$, let v be the last vertex used by P in $P_U \cap P_L^T$, and let v^T be the first vertex used by P in $P_L \cap P_U^T$. Now we may define

$$P_1 : i \to v, \qquad P_2 : v \to v^T, \qquad P_3 : v^T \to i$$

and write $P = P_1 \circ P_2 \circ P_3$. We may then set

$$\tau(P) = P_1 \circ P_2^T \circ P_3.$$

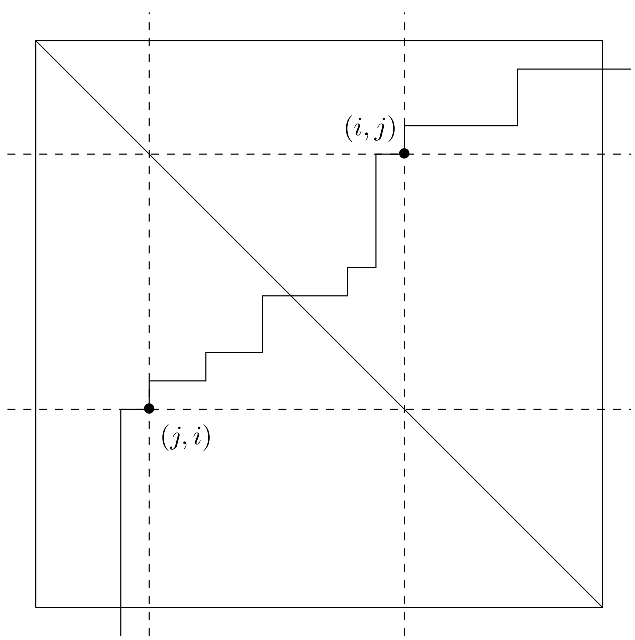

Figure 4.1. A diagram demonstrating the parallel lines argument in the proof of Proposition 4.1.

Proposition 4.1. *Given $P \in \Gamma_D(r,d)$ such that P is not transpose-disjoint, we have $w(P) + w(\tau(P)) = 0$.*

Proof. First note that the cross-edge of P will necessarily be used by P_2, and furthermore that $P_2 \cap P_2^T = \{v, v^T\}$. From now on we set $v = (i,j)$.

We examine the case where the cross-edge of P, the edge going from (i_c, j_c) to (i_{c+1}, j_{c+1}), is vertical. This gives us that $w(P_2)$ has the form

$$w(P_2) = t_{i,j}^{-1} t_{i_1,j_1} t_{i_2,j_2}^{-1} \cdots t_{i_c,j_c} t_{j_{c+1},i_{c+1}}^{-1} t_{j_{c+2},i_{c+2}} \cdots t_{j_k,i_k}^{-1} t_{i,j},$$

where for $1 \leq \ell \leq k$ the indices (i_ℓ, j_ℓ) represent the subsequence of turns in P_2, and where we add the $t_{i,j}$ and $t_{i,j}^{-1}$ since P_2 begins at (i,j) and ends at (j,i). Similarly, the transpose has weight of the form

$$w(P_2^T) = -qt_{j_k,i_k}^{-1} t_{j_{k-1},i_{k-1}} \cdots t_{j_{c+2},i_{c+2}} t_{j_{c+1},i_{c+1}}^{-1} t_{j_c,i_c} \cdots t_{i_1,j_1}.$$

We now want to show that we can reverse the order of the generators in $w(P_2)$ to obtain $w(P_2) = -w(P_2^T)$. Our general strategy will be to commute each generator left of t_{j_k,i_k}^{-1} just to the right of t_{j_k,i_k}^{-1}. That is, we will start with $t_{i,j}^{-1}$ and commute it just to the right of t_{j_k,i_k}^{-1}, and then take t_{i_1,j_1} and commute it just to the right of t_{j_k,i_k}^{-1}, and so on.

To start the process, we take the first generator in the list $t_{i,j}^{-1}$ and note that

$$t_{i,j}^{-1} t_{i_1,j_1} = qt_{i_1,j_1} t_{i,j}^{-1}$$

since we have $i = i_1, j > j_1$. Then see that for $2 \leq \ell \leq k-1$ we have

$$t_{i,j}^{-1} t_{i_\ell,j_\ell}^{\delta_\ell} = t_{i_\ell,j_\ell}^{\delta_\ell} t_{i,j}^{-1}$$

where $\delta_\ell \in \{\pm 1\}$ depends on the parity of ℓ. For $\ell = k$, we have

$$t_{i,j}^{-1} t_{j_k,i_k}^{-1} = qt_{j_k,i_k}^{-1} t_{i,j}^{-1}$$

since $j_k = j, i < i_k$. So commuting $t_{i,j}^{-1}$ past t_{j_k,i_k}^{-1} gives us a q^2.

Now for $\ell \in [k-1]$, in order to commute $t_{i_\ell,j_\ell}^{\delta_\ell}$ (or $t_{j_\ell,i_\ell}^{\delta_\ell}$ if $\ell > c$) just past t_{j_k,i_k}^{-1}, we need only take into account when $t_{i_\ell,j_\ell}^{\delta_\ell}$ q-commutes with another generator $t_{i_m,j_m}^{\delta_m}$. This occurs if $\{i_\ell, j_\ell\} \cap \{i_m, j_m\} \neq \emptyset$, and visually, this first happens for the generators that lie on the pairs of parallel lines defined by the coordinates of (i_ℓ, j_ℓ) and (j_ℓ, i_ℓ). From this we get a q-commutation with the first swap. Since P_2 is disjoint from P_2^T except at (i,j) and (j,i), we won't encounter the set of parallel lines defined by (i_ℓ, j_ℓ) and (j_ℓ, i_ℓ) again. But if we encounter a turn $t_{i_m,j_m}^{\delta_m}$ that lies beyond the (i_ℓ, j_ℓ) and (j_ℓ, i_ℓ) lines that q-commutes with t_{i_ℓ,j_ℓ}, then we need to consider the set of parallel lines defined by (i_m, j_m) and (j_m, i_m). That is, we encounter one turn on the i_m, j_m lines. In fact we also must have a second turn on this set of lines since $\{i_\ell, j_\ell\} \cap \{i_m, j_m\} \neq \emptyset$, and P_2 contains vertices beyond the i_m, j_m lines. For a visualisation of this, see Figure 4.1.

Ultimately this means that the net factor of q obtained from commuting a single generator through the line is determined only by the first swap, and this depends only on the kind of turn that we start with. For a Γ-turn we have

$$t_{i_a,j_a}^{\delta_a} \cdots t_{i_\ell,j_\ell} \cdots t_{j_k,i_k}^{-1} = q^{-1} \cdots t_{j_k,i_k}^{-1} t_{i_\ell,j_\ell},$$

and for a ⅃-turn, we have

$$t_{i_a,j_a}^{\delta_a} \cdots t_{i_\ell,j_\ell}^{-1} \cdots t_{j_k,i_k}^{-1} = q \cdots t_{j_k,i_k}^{-1} t_{i_\ell,j_\ell}^{-1},$$

where $t_{i_a,j_a}^{\delta_a}$ varies based on where we are in the process of moving all of the generators. There are corresponding but nearly identical relations that we get in the case $\ell > c$, and we omit them here. Importantly during this process we do not need to move t_{j_k,i_k}^{-1} at all, which means that we commute one more Γ-turn than ⅃-turn, which implies that in reversing the order of the generators, the resulting combination of q-factors is a q^2 from moving the $t_{i,j}^{-1}$ and a q^{-1} from the other $k-1$ generators.

This leaves us with

$$\begin{aligned}
w(P_2) &= t_{i,j}^{-1} t_{i_1,j_1} t_{i_2,j_2}^{-1} \cdots t_{i_c,j_c} t_{j_{c+1},i_{c+1}}^{-1} t_{j_{c+2},i_{c+2}} \cdots t_{j_k,i_k}^{-1} t_{i,j} \\
&= q^2 t_{i_1,j_1} t_{i_2,j_2}^{-1} \cdots t_{i_c,j_c} t_{j_{c+1},i_{c+1}}^{-1} t_{j_{c+2},i_{c+2}} \cdots t_{j_k,i_k}^{-1} t_{i,j}^{-1} t_{i,j} \\
&= q^2 t_{i_1,j_1} t_{i_2,j_2}^{-1} \cdots t_{i_c,j_c} t_{j_{c+1},i_{c+1}}^{-1} t_{j_{c+2},i_{c+2}} \cdots t_{j_k,i_k}^{-1} \\
&= qt_{j_k,i_k}^{-1} t_{j_{k-1},i_{k-1}} \cdots t_{j_{c+2},i_{c+2}} t_{j_{c+1},i_{c+1}}^{-1} t_{j_c,i_c} \cdots t_{i_1,j_1} \\
&= -w(P_2^T).
\end{aligned}$$

Now we have that

$$w(P) = w(P_1)w(P_2)w(P_3) = -w(P_1)w(P_2^T)w(P_3) = -w(\tau(P)),$$

so

$$w(P) + w(\tau(P)) = 0.$$

Resolving the case where P's cross-edge is horizontal is nearly identical to the above argument. □

This proposition gives us that for any $i \in [n]$, the sum over edge-weights of paths in $\Gamma_D(i,i)$ is 0, since no path in $\Gamma_D(i,i)$ is transpose-disjoint.

5. Proofs by Induction

Theorem 5.1. $\mathcal{O}_q(Sk_n) \cong A_n$

This is the major result of our work on skew-symmetric quantum matrices. The isomorphism is the map $\phi : \mathcal{O}_q(Sk_n) \to A_n$ which is the homomorphism defined by sending each generator of $\mathcal{O}_q(Sk_n)$ to the corresponding x_{ij}. This clearly is surjective. To prove injectivity, one can use techniques of GK-dimension theory. To prove that ϕ is well-defined, we need to show that the generators of A_n satisfy the same relations as $\mathcal{O}_q(Sk_n)$. We do this in the following sections.

Definition 5.2. A path P_{ij} in an $n \times n$ graph has the *parent path* P_{kl} in an $n-1 \times n-1$ graph, if P_{ij} has the form:

$$P_{ij} = t_{in} t_{kn}^{-1} P_{kl} t_{ln}^{-1} t_{jn}$$

P_{ij} is then considered a *child path* of P_{kl}.

Lemma 5.3. *All paths in the $n \times n$ graph either have a parent path or are of the form*

$$P_{ij} = \begin{cases} t_{in} & \text{if } i < j, \\ t_{nj} & \text{if } j < i. \end{cases}$$

Proof. Given a path P_{ij} with $i < j$ the first vertex in P_{ij} must be of the form t_{in}. If the path ends in column n then that is the entire path and we are done.

If not, the path must leave the n column at some coordinate k. Thus the next turning vertex must be of the form t_{kn}^{-1}. The only possible way this is not the next turning vertex is if they are the same vertex, in which case we still have $t_{in}t_{kn}^{-1} = t_{in}t_{in}^{-1}$, which simply cancel out in the simplified form. If $j \neq n$, there must exist turns $(-t_{ln}^{-1})$ and $(-t_{jn})$ to reach the end of the path. (Note that, again, we could have $l = j$). A very similar argument applies for when $j < i$. □

Lemma 5.4. *Given P_{ij} with $i < n$ and $j < n$ then:*

$$P_{ij} t_{kn} = \begin{cases} t_{kn} P_{ij} & \text{if } i \neq k \neq j, \\ q t_{kn} P_{ij} & \text{else.} \end{cases}$$

Proof. We track the q's created as t_{kn} is moved through P_{ij}. As long as none of the indices are equal t_{kn} will simply commute with any $t_{ab} \in P_{ij}$. Thus we only worry about when t_{kn} meets some $t \in P_{ij}$ with indices equal to k. If k is met in the middle of the path rather than the edge then it will be met again right afterwards with an inverse, due to the grid pattern of the path. The only fear is that the relation of the other indices to k will change, making both meetings create the same q type. This will only occur when the diagonal cross happens in between these two points. However, this is not actually a problem since if P_{ij} crosses the diagnol horizontally we encounter t_{ka}^{-1} and t_{bk} with $b < k < a$. Which gives us $t_{ka}^{-1} t_{kn} = q^{-1} t_{kn} t_{ka}^{-1}$ and $t_{bk} t_{kn} = q t_{kn} t_{bk}$. If P_{ij} crosses the diagnol vertically we encounter t_{ak} and t_{kb}^{-1} with $a < k < b$. But this is in fact no problem since $t_{ak} t_{kn} = q t_{kn} t_{ak}$ and $t_{kb}^{-1} t_{kn} = q^{-1} t_{kn} t_{kb}^{-1}$ thus they cancel.

Now for the edge cases. If $k < i$ then the edge will have nothing in common with t_{kn} and no q will be created. If on the other hand $k = i$ then it will have the form $t_{ka} t_{kn}$ at the edge which will create a q. The same happens if $k = j$.Therefore if $i \neq k \neq j$ then no q will be created and otherwise a single q will be created. □

Lemma 5.5. *All paths P_{ij} in the $n \times n$ graph will fall into one of two categories:*

a. There exists P'_{ij} such that $w(P_{ij}) = -w(P'_{ij})$.

b. There exists $P_{ji} = P_{ij}^T$ such that P_{ij} is transpose disjoint and

$$w(P_{ji}) = \begin{cases} -q w(P_{ij}) & \text{if } i < j, \\ -q^{-1} w(P_{ij}) & \text{if } j < i. \end{cases}$$

Proof. We will prove this by induction.

Base case: A 1×1 graph has no paths and thus the lemma is vacuously true.

Inductive assumption:

Assume that for some $n-1$ the lemma holds for all paths P_{kl} in the $n-1 \times n-1$ graph.

Given a path P_{ij} in the $n \times n$ graph, P_{ij} must fall into one of the following cases:

(1) P_{ij} has no parent path
(2) P_{ij} has a parent path P_{kl}
  (a) P_{kl} satisfies part a of the lemma
  (b) P_{kl} satisfies part b of the lemma
    (i) $i \leq j$ and $k < j$
    (ii) $j \leq i$ and $l < i$
    (iii) $j \leq k$ and $i \leq l$

Either P_{ij} has a parent path or it does not. If it has a parent path that parent path is in an $n-1 \times n-1$ graph and must therefore satisfy either part a. or part b. of the lemma by our inductive hypothesis. Since the relations of $i \leq k$ and $j \leq l$ are fixed by the construction of the parent path, the conditions outlined above will cover all possible relations between i, j, k, and l.

By Lemma 5.3, if P_{ij} does not have a parent path P_{kl} it is of the form

$$P_{ij} = \begin{cases} P_{in} = t_{in} & \text{if } i < j, \\ P_{ni} = t_{ni} & \text{if } j < i. \end{cases}$$

Thus we see that

$$\begin{aligned} w(P_{in}) &= t_{in} \\ w(P_{ni}) &= -qt_{in} \\ w(P_{ni}) &= -qw(P_{in}) \end{aligned}$$

Note that $P_{ni} = P_{in}^T$ and is transpose disjoint. Thus P_{in} fall into case b. and satisfies the lemma as does its transpose P_{ni}. Going forward we can thus assume that P_{ij} has a parent path P_{kl} which satisfies the lemma.

If P_{kl} satisfies part a. of the lemma then there exists some P'_{kl} such that $w(P'_{kl}) = -w(P_{kl})$.

Consider

$$P'_{ij} = t_{in}t_{kn}^{-1}P'_{kl}t_{ln}^{-1}t_{jn}$$

Recall that

$$P_{ij} = t_{in}t_{kn}^{-1}P_{kl}t_{ln}^{-1}t_{jn}$$

Since $w(P'_{kl}) = -w(P_{kl})$, we can calculate the weights of P_{ij} and P'_{ij}:

$$\begin{aligned} w(P_{ij}) &= t_{in}t_{kn}^{-1}w(P_{kl})t_{ln}^{-1}t_{jn} \\ w(P'_{ij}) &= t_{in}t_{kn}^{-1}w(P'_{kl})t_{ln}^{-1}t_{jn} \\ w(P'_{ij}) &= t_{in}t_{kn}^{-1} - qw(P_{kl})t_{ln}^{-1}t_{jn} \\ w(P'_{ij}) &= -qw(P_{ij}) \end{aligned}$$

Thus if the parent path satisfies part a. then so too does the child path. All that remains is the case where the parent path satisfies part b. To do this we want to first pin down the possible relationships between i, j, k, and l.

So far we know: $i \leq k$ and $j \leq l$.

All possible relations can be covered with three cases:

(i) $i \leq j$ and $k < j$
(ii) $j \leq i$ and $l < i$
(iii) $j \leq k$ and $i \leq l$

The only fear is if it is possible to be in none of these. Let $j \leq i$. Either $l < i$ and we are in case 2, or $l \geq i$, and we are in case 3 because $j \leq i \leq k$. Now let $i \leq j$. Then either $k < j$ and we are in case 1, or $k \geq j$, and we are in case 3 because $i \leq j \leq l$. Thus these three cases will cover all possible relations.

Now for case i (and ii):

Assume that $i \leq j$ and $k < j$. This means that $k < l$ too. Giving us the relationship: $i \leq k < j \leq l$.

Now consider $P_{ji} = t_{jn}t_{ln}^{-1}P_{lk}t_{kn}^{-1}t_{in}$. In order to compare it to $P_{ij} = t_{in}t_{kn}^{-1}P_{kl}t_{ln}^{-1}t_{jn}$ we need to move the t_{ab} terms into like positions. To do this we will move each term in P_{ij} through the path in order.

(Note that both P_{ij} and P_{ji} can be built from the other so this process works for case ii, which is satisfied by P_{ji}, as well.)

Begin with t_{in}:

$$t_{in}t_{kn}^{-1} = \begin{cases} q^{-1}t_{kn}^{-1}t_{in} & \text{if } i < k, \\ t_{kn}^{-1}t_{in} & \text{if } i = k. \end{cases}$$

$$t_{in}P_{kl} = \begin{cases} P_{kl}t_{in} & \text{if } i < k, \\ q^{-1}P_{kl}t_{in} & \text{if } i = k. \end{cases}$$

$$t_{in}t_{ln}^{-1} = q^{-1}t_{ln}^{-1}t_{in}$$

$$t_{in}t_{jn} = qt_{jn}t_{in}$$

These equalities can be derived from the given relations and Lemma 5.4.

Next consider t_{kn}^{-1}:

$$t_{kn}^{-1}P_{kl} = qP_{kl}t_{kn}^{-1}$$

$$t_{kn}^{-1}t_{ln}^{-1} = qt_{ln}^{-1}t_{kn}^{-1}$$

$$t_{kn}^{-1}t_{jn} = q^{-1}t_{jn}t_{kn}^{-1}$$

Once again, we derived the equalities from the relations, and Lemma 5.4.

Next consider t_{jn} as it passes in the opposite direction:

$$t_{ln}^{-1}t_{jn} = \begin{cases} qt_{jn}t_{ln}^{-1} & \text{if } j < l, \\ t_{jn}t_{ln}^{-1} & \text{if } j = l. \end{cases}$$

$$P_{kl}t_{jn} = \begin{cases} t_{jn}P_{kl} & \text{if } j < l, \\ qt_{jn}P_{kl} & \text{if } j = l. \end{cases}$$

Finally t_{ln}^{-1} must be moved to the other side of P_{kl} which by Lemma 5.4 will create a q^{-1}. Thus we have:

$$t_{ln}^{-1}P_{kl} = q^{-1}P_{kl}t_{ln}^{-1}$$

Now we see that after moving all terms to their new places all the q's and q^{-1}'s cancel. Thus we have:

$$w(P_{ij}) = t_{in}t_{kn}^{-1}w(P_{kl})t_{ln}^{-1}t_{jn} = t_{jn}t_{ln}^{-1}w(P_{kl})t_{kn}^{-1}t_{in}$$

and since we have a formula for $w(P_{kl})$ we can substitute to get:

$$w(P_{ij}) = t_{jn}t_{ln}^{-1}(-q^{-1})w(P_{lk})t_{kn}^{-1}t_{in}$$

which can then be seen as:

$$w(P_{ij}) = -q^{-1}w(P_{ji})$$

or

$$w(P_{ji}) = -q(P_{ij})$$

It is also true, as we can see from their formulas and the relations of i, j, k and l that $P_{ji} = P_{ij}^T$ and is transpose disjoint. This means P_{ij} satisfies part b. of the lemma, as does P_{ji} which takes care of the case where $j < i$ and $i > l$ satisfying case ii.

For case iii: If $j \leq k$ and $i \leq l$, then we are still unsure about the relation of k to l and i to j. Fortunately, we do not care about the relation of i to j and it is sufficient to know that they are both less than or equal to k and l.

As for k and l, since P_{kl} is transpose disjoint, we know that $k \neq l$.

So now we consider $P'_{ij} = t_{in}t_{ln}^{-1}P_{lk}t_{kn}^{-1}t_{jn}$.

This time we need only swap t_{ln}^{-1} and t_{kn}^{-1} in P_{ij} to get it in the proper form. Our swapping relations are:

$$t_{kn}^{-1}P_{kl} = qP_{kl}t_{kn}^{-1}$$
$$t_{kn}^{-1}t_{ln}^{-1} = \begin{cases} qt_{ln}^{-1}t_{kn}^{-1} & \text{if } k < l, \\ q^{-1}t_{ln}^{-1}t_{kn}^{-1} & \text{if } l < k. \end{cases}$$
$$P_{kl}t_{ln}^{-1} = q^{-1}t_{ln}^{-1}P_{lk}$$

These equalities can once more be seen from the given realtions and Lemma 5.4.

We can now see that carrying out these swaps will result in the addition of a single q if $k < l$ and a single q^{-1} if $l < k$. Thus we have:

$$w(P_{ij}) = \begin{cases} qt_{in}t_{ln}^{-1}w(P_{kl})t_{kn}^{-1}t_{jn} & \text{if } k < l, \\ q^{-1}t_{in}t_{ln}^{-1}w(P_{kl})t_{kn}^{-1}t_{jn} & \text{if } l < k. \end{cases}$$

which after we substitute in our known formula for $w(P_{kl})$, which adds a q^{-1} if $k < l$ and a q if $l < k$, we attain:

$$w(P_{ij}) = -qq^{-1}t_{in}t_{ln}^{-1}w(P_{lk})t_{kn}^{-1}t_{jn}$$

Which after the q's cancel finally becomes:

$$w(P_{ij}) = -w(P'_{ij})$$

and thus P_{ij} satisfies part a. of the lemma.

Therefore all possible cases are proven. Thus we find that the lemma holds for $n = 1$ and if the lemma holds for $n - 1$ it will hold for n, thus by induction the lemma holds for all $n \geq 1$. □

Theorem 5.6. *$x_{ji} = -qx_{ij}$ where $i \leq j$.*

Proof. By Lemma 5.5 all $w(P_{ij})$ terms in the sum which make up x_{ij} will either have a corresponding $w(P'_{ij})$ with which it will cancel or will have a corresponding $w(P_{ji})$ in x_{ji} from which we can remove a q term, to get $w(P_{ji}) = -qw(P_{ij})$. Thus when we add them together we get:

$$x_{ji} = \sum_{P_{j\to i}} w(P_{ji}) = \sum -qw(P_{ij}) = -q \sum_{P_{j\to i}} w(P_{ij}) = -qx_{ij}$$

□

Corollary 5.7. *$x_{ii} = 0$ for all i.*

Proof. Proof by contradiction: By Theorem 5.6, $x_{ii} = -qx_{ii}$. Thus if $x_{ii} \neq 0$, then we have $1 = -q$ which has been forbidden by our assumptions about q. □

Corollary 5.8. *All paths whose wieghts in x_{ij} do not cancel out are transpose disjoint.*

Proof. Given a non-transpose disjoint path P_{ij} from $i \to j$, P_{ij} must satisfy one of the two parts of the lemma. Since P_{ij} is not transpose disjoint it cannot satisfy part b of the lemma and must therefore satisfy part a. Thus there exists P'_{ij} such that $w(P_{ij}) = -w(P'_{ij})$. Thus the wieght of P_{ij} will cancel out in x_{ij}. Therefore if a path does not cancel out, it cannot be non-transpose disjoint and must therefore be transpose disjoint. □

5.1. **Setting up for Inductive Relations.** Next we turn to proving the relations:

(1) $x_{ij}x_{il} = qx_{il}x_{ij}$ for $i < j < l$

(2) $x_{ij}x_{jl} = qx_{jl}x_{ij}$ for $i < j < l$

(3) $x_{ij}x_{kj} = qx_{kj}x_{ij}$ for $i < k < j$

(4) $x_{ij}x_{kl} = qx_{kl}x_{ij}$ for $i < k < l < j$

(5) $x_{ij}x_{kl} = qx_{kl}x_{ij} + (q - q^{-1})x_{il}x_{kj}$ for $i < k < j < l$

(6) $x_{ij}x_{kl} = qx_{kl}x_{ij} + (q - q^{-1})x_{ik}x_{jl} - q(q - q^{-1})x_{il}x_{jk}$ for $i < j < k < l$

We first prove the relations in the case where one of the paths has no parent path. As previously discussed, a path has no parent path if and only if it starts or ends at n. For these cases we cannot use induction.

Relation 1: $x_{ab}x_{ad} = qx_{ad}x_{ab}$ for $a < b < d$

If one path has no parent path, then $d = n$ and $x_{ad} = t_{an}$. Take any path from a to b,

$$P_{ab} = t_{an}t_{a'n}^{-1}P_{a'b'}t_{b'n}^{-1}t_{bn}$$

Consider the product

$$(t_{an}t_{a'n}^{-1}P_{a'b'}t_{b'n}^{-1}t_{bn})(x_{in}) = (t_{an}t_{a'n}^{-1}P_{a'b'}t_{b'n}^{-1}t_{bn})(t_{in})$$

If we show that

$$(t_{an}t_{a'n}^{-1}P_{a'b'}t_{b'n}^{-1}t_{bn})(t_{in}) = (t_{in})(t_{an}t_{a'n}^{-1}P_{a'b'}t_{b'n}^{-1}t_{bn})$$

for all paths from a to b then we are done. So, we commute t_{in}. If $b' \neq b$, then $t_{b'n}^{-1}t_{bn}t_{an} = t_{an}t_{b'n}t_{bn}$ by a simple application of commutativity rules. If $b' = b$, then either $P_{ab} = t_{ab}$ (in which case the result is trivial), or $P_{a'b'}$ has its last turn at some t_{bj}, for $b > a$, in which case commuting t_{bn} this far still gives us $q^{-1} \cdot q$. Similarly, when we commute t_{in} the rest of the way through, we will always obtain a q, because if $a' \neq a$, $t_{a'n}^{-1}t_{an} = qt_{an}t_{a'n}^{-1}$, and if $a' \neq a$, $P_{a'b'}$ will first turn at some t_{ai}, $i < n$, and we will get a q. We have dealt with the nth row and column and the ath row, and P_{ab} cannot pass through the ath column. So we are done with this case.

Relation 2: $x_{ab}x_{bd} = qx_{bd}x_{ab}, a < b < d$

If one path has no parent path, then $d = n$ and $x_{bd} = t_{bn}$. Take any path from a to b,

$$P_{ab} = t_{an}t_{a'n}^{-1}P_{a'b'}t_{b'n}^{-1}t_{bn}$$

Recall from section 2.1 that if P_{ab} is not transpose disjoint, there exists another non-transpose disjoint path $\tau(P_{ab})$ such that $w(\tau(P_{ab})) = -w(P_{ab})$. So we can ignore all non transpose disjoint paths. It is a simple computation, similar to the computation in case 1, to show that for any transpose disjoint path,

$$w(P_{ab})t_{bn} = qt_{bn}w(P_{ab})$$

This gives the desired result.

Relation 3: $x_{ab}x_{cb} = qx_{cb}x_{ab}, a < c < b$

If a path has no parent path, then $b = n$ and $x_{ab} = t_{an}$ and $x_{cd} = t_{cn}$. From the commutativity rules,

$$t_{an}t_{cn} = qt_{cn}t_{an}$$

So we are done.

Relation 4: $x_{ab}x_{cd} = x_{cd}x_{ab}, a < c < d < b$

If one path has no parent path, then $b = n$ and $x_{ab} = t_{an}$. Take any path from c to d,

$$P_{cd} = t_{cn}t_{c'n}^{-1}P_{c'd'}t_{d'n}^{-1}t_{dn}$$

It is simple to show with the commutativity relations that

$$t_{an}(w(P_{cd})) = (w(P_{cd}))t_{an}$$

The result follows.

Relation 5: $x_{ab}x_{cd} = x_{cd}x_{ab} + (q - q^{-1})x_{ad}x_{cb}, a < c < b < d$

If a path has no parent path, then $d = n$, $x_{cd} = t_{cn}$ and $x_{ad} = t_{an}$. We can divide the paths from a to b into two cases depending on the relationship between a', b and c.

Let X_1 be the sum of the weights of paths with $a' < c$.

Let X_2 be the sum of the weights of paths with $a' \geq c$.

Note that $x_{ab}x_{cd} = (X_1 + X_2)t_{cn}$. Similarly, $x_{cd}x_{ab} = t_{cn}(X_1 + X_2)$. It is easily verified through commutativity relations that

$$t_{cn}(X_1 + X_2) = X_1 t_{cn} + q^{-2}(X_2)t_{cn}$$

Notice that any path from c to b by starting with with c_n and then following a path from a to b with $a' \geq c$. So a path from c to b is of the form $t_{cn}t_{a'n}^{-1}t_{a'b'}t_{b'n}^{-1}t_{bn}$, where $a' > c$. When we move the c_n over to the end, we get a q^{-1} in front. So

$$(q - q^{-1})x_{an}x_{cb} = (q - q^{-1})q^{-1}(X_2)t_{cn} = X_2 t_{cn} - q^{-2}(X_2)t_{cn}$$

This gives us

$$x_{cd}x_{ab} + (q - q^{-1})x_{ad}x_{cb} = X_1 t_{cn} + q^{-2}(X_2)t_{cn} + X_2 t_{cn} - q^{-2}(X_2)t_{cn} = x_{ab}x_{cd}$$

Relation 6: $x_{ab}x_{cd} = x_{cd}x_{ab} + (q - q^{-1})x_{ac}x_{bd} - q(q - q^{-1})x_{ad}x_{bc}, a < b < c < d$

If one path has no parent path, then $d = n, x_{cd} = x_{cn}, x_{bd} = x_{bn}$, and $x_{ad} = x_{an}$. We now divide the paths into cases.

Let X_1 be the sum of weights of paths from a to b with $a' < b$, $b' < c$.
Let X_2 be the sum of weights of paths from a to b with $a' < b$, $b' \geq c$.
Let X_2' be the sum of weights of paths from a to c with $a' < b$.
Let X_3 be the sum of weights of paths from a to b with $b \leq a' < c$, $b' < c$.
Let X_4 be the sum of weights of paths from a to b with $b \leq a' < c$, $b' \geq c$.
Let X_4' be the sum of weights of paths from a to c with $b \leq a' < c$.
Let X_5 be the sum of weights of paths from a to b with $a' \geq c$, $b' < c$.
Let X_6 be the sum of weights of paths from a to b with $a' \geq c$, $b' \geq c$.
Let X_6' be the sum of weights of paths from a to c with $a' \geq c$.

Note that every path in X_6 is non-transpose disjoint, and so there exists a path whose weight is its negative, as previously discussed. That path is also in X_6. So $X_6 = 0$. Similarly, $X_6' = 0$.

We have

$$x_{ab}x_{cn} = (X_1 + X_2 + X_3)t_{cn}$$

By commuting t_{cn}, we get

$$x_{ab}x_{cn} = t_{cn}(X_1 + X_3) + q^2 t_{cn}(X_2 + X_4 + X_5) + q^4 t_{cn}X_6$$

$$x_{ab}x_{cn} = x_{cn}x_{ab} + (q^2 - 1)x_{cn}(X_2 + X_4 + X_5)$$

(We get rid of X_6 for now since it equals 0)

Note that $x_{ac}x_{bn} = (X_2' + X_4' + X_6')t_{bn}$. So we can commute t_{bn} through to the end of X_2 and X_4 to get

$$x_{ab}x_{cn} = x_{cn}x_{ab} + q^{-1}(q^2 - 1)(X_2' + X_4')t_{bn} + (q^2 - 1)t_{cn}X_5$$

$$x_{ab}x_{cn} = x_{cn}x_{ab} + (q - q^{-1})(X_2' + X_4' + X_6')t_{bn} + q(q - q^{-1})t_{cn}X_5$$

$$x_{ab}x_{cn} = x_{cn}x_{ab} + (q - q^{-1})x_{ac}x_{bn} + q(q - q^{-1})t_{cn}X_5$$

$$x_{ab}x_{cn} = x_{cn}x_{ab} + (q - q^{-1})x_{ac}x_{bn} + q(q - q^{-1})t_{cn}(X_5 + X_6)$$

Note that $x_{an}x_{cb} = qx_{cn}(X_5 + X_6)$ because we can just switch the t_{an} and the t_{cn}. So:

$$x_{ab}x_{cn} = x_{cn}x_{ab} + (q - q^{-1})x_{ac}x_{bn} + (q - q^{-1})x_{an}x_{cb}$$

Recall we proved that $x_{ji} = -qx_{ij}$ if $i < j$. So $x_{cb} = -qx_{bc}$. So:

$$x_{ab}x_{cn} = x_{cn}x_{ab} + (q - q^{-1})x_{ac}x_{bn} - q(q - q^{-1})x_{an}x_{bc}$$

Now we are done with the cases where one path has no parent path.

Now, to use induction we must find the relation between x_{ij}'s in $n \times n$ and those in $n - 1 \times n - 1$. We begin with

$$x_{ij} = \sum_{P_{i \to j}} w(P_{ij})$$

Next we note that every path P_{ij} which is not one of the cases prviously dealt with has a parent path P_{kl} in $n-1 \times n-1$ thus we have:

$$x_{ij} = \sum_{k,l} \sum_{P_{k \to l}} t_{in} t_{kn}^{-1} w(P_{kl}) t_{ln}^{-1} t_{jn}$$

Where k and l can range $i \leq k < j \leq l < n$ since the relevant paths are transpose disjoint and thus k cannot be larger than j. If it were then the path would cross it's transpose at t_{kn}.

Now we observe that the t's can be pulled out of the inner sum to get the formula for x_{kl} within the formula for x_{ij}.

$$x_{ij} = \sum_{k,l} t_{in} t_{kn}^{-1} (\sum_{P_{k \to l}} w(P_{kl})) t_{ln}^{-1} t_{jn} = \sum_{k,l} t_{in} t_{kn}^{-1} x_{kl} t_{ln}^{-1} t_{jn}$$

So now we examine the product of two such x_{ij} sums. We refer to the summand in the expression on the right side in the above display as *a term*, and when we speak about "two terms" or "both terms" we are referring to the product of two such expressions. For convenience we will now deal with slightly new notation. Where before we had $i \leq k < j \leq l < n$, we will now look at $a \leq a' < b \leq b' < n$. So our product looks like this:

$$\begin{aligned} x_{ab} x_{cd} &= \sum_{a',b'} t_{an} t_{a'n}^{-1} x_{a'b'} t_{b'n}^{-1} t_{bn} \sum_{c',d'} t_{cn} t_{c'n}^{-1} x_{c'd'} t_{d'n}^{-1} t_{dn} \\ &= \sum_{a',b',c',d'} t_{an} t_{a'n}^{-1} x_{a'b'} t_{b'n}^{-1} t_{bn} t_{cn} t_{c'n}^{-1} x_{c'd'} t_{d'n}^{-1} t_{dn} \end{aligned}$$

This is the form we will use from now on. Next we must find a rule for commuting t's with x's. This turns out to be relatively simple since x is made up of the wieghts of paths on which we we can use Lemma 5.4. Thus simple manipulation gives us:

$$\begin{aligned} t_{in} x_{jk} &= t_{in} \sum_{P_{j \to k}} w(P_{jk}) \\ &= \sum_{P_{j \to k}} t_{in} w(P_{jk}) \\ &= \sum_{P_{j \to k}} q^{\delta} w(P_{jk}) t_{in} \\ &= q^{\delta} x_{jk} t_{in} \end{aligned}$$

Where

$$\delta = \begin{cases} 0 & \text{if } j \neq i \neq k, \\ -1 & \text{else.} \end{cases}$$

Thus we have

$$t_{in} x_{jk} = \begin{cases} x_{kl} t_{ij} & \text{if } j \neq i \neq k, \\ q^{-1} x_{jk} t_{in} & \text{else.} \end{cases}$$

Now having a way to manipulate all elements in our parent form of x_{ab} we can attempt to verify the relations by adding both sides of the equality together and using the expanded form. Since both terms have the same possible values of a', b', c' and d', we can compare the terms with the same values of a', b', c' and d'.

But in order to compare the terms we will need a convenient way to get the two terms in the same, or at least similar, order.

Notation 5.9 (Standard Form)**.** The order we will want to work with is

$$t_{cn} t_{c'n}^{-1} t_{dn} t_{d'n}^{-1} x_{ij} x_{kl} t_{a'n}^{-1} t_{an} t_{b'n}^{-1} t_{bn}$$

Where the pair of x terms in the middle may be arranged $x_{ij} x_{kl}$ or $x_{kl} x_{ij}$. From now on we will refer to this form as *the Standard Form*, in some cases using the following shorthand expression:

$$SF(x_{ij} x_{kl}) := t_{cn} t_{c'n}^{-1} t_{dn} t_{d'n}^{-1} x_{ij} x_{kl} t_{a'n}^{-1} t_{an} t_{b'n}^{-1} t_{bn}.$$

Getting both terms into Standard Form will create a lot of q's which we will now attempt to organize. To get the term from $x_{cd}x_{ab}$ in the proper form we need to swap the positions of:

$$x_{c'd'}t_{d'n}^{-1} \quad (1)$$
$$t_{a'n}^{-1}x_{a'b'} \quad (2)$$
$$t_{an}x_{a'b'} \quad (3)$$
$$t_{an}t_{a'n}^{-1} \quad (4)$$
$$x_{c'd'}t_{dn} \quad (5)$$
$$t_{d'n}^{-1}t_{dn} \quad (6)$$

The first two will cancel with each other as will the last four. This is because (1) will create a q^{-1} and (2) will create a q. Between them, (3) and (4) will create a q^{-1} with (3) creating it if $a = a'$ and (4) creating it otherwise. By the same logic (5) and (6) create a q which then cancels with the q^{-1} from (3) and (4). Thus $x_{cd}x_{ab}$ can be put into standard form without creating any extra q's.

On the other hand the term from $x_{ab}x_{cd}$ will need 30 different swaps. Six of these are the same as the six above. Four more ($x_{a'b'}t_{c'n}^{-1}$, $x_{a'b'}t_{d'n}^{-1}$, $t_{a'n}^{-1}x_{c'd'}$, $t_{b'n}^{-1}x_{c'd'}$) will cancel with each other, leaving 20 swaps which can be grouped into four groups of six with each group looking like this:

$$t_{\alpha n}t_{\beta n}$$
$$t_{\alpha' n}^{-1}t_{\beta n}$$
$$t_{\alpha n}t_{\beta' n}^{-1}$$
$$t_{\alpha' n}^{-1}t_{\beta' n}^{-1}$$
$$x_{\alpha'}t_{\beta n}$$
$$t_{\alpha n}x_{\beta'}$$

Where $\alpha \in \{a, b\}$ and $\beta \in \{c, d\}$. The final two terms are included in two seperate groups each. However, if those terms create a q, that q will only be counted in one group since we are only considering one of x's indices per group.

The q's created by the group as a whole will depend on the relations of α, β, α' and β'.

Let us assume that $\alpha \leq \beta$, (we can then generalize by inverting whatever q's we create in the case of $\beta \leq \alpha$). This assumption gives us three possible general positionings:

$$\alpha \leq \alpha' \leq \beta \leq \beta' \quad (1)$$
$$\alpha \leq \beta \leq \alpha' \leq \beta' \quad (2)$$
$$\alpha \leq \beta' \leq \beta \leq \alpha' \quad (3)$$

Depending on whether each $\leq$ is $<$ or is $=$ each of these three can be expanded into 8 different orderings. However these 24 orderings will have much overlap when an equality allows a swap in position, thus let us take positioning (2) as the default positioning. Now only those positionings from (1) which have $\alpha' < \beta$ are unique. In addition the ordering between α and α' does not make a difference, the same goes for β. Thus positioning (1) only results in one relevant ordering: $\alpha \leq \alpha' < \beta \leq \beta'$. By similar logic positioning (3) only results in one relevant positioning: $\alpha < \beta' \leq \beta < \alpha'$. Thus we have 10 possible relevant and unique orderings.

(1) $\alpha \leq \alpha' < \beta \leq \beta'$
(2) $\alpha < \beta' \leq \beta < \alpha'$
(3) $\alpha = \beta = \alpha' = \beta'$
(4) $\alpha = \beta < \alpha' = \beta'$
(5) $\alpha = \beta = \alpha' < \beta'$
(6) $\alpha = \beta < \alpha' < \beta'$
(7) $\alpha < \beta = \alpha' = \beta'$
(8) $\alpha < \beta < \alpha' = \beta'$
(9) $\alpha < \beta = \alpha' < \beta'$
(10) $\alpha < \beta < \alpha' < \beta'$

We can reduce this even further by noting that if $\beta = \alpha'$ then $x_{\alpha'}t_{\beta n}$ will create a q in its swap while $t_{\alpha' n}t_{\beta n}$ will create nothing in its swap. On the other hand if $\beta < \alpha'$ then the situation will be reversed and $x_{\alpha'}t_{\beta n}$ will create nothing while $t_{\alpha' n}t_{\beta n}$ will create a q. Thus β's relation to α' is irrelevant in (3) through (10), leaving us with only 6 cases to check in Table 5.1:

(1) $\alpha \leq \alpha' < \beta \leq \beta'$
(2) $\alpha < \beta' \leq \beta < \alpha'$
(3) $\alpha = \beta \leq \alpha' = \beta'$
(4) $\alpha = \beta \leq \alpha' < \beta'$
(5) $\alpha < \beta \leq \alpha' = \beta'$
(6) $\alpha < \beta \leq \alpha' < \beta'$

TABLE 5.1. A list of the q's created by each swap with the given ordering. Three orderings cancel out while three do not.

Swap	$\alpha \leq \alpha' < \beta \leq \beta'$	$\alpha < \beta' \leq \beta < \alpha'$	$\alpha = \beta \leq \alpha' = \beta'$
$t_{\alpha n}t_{\beta n}$	q	q	-
$t^{-1}_{\alpha' n}t_{\beta n}/x_{\alpha'}t_{\beta n}$	q^{-1}	q	q
$t_{\alpha n}t^{-1}_{\beta' n}$	q^{-1}	q^{-1}	q^{-1}
$t^{-1}_{\alpha' n}t^{-1}_{\beta' n}$	q	q^{-1}	-
$t_{\alpha n}x_{\beta' n}$	-	-	-

Swap	$\alpha = \beta \leq \alpha' < \beta'$	$\alpha < \beta \leq \alpha' = \beta'$	$\alpha < \beta \leq \alpha' < \beta'$
$t_{\alpha n}t_{\beta n}$	$-$	q	q
$t^{-1}_{\alpha' n}t_{\beta n}/x_{\alpha'}t_{\beta n}$	q	q	q
$t_{\alpha n}t^{-1}_{\beta' n}$	q^{-1}	q^{-1}	q^{-1}
$t^{-1}_{\alpha' n}t^{-1}_{\beta' n}$	q	-	q
$t_{\alpha n}x_{\beta' n}$	-	-	-

We can see then that the only cases where the q's don't cancel are:

I $\alpha < \beta \leq \alpha' < \beta'$
II $\alpha < \beta \leq \alpha' = \beta'$
III $\alpha = \beta \leq \alpha' < \beta'$

We will now use these relation types to expedite the analysis of terms. Each relation type will be henceforth referred to using its roman numeral type here. Thus an ac problem of type II will indicate that $a < c \leq a' = c'$. To reintroduce the possibility of $\beta \leq \alpha$ we add inverse types, where a bd problem of type I^{-1} would indicate that $d < b \leq d' < b'$.

5.2. **In Depth Relation 2: $a < b = c < d$.** Now let us take an in-depth look to the case where $a < b = c < d$. Verify the given relation by adding both sides of the equality together:

$$x_{ab}x_{cd} = \sum_{a',b',c',d'} t_{an}t_{a'n}^{-1}x_{a'b'}t_{b'n}^{-1}t_{bn}t_{cn}t_{c'n}^{-1}x_{c'd'}t_{d'n}^{-1}t_{dn} + q\sum_{a',b',c',d'} t_{cn}t_{c'n}^{-1}x_{c'd'}t_{d'n}^{-1}t_{dn}t_{an}t_{a'n}^{-1}x_{a'b'}t_{b'n}^{-1}t_{bn}$$

We will now split these up into cases based on the possible relations between a', b', c' and d'.

Case 1: $a' = c' < b' < d'$

Since we know that $a < b = c < d$ and that $a' < b$, we know that a' is strictly less than all the other prime terms. Thus this case and all other cases where a' is not the smallest are impossible.

Case 2: $a' < b' = c' < d'$

Along with the given relations this gives us the total ordering $a \leq a' < b = c \leq b' = c' < d \leq d'$. In this case none of the problem cases arise in putting the terms in Standard form, the only possible q arises from $x_{a'b'}x_{c'd'}$ which given the relations satisfy Relation 2 we know adds a single q to the left term which is what we wanted in order to now cancel with the equal and opposite term from the right.

Case 3: $a' < c' < b' = d'$

This case gives us the total ordering $a \leq a' < b = c \leq c' < d < b' = d'$, which has a problem case with bc and with bd. The bc is a type III^{-1} problem which creates an extra q^{-1} while the bd is a type II problem which creates an extra q. These cancel leaving us with nothing extra except the q created by the switching of $x_{a'b'}$ and $x_{c'd'}$. A single q being what we wanted to cancel with its equal and opposite term on the right.

Case 4: $a' < c' < d' < b'$

This case creates the total ordering $a \leq a' < b = c \leq c' < d \leq d' < b'$. This ordering has a type II^{-1} problem with bc and no others. Also satisfying Relation 4 means no q is created when the x's are swapped. Thus we have an extra q^{-1} and this term will incompletely cancel leaving a term which looks like:

$$(q^{-1} - q)t_{cn}t_{c'n}^{-1}t_{dn}t_{d'n}^{-1}x_{c'd'}x_{a'b'}t_{a'n}^{-1}t_{an}t_{b'n}^{-1}t_{bn}$$

Case 5: $a' < c' < b' < d'$

This case forks into two possible total orderings:

a) $a \leq a' < b = c \leq c' < d < b' < d'$
b) $a \leq a' < b = c \leq c' < b' < d \leq d'$

Subcase a:

This has a type III^{-1} problem with bc as well as a type I problem with bd, which creates a q^{-1} and a q^2 leaving a single extra q before the swapping of the x's. Since the primes satisfy relation 5, switching the x's breaks it into two terms:

$$qt_{cn}t_{c'n}^{-1}t_{dn}t_{d'n}^{-1}x_{c'd'}x_{a'b'}t_{a'n}^{-1}t_{an}t_{b'n}^{-1}t_{bn}$$

which cancels with the corresponding term on the right and:

$$(q^2 - 1)t_{cn}t_{c'n}^{-1}t_{dn}t_{d'n}^{-1}x_{a'd'}x_{c'b'}t_{a'n}^{-1}t_{an}t_{b'n}^{-1}t_{bn}$$

which remains (and will end up canceling with the left over term from case 4, more on that later).

Subcase b:

This has a type III^{-1} problem with bc and no other problems leaving a q^{-1} before the x swap, which breaks it into two terms:

$$(q^{-1})t_{cn}t_{c'n}^{-1}t_{dn}t_{d'n}^{-1}x_{c'd'}x_{a'b'}t_{a'n}^{-1}t_{an}t_{b'n}^{-1}t_{bn}$$

which will cancel incompletely with the corresponding term from the right to become:

$$(q^{-1} - q)t_{cn}t_{c'n}^{-1}t_{dn}t_{d'n}^{-1}x_{c'd'}x_{a'b'}t_{a'n}^{-1}t_{an}t_{b'n}^{-1}t_{bn}$$

and the second term is:

$$(1 - q^{-2})t_{cn}t_{c'n}^{-1}t_{dn}t_{d'n}^{-1}x_{a'd'}x_{c'b'}t_{a'n}^{-1}t_{an}t_{b'n}^{-1}t_{bn}$$

(later we will show that these will cancel with another left over term from the upcoming case 6).

Case 6: $a' < b' < c' < d'$

This case results in a total ordering of $a \leq a' < b = c \leq b' < c' < d \leq d'$. This time we have a type III problem with bc but no other problems before the x swap. Thus upon adding in the q and breaking it up we get these three terms:

$$(q)t_{cn}t_{c'n}^{-1}t_{dn}t_{d'n}^{-1}x_{c'd'}x_{a'b'}t_{a'n}^{-1}t_{an}t_{b'n}^{-1}t_{bn}$$

which cancels with the corresponding term on the right.

$$(q^2-1)t_{cn}t^{-1}_{c'n}t_{dn}t^{-1}_{d'n}x_{a'c'}x_{b'd'}t^{-1}_{a'n}t_{an}t^{-1}_{b'n}t_{bn}$$

and

$$-(q^3-q)t_{cn}t^{-1}_{c'n}t_{dn}t^{-1}_{d'n}x_{a'd'}x_{b'c'}t^{-1}_{a'n}t_{an}t^{-1}_{b'n}t_{bn}$$

So now let us examine what happens to the left over terms from cases 4, 5 and 6.

If we have a case 4, then we know from the total ordering that $d < b'$, and that $b < d'$. This means that the values of b' and d' appear in another term in switched places. Thus we have one term with:

$$a \le a' < b = c \le c' < d < d'_1 < b'_1$$

and another with:

$$a \le a' < b = c \le c' < d < b'_2 < d'_2$$

where $b'_1 = d'_2$ and $d'_1 = b'_2$.

The same logic applies in the opposite direction so the existence of either one of these terms implies the existence of the other and they must come in these pairs. One of these terms then is a case 4 and the other is a case 5a. Thus from both these terms together we have:

$$(q^{-1}-q)t_{cn}t^{-1}_{c'n}t_{dn}t^{-1}_{d'_1n}x_{c'd'_1}x_{a'b'_1}t^{-1}_{a'n}t_{an}t^{-1}_{b'_1n}t_{bn}$$

and

$$(q^2-1)t_{cn}t^{-1}_{c'n}t_{dn}t^{-1}_{d'_2n}x_{a'd'_2}x_{c'b'_2}t^{-1}_{a'n}t_{an}t^{-1}_{b'_2n}t_{bn}$$

Now use the given equalities to substitute and make the second term be:

$$(q^2-1)t_{cn}t^{-1}_{c'n}t_{dn}t^{-1}_{b'_1n}x_{a'b'_1}x_{c'd'_1}t^{-1}_{a'n}t_{an}t^{-1}_{d'_1n}t_{bn}$$

Since the primes in 1 satisfy relation 4 the x's can be simply commuted. Next we need to switch the positions of $t^{-1}_{b'_1n}$ and $t^{-1}_{d'_1n}$.

Following the creation and cancellation of q's we see that:

$$t^{-1}_{b'_1n}x_{a'b'_1}x_{c'd'_1}t_{an}t^{-1}_{a'n}t^{-1}_{d'_1n} = q^{-1}t^{-1}_{d'_1n}x_{a'b'_1}x_{c'd'_1}t_{an}t^{-1}_{a'n}t^{-1}_{b'_1n}$$

Thus the second term becomes:

$$q^{-1}(q^2-1)t_{cn}t^{-1}_{c'n}t_{dn}t^{-1}_{d'_1n}x_{c'd'_1}x_{a'b'_1}t^{-1}_{a'n}t_{an}t^{-1}_{b'_1n}t_{bn}$$

Which is the equal and opposite term to the first and they cancel leaving only the terms from 5b and 6. By the same logic as before, only this time switching b' with c', a term of type 5b:

$$a \le a' < b = c \le c'_1 < b'_1 < d \le d'$$

will exist if and only if there exists a corresponding term of type 6:

$$a \le a' < b = c \le c'_2 < b'_2 < d \le d'$$

where $c'_1 = d'_2$ and $d'_1 = c'_2$.

Thus we have four leftover terms:

$$(q^{-1}-q)t_{cn}t^{-1}_{c'_1n}t_{dn}t^{-1}_{d'n}x_{c'_1d'}x_{a'b'_1}t^{-1}_{a'n}t_{an}t^{-1}_{b'_1n}t_{bn} \tag{1}$$

$$(1-q^{-2})t_{cn}t^{-1}_{c'_1n}t_{dn}t^{-1}_{d'n}x_{a'd'}x_{c'_1b'_1}t^{-1}_{a'n}t_{an}t^{-1}_{b'_1n}t_{bn} \tag{2}$$

$$(q^2-1)t_{cn}t^{-1}_{c'_2n}t_{dn}t^{-1}_{d'n}x_{a'c'_2}x_{b'_2d'}t^{-1}_{a'n}t_{an}t^{-1}_{b'_2n}t_{bn} \tag{3}$$

$$-(q^3-q)t_{cn}t^{-1}_{c'_2n}t_{dn}t^{-1}_{d'n}x_{a'd'}x_{b'_2c'_2}t^{-1}_{a'n}t_{an}t^{-1}_{b'_2n}t_{bn} \tag{4}$$

which after substitution become

$$(q^{-1}-q)t_{cn}t^{-1}_{c'_1n}t_{dn}t^{-1}_{d'n}x_{c'_1d'}x_{a'b'_1}t^{-1}_{a'n}t_{an}t^{-1}_{b'_1n}t_{bn} \tag{1}$$

$$(1-q^{-2})t_{cn}t^{-1}_{c'_1n}t_{dn}t^{-1}_{d'n}x_{a'd'}x_{c'_1b'_1}t^{-1}_{a'n}t_{an}t^{-1}_{b'_1n}t_{bn} \tag{2}$$

$$(q^{2}-1)t_{cn}t^{-1}_{b'_1n}t_{dn}t^{-1}_{d'n}x_{a'b'_1}x_{c'_1d'}t^{-1}_{a'n}t_{an}t^{-1}_{c'_1n}t_{bn} \tag{3}$$

$$-(q^{3}-q)t_{cn}t^{-1}_{b'_1n}t_{dn}t^{-1}_{d'n}x_{a'd'}x_{c'_1b'_1}t^{-1}_{a'n}t_{an}t^{-1}_{c'_1n}t_{bn} \tag{4}$$

We can see that (1) and (3) resemble each other as do (2) and (4). However to complete the resemblance we still need to swap the x's in (3) as well as certain t's in (3) and (4). After these transformations are acomplished we end up with five terms:

$$\begin{aligned}
(q^{-1}-q)t_{cn}t^{-1}_{c'_1n}t_{dn}t^{-1}_{d'n}x_{c'_1d'}x_{a'b'_1}t^{-1}_{a'n}t_{an}t^{-1}_{b'_1n}t_{bn}\\
(1-q^{-2})t_{cn}t^{-1}_{c'_1n}t_{dn}t^{-1}_{d'n}x_{a'd'}x_{c'_1b'_1}t^{-1}_{a'n}t_{an}t^{-1}_{b'_1n}t_{bn}\\
(q^{-1})(q^{2}-1)t_{cn}t^{-1}_{c'_1n}t_{dn}t^{-1}_{d'n}x_{c'_1d'}x_{a'b'_1}t^{-1}_{a'n}t_{an}t^{-1}_{b'_1n}t_{bn}\\
(q^{-1})(q-q^{-1})(q^{2}-1)t_{cn}t^{-1}_{c'_1n}t_{dn}t^{-1}_{d'n}x_{a'd'}x_{c'_1b'_1}t^{-1}_{a'n}t_{an}t^{-1}_{b'_1n}t_{bn}\\
-(q^{-1})(q^{3}-q)t_{cn}t^{-1}_{c'_1n}t_{dn}t^{-1}_{d'n}x_{a'd'}x_{c'_1b'_1}t^{-1}_{a'n}t_{an}t^{-1}_{b'_1n}t_{bn}
\end{aligned}$$

Now all that remains is to add the coefficeints of like terms:

$$\begin{aligned}
(q^{-1}-q)+(q^{-1})(q^{2}-1) &= 0\\
(1-q^{-2})+(q^{-1})(q-q^{-1})(q^{2}-1)-(q^{-1})(q^{3}-q) &=\\
&= (1-q^{-2})+(1-q^{-2})(q^{2}-1)-(q^{2}-1)\\
&= (1-q^{-2})+(q^{2}-1-1+q^{-2})-(q^{2}-1)\\
&= 0
\end{aligned}$$

Thus all terms cancel within our original sum and we now have that

$$x_{ab}x_{cd}-qx_{cd}x_{ab}=0,$$

which proves our second relation.

5.3. **Pairing Cancellations.** In order to prove all the relations we will have to do this same process for every possible ordering in every relation. Unfortunately no other relation has as few cases as the second. Thus we will not be going over every possible case, only describing the necessary steps for proving that two terms cancel out. Tables describing all possible terms and which terms they cancel with are provided.

Some of the terms will cancel completely with their corresponding terms with the same ordering, like cases 2 and 3 in relation 2. Others will need pair terms in order to finish canceling like cases 4, 5, and 6. We describe that process here.

If two primes are not equal but are next to each other in the ordering, then they have the same relation to all other elements in the order with the possible exception of one being equal to a non-prime. In this case there must exist another term with the same ordering but with the values of the those two primes switched. We must examine these terms together. When placed in standard form the two terms will differ by q^2 since either we have a Type I problem and no problem or a type III problem and a type III^{-1} problem. Let the term with $\gamma' \in \{a', b'\} \leq \delta' \in \{c', d'\}$ have a coefficient of α which makes its paired term have a coefficient of αq^{-2}. Next commute the x's in both terms and subtract the corresponding term. Now substitute the values of the primes in the first term for the values in the second. That is, we know that $\gamma_1 = \delta_2$ and $\gamma_2 = \delta_1$ so make all the primes have matching subscripts. In order to get it back into standard form we must move the two swapped primes in opposite directions. Since they have the same relations to all other elements, any q created by the passing of one will cancel with the q^{-1} created by the passing of the other in the opposite direction. The only possible problem is if one is equal to a non-prime, however the non-prime which it is equal to will not be one of the elements with which they need to switch and thus is not a problem. The only q left over will be from the passing of the two primes themselves, which will create a q. Now, if needed, swap the x's again to match with the other term. Finally add up the coefficients.

Using these steps we find the formulae for the left over terms for each swappable case, described in Table 5.2.

TABLE 5.2. With a given ordering of the primed indices, the terms left over after a term with that ordering has been pair canceled, along with the α requirement for complete cancelation. Where α is as described above and β is the coefficient of the corresponding term on the right side of the relation. The subscripts indicate which primes are being swapped.

Primed Ordering	Leftover Terms	Canceling Requirement
$a' = c' < b'_1 < d'_1$	$(\alpha q^{-1} - \beta + \alpha q - \beta q^2) SF(x_{c'd'} x_{a'b'})$	$\alpha = \beta q$
$a'_1 < c'_1 < b' = d'$	$(\alpha q^{-1} - \beta + \alpha q^{-3} - \beta q^2) SF(x_{c'd'} x_{a'b'})$	$\alpha = \beta q$
$a'_1 < c'_1 < b'_1 < d'_1$	$(\alpha - \beta + \alpha q^{-2} - \beta q^2) SF(x_{c'd'} x_{a'b'})$ $(\alpha q + \beta q - \alpha q^{-1} - \beta q^3) SF(x_{a'd'} x_{c'b'})$	$\alpha = \beta q^2$
$a'_1 < c'_1 < d' < b'$	$(\alpha q^{-2} - \beta) SF(x_{c'd'} x_{a'b'})$ $(\alpha q^{-2} - \beta) SF(x_{a'd'} x_{c'b'})$	$\alpha = \beta q^2$
$c' < a'_1 < d'_1 < b'$	$(\alpha q^{-2} - \beta) SF(x_{c'd'} x_{a'b'})$ $(\alpha q^{-2} - \beta) SF(x_{c'a'} x_{d'b'})$	$\alpha = \beta q^2$
$c' < a' < b'_1 < d'_1$	$(\alpha q^{-2} - \beta) SF(x_{c'd'} x_{a'b'})$ $(\alpha q^{-1} - \beta q) SF(x_{c'b'} x_{a'd'})$	$\alpha = \beta q^2$
$a' < b'_1 < c'_1 < d'$	$(\alpha - \beta) SF(x_{c'd'} x_{a'b'})$ $(\alpha q - \beta q) SF(x_{a'c'} x_{b'd'})$ $(\alpha - \beta + \alpha q^2 - \beta q^2) SF(x_{a'd'} x_{b'c'})$	$\alpha = \beta$
$a' < c' < b'_1 < d'_1$	$(\alpha - \beta) SF(x_{c'd'} x_{a'b'})$ $(\alpha q - \beta q) SF(x_{a'd'} x_{c'b'})$	$\alpha = \beta$
$a'_1 < c'_1 < b' < d'$	$(\alpha - \beta) SF(x_{a'd'} x_{c'b'})$ $(\alpha q - \beta q) SF(x_{c'd'} x_{a'b'})$	$\alpha = \beta$
$a'_1 < c'_1 < d'_1 < b'_1$	$(2\alpha - 2\beta) SF(x_{c'd'} x_{a'b'})$	$\alpha = \beta$

6. Cancellation Cases

6.1. **Relations 1,2,3, and 4.** Unfortunately in order to check the relations we need to check all possible orderings of a, b, c, d and their primes. We use the same methods used in the in-depth checking of Relation 2 and those described in the pair cancellation section in order to pair up and cancel out all terms. Tables 6.1 through 6.4 give an outline of which terms match up and how for relation 1 through 4.

TABLE 6.1. Relation 1: $a = c < b < d$. A list of all possible orderings when a, b, c and d are in relation 1 and how they cancel out individually or in pairs.

Ordering		
Non-Paired Terms	Left Term	Right Term
$a = c \le a' = c' < b < d \le d' = b'$	$qSF(x_{c'd'}x_{a'b'})$	$qSF(x_{c'd'}x_{a'b'})$
$a = c \le a' = c' < b \le b' < d \le d'$	$qSF(x_{c'd'}x_{a'b'})$	$qSF(x_{c'd'}x_{a'b'})$
$a = c \le a' < b \le b' = c' < d \le d'$	$qSF(x_{c'd'}x_{a'b'})$	$qSF(x_{c'd'}x_{a'b'})$
$a = c \le a' < b \le c' < d \le b' = d'$	$qSF(x_{c'd'}x_{a'b'})$	$qSF(x_{c'd'}x_{a'b'})$
Paired Terms	α	Required α
$a = c \le a' < c' < b < d \le d' < b'$ $a = c \le c' < a' < b < d \le b' < d'$	q^3	q^3
$a = c \le a' < c' < b < d \le b' = d'$ $a = c \le c' < a' < b < d \le b' = d'$	q^2	q^2
$a = c \le a' < c' < b < d \le b' < d'$ $a = c \le c' < a' < b < d \le d' < b'$	q	q
$a = c \le a' = c' < b < d \le b' < d'$ $a = c \le a' = c' < b < d \le d' < b'$	q^2	q^2
$a = c \le a' < c' < b \le b' < d \le d'$ $a = c \le c' < a' < b \le b' < d \le d'$	q	q
$a = c \le a' < b \le b' < c' < d \le d'$ $a = c \le a' < b \le c' < b' < d \le d'$	q	q
$a = c \le a' < b \le c' < d \le b' < d'$ $a = c \le a' < b \le c' < d \le d' < b'$	q	q

TABLE 6.2. Relation 2: $a < b = c < d$. A list of all possible orderings when a, b, c and d are in relation 2 and how they cancel out individually or in pairs.

Ordering		
Non-Paired Terms	Left Term	Right Term
$a \le a' < b = c \le b' = c' < d \le d'$	$qSF(x_{c'd'}x_{a'b'})$	$qSF(x_{c'd'}x_{a'b'})$
$a \le a' < b = c \le c' < d \le d' = b'$	$qSF(x_{c'd'}x_{a'b'})$	$qSF(x_{c'd'}x_{a'b'})$
Paired Terms	α	Required α
$a \le a' < b = c \le c' < d \le b' < d'$ $a \le a' < b = c \le c' < d \le d' < b'$	q	q
$a \le a' < b = c \le b' < c' < d \le d'$ $a \le a' < b = c \le c' < b' < d \le d'$	q	q

TABLE 6.3. Relation 3: $a < c < b = d$. A list of all possible orderings when a, b, c and d are in relation 3 and how they cancel out individually or in pairs.

Ordering		
Non-Paired Terms	Left Term	Right Term
$a \leq a' < c \leq c' < b = d \leq b' = d'$	$qSF(x_{c'd'}x_{a'b'})$	$qSF(x_{c'd'}x_{a'b'})$
Paired Terms	α	Required α
$a < c \leq a' = c' < b = d \leq b' < d'$ $a < c \leq a' = c' < b = d \leq d' < b'$	q^2	q^2
$a < c \leq a' < c' < b = d \leq b' = d'$ $a < c \leq c' < a' < b = d \leq b' = d'$	q^2	q^2
$a \leq a' < c \leq c' < b = d \leq b' < d'$ $a \leq a' < c \leq c' < b = d \leq d' < b'$	q	q
$a < c \leq a' < c' < b = d \leq d' < b'$ $a < c \leq c' < a' < b = d \leq b' < d'$	q	q
$a < c \leq a' < c' < b = d \leq b' < d'$ $a < c \leq c' < a' < b = d \leq d' < b'$	q^3	q^3

TABLE 6.4. Relation 4: $a < c < d < b$. A list of all possible orderings when a, b, c and d are in relation 4 and how they cancel out individually or in pairs.

Ordering		
Non-Paired Terms	Left Term	Right Term
$a < c \le a' = c' < d < b \le d' = b'$	$qSF(x_{c'd'}x_{a'b'})$	$qSF(x_{c'd'}x_{a'b'})$
$a < c \le a' = c' < d \le d' < b \le b'$	$qSF(x_{c'd'}x_{a'b'})$	$qSF(x_{c'd'}x_{a'b'})$
$a < c \le c' < d \le a' = d' < b \le b'$	$qSF(x_{c'd'}x_{a'b'})$	$qSF(x_{c'd'}x_{a'b'})$
$a \le a' < c \le c' < d < b \le b' = d'$	$qSF(x_{c'd'}x_{a'b'})$	$qSF(x_{c'd'}x_{a'b'})$
$a < c \le c' < d \le a' < b \le b' = d'$	$qSF(x_{c'd'}x_{a'b'})$	$qSF(x_{c'd'}x_{a'b'})$
$a \le a' < c \le c' < d \le d' < b \le b'$	$qSF(x_{c'd'}x_{a'b'})$	$qSF(x_{c'd'}x_{a'b'})$
Paired Terms	α	Required α
$a < c \le a' = c' < d < b \le b' < d'$ $a < c \le a' = c' < d < b \le d' < b'$	q	q
$a < c \le a' < c' < d < b \le b' = d'$ $a < c \le c' < a' < d < b \le b' = d'$	q	q
$a \le a' < c \le c' < d < b \le b' < d'$ $a \le a' < c \le c' < d < b \le d' < b'$	1	1
$a < c \le a' < c' < d \le d' < b \le b'$ $a < c \le c' < a' < d \le d' < b \le b'$	q^2	q^2
$a < c \le a' < c' < d < b \le b' < d'$ $a < c \le c' < a' < d < b \le d' < b'$	q^2	q^2
$a < c \le a' < c' < d < b \le d' < b'$ $a < c \le c' < a' < d < b \le b' < d'$	1	1
$a < c \le c' < d \le a' < b \le b' < d'$ $a < c \le c' < d \le a' < b \le d' < b'$	q^2	q^2
$a < c \le c' < d \le a' < d' < b \le b'$ $a < c \le c' < d \le d' < a' < b \le b'$	q^2	q^2

6.2. **Relations 5 and 6.** For relations 5 and 6, we note that the right side of the relation has multiple terms, thus the cancellation rules must take the extra term into account. The first step is to find the rules for putting the extra term into standard form.

For relation 5 the original form of the new term is:

$$t_{an}t_{a'n}^{-1}x_{a'd'}t_{d'n}^{-1}t_{dn}t_{cn}t_{c'n}^{-1}x_{c'b'}t_{b'n}^{-1}t_{bn}$$

Thus to get it into standard form we need to make these swaps:

(1)	$t_{a'n}^{-1}t_{dn}$	(9)	$t_{an}x_{a'd'}$	(17)	$t_{a'n}^{-1}t_{c'n}^{-1}$
(2)	$t_{an}t_{d'n}^{-1}$	(10)	$x_{a'd'}t_{dn}$	(18)	$t_{a'n}t_{cn}$
(3)	$t_{a'n}^{-1}t_{d'n}^{-1}$	(11)	$t_{dn}t_{c'n}^{-1}$	(19)	$t_{an}t_{c'n}^{-1}$
(4)	$t_{an}t_{dn}$	(12)	$t_{d'n}^{-1}t_{cn}$	(20)	$t_{a'n}^{-1}t_{cn}$
(5)	$t_{a'n}^{-1}x_{a'd'}$	(13)	$t_{d'n}^{-1}t_{c'n}^{-1}$	(21)	$t_{a'n}^{-1}x_{c'b'}$
(6)	$x_{a'd'}t_{d'n}^{-1}$	(14)	$t_{dn}t_{cn}$	(22)	$x_{a'd'}t_{c'n}^{-1}$
(7)	$t_{d'n}^{-1}t_{dn}$	(15)	$t_{an}x_{c'b'}$		
(8)	$t_{an}t_{a'n}^{-1}$	(16)	$x_{a'd'}t_{cn}$		

Since for this term $a \le a' < d \le d'$, (1)-(4) will cancel. Similarly to what we have seen previously (5) and (6) will cancel with each other, and (7) through (10) will cancel. Since in this term $c' < b < d$, (11) through

(14) will cancel. (15) through (20) form one of the groups we outlined before, thus we can describe the q's created by looking at the formation of a, a', c and c'. Finally we note that the last two terms, (21) and (22), will cancel except in the case of $a' = b'$ when they create a q. Thus we can now check what happends to each possible ordering in relation 5, taking both the first and the second term on the right into account. This is done in Tables 6.5 through 6.11.

TABLE 6.5. The set of terms which will cancel with the corresponding first term on the left.

Single First Only Terms	Left Term	First Right Term
$a \le a' < c < b \le b' = c' < d \le d'$	$SF(x_{c'd'}x_{a'b'})$	$SF(x_{c'd'}x_{a'b'})$
$a \le a' < c < b \le c' < d \le d' = b'$	$SF(x_{c'd'}x_{a'b'})$	$SF(x_{c'd'}x_{a'b'})$

TABLE 6.6. The set of terms which pair cancel with the corresponding pair on the first right.

Paired First Only Terms	α	Required α
$a \le a' < c < b \le c' < d \le b' < d'$ $a \le a' < c < b \le c' < d \le d' < b'$	1	1
$a \le a' < c < b \le b' < c' < d \le d'$ $a \le a' < c < b \le c' < b' < d \le d'$	1	1

TABLE 6.7. The set of terms which cancel individually but need both the first and second term on the right to do it. In some cases the equality of certain primes is used.

Single 1st and 2nd Term	Left Term	First Right Term	Second Right Term
$a < c \le a' = c' < b < d \le b' = d'$	$q^2 SF(x_{c'd'}x_{a'b'})$	$SF(x_{c'd'}x_{a'b'})$	$(q^2-1)SF(x_{c'd'}x_{a'b'})$
$a < c \le a' = c' < b \le b' < d \le d'$	$q^2 SF(x_{c'd'}x_{a'b'})$	$SF(x_{c'd'}x_{a'b'})$	$(q^2-1)SF(x_{c'd'}x_{a'b'})$
$a \le a' < c \le c' < b < d \le d' = b'$	$q^2 SF(x_{c'd'}x_{a'b'})$	$SF(x_{c'd'}x_{a'b'})$	$(q^2-1)SF(x_{c'd'}x_{a'b'})$
$a \le a' < c \le c' < b \le b' < d \le d'$	$SF(x_{c'd'}x_{a'b'})$ $(q-q^{-1})SF(x_{a'd'}x_{c'b'})$	$SF(x_{c'd'}x_{a'b'})$	$(q-q^{-1})SF(x_{c'd'}x_{a'b'})$

TABLE 6.8. The set of terms which are paired orderings which do not meet the α requirement to cancel with the first right pair terms alone but which have their remainder terms cancel with the paired second term.

Pair 1st and 2nd Term	α	Remainder Term	Second Right Term
$a < c \leq a' = c' < b < d \leq b' < d'$ $a < c \leq a' = c' < b < d \leq d' < b'$	q^3	$(1+q^4)SF(x_{c'd'}x_{a'b'})$	$(1+q^4)SF(x_{c'd'}x_{a'b'})$
$a < c \leq a' < c' < b < d \leq d' = b'$ $a < c \leq c' < a' < b < d \leq d' = b'$	q^3	$(1+q^4)SF(x_{c'd'}x_{a'b'})$	$(1+q^4)SF(x_{c'd'}x_{a'b'})$
$a \leq a' < c \leq c' < b < d \leq b' < d'$ $a \leq a' < c \leq c' < b < d \leq d' < b'$	q^2	$(q^2-1)SF(x_{c'd'}x_{a'b'})$ $(q^3-q)SF(x_{a'd'}x_{c'b'})$	$(q^2-1)SF(x_{c'd'}x_{a'b'})$ $(q^3-q)SF(x_{a'd'}x_{c'b'})$
$a < c \leq a' < c' < b \leq b' < d \leq d'$ $a < c \leq c' < a' < b \leq b' < d \leq d'$	q^2	$(q^3-q)SF(x_{c'd'}x_{a'b'})$ $(q^2-1)SF(x_{a'd'}x_{c'd'})$	$(q^3-q)SF(x_{c'd'}x_{a'b'})$ $(q^2-1)SF(x_{a'd'}x_{c'd'})$

TABLE 6.9. Four separate orderings which must be considered together to cancel.

Four-way Cancel	Left Term	1st Term	2nd Term
$a < c \leq a' < c' < b < d \leq b' < d'$	$q^4SF(x_{c'd'}x_{a'b'})$ $(q^5-q^3)SF(x_{c'b'}x_{a'd'})$	$SF(x_{c'd'}x_{a'b'})$	$(q^3-q)SF(x_{c'b'}x_{a'd'})$
$a < c \leq a' < c' < b < d \leq d' < b'$	$q^3SF(x_{c'b'}x_{a'd'})$	$qSF(x_{c'b'}x_{a'd'})$	$(q^5-q^3-q^3-q)SF(x_{c'b'}x_{a'd'})$ $(q^4-q^2)SF(x_{c'd'}x_{a'b'})$
$a < c \leq c' < a' < b < d \leq b' < d'$	$qSF(x_{c'b'}x_{a'd'})$	$q^3SF(x_{c'b'}x_{a'd'})$	$(q^2-1)SF(x_{c'd'}x_{a'b'})$
$a < c \leq c' < a' < b < d \leq d' < b'$	$q^2SF(x_{c'd'}x_{a'b'})$ $(q^3-q)SF(x_{c'b'}x_{a'd'})$	$q^2SF(x_{c'd'}x_{a'b'})$	$(q^3-q)SF(x_{c'b'}x_{a'd'})$

TABLE 6.10. The set of terms which do not cancel with the first term and have no direct second term to cancel with, but which can cancel with the term found in the second term if a' and c' are switched (a q is added in the process just like the pair cancellation switch they have the same relation to all the elements they need to swap with).

a', c' Swap Singles	Left Term	1st Term	Swapped 2nd Term
$a < c \leq a' < b \leq b' = c' < d \leq d'$ $(a < c \leq c' < b \leq b' = a' < d \leq d')$	$q^2SF(x_{c'd'}x_{a'b'})$	$SF(x_{c'd'}x_{a'b'})$	$(q^2-1)SF(x_{c'd'}x_{a'b'})$
$a < c \leq a' < b \leq c' < d \leq d' = b'$ $(a < c \leq c' < b \leq a' < d \leq d' = b')$	$q^2SF(x_{c'd'}x_{a'b'})$	$SF(x_{c'd'}x_{a'b'})$	$(q^2-1)SF(x_{c'd'}x_{a'b'})$

TABLE 6.11. The pair cancellation version of Table 6.10.

a', c' Swap Pairs	α	Remainder Term	Swapped 2nd Term
$a < c \leq a' < b \leq c' < d \leq b' < d'$ $a < c \leq a' < b \leq c' < d \leq d' < b'$ $(a < c \leq c' < b \leq a' < d \leq d' < b')$ $(a < c \leq c' < b \leq a' < d \leq b' < d')$	q^2	$(q^2-1)SF(x_{c'd'}x_{a'b'})$ $(q^3-q)SF(x_{a'd'}x_{c'b'})$	$(q^2-1)SF(x_{c'd'}x_{a'b'})$ $(q^3-q)SF(x_{a'd'}x_{c'b'})$
$a < c \leq a' < b \leq b' < c' < d \leq d'$ $a < c \leq a' < b \leq c' < b' < d \leq d'$ $(a < c \leq c' < b \leq b' < a' < d \leq d')$ $(a < c \leq c' < b \leq a' < b' < d \leq d')$	q^2	$(q^2-1)SF(x_{c'd'}x_{a'b'})$ $(q^3-q)SF(x_{a'd'}x_{c'b'})$	$(q^2-1)SF(x_{c'd'}x_{a'b'})$ $(q^3-q)SF(x_{a'd'}x_{c'b'})$

In relation 6 we notice that there are in fact three terms to be aware of. We use the same process as in relation 5 to determine that the second term can be placed in Standard Form with a q created when $a' = b'$ or $c' = d$ and a q^{-1} when $c' = d'$. The third term can be placed in Standard Form by looking at the relations of a, c, a', c' and d, c, d', c' in addition a q^{-1} is created when $d' = c'$ and q is created when $a' = b'$ or $c' = d$. This last q is a possible problem but can ignored when comparing a second and a third term since both will have it. It can also be ignored when pair and swap comparing since when we substitute into the same case the q will work out to being the same created in both terms. Thus we are ready to check all possible terms for relation 6 in Talbes 6.12 through 6.20.

Table 6.12. One of the terms from the left side will cancel singly with the three corresponding terms on the right.

Ordering	Left Term	Right Term	
$a \leq a' < b \leq b' < c \leq c' < d \leq d'$	$SF(x_{c'd'}x_{a'b'})$ $(q - q^{-1})SF(x_{a'c'}x_{b'd'})$ $-(q^2 - 1)SF(x_{a'd'}x_{b'c'})$	$SF(x_{c'd'}x_{a'b'})$ $(q - q^{-1})SF(x_{a'c'}x_{b'd'})$ $-(q^2 - 1)SF(x_{a'd'}x_{b'c'})$	1st Term 2nd Term 3rd Term

Table 6.13. The term on the left side which will cancel with the corresponding first and second terms on the left.

Single Canceling	Left Term	1st	2nd
$a \leq a' < b < c \leq b' = c' < d \leq d'$	$q^2SF(x_{c'd'}x_{a'b'})$	$SF(x_{c'd'}x_{a'b'})$	$(q^2 - 1)SF(x_{c'd'}x_{a'b'})$

Table 6.14. The terms which will pair cancel with the first and the remaining term will cancel with the corresponding 2nd term ordering.

Pair Canceling	α	Remainder Term	2nd
$a \leq a' < b < c \leq b' < c' < d \leq d'$ $a \leq a' < b < c \leq c' < b' < d \leq d'$	q^2	$(q^2 - 1)SF(x_{c'd'}x_{a'b'})$ $(q^3 - q)SF(x_{a'c'}x_{b'd'})$ $(q^4 - 1)SF(x_{a'd'}x_{b'c'})$	$(q^2 - 1)SF(x_{c'd'}x_{a'b'})$ $(q^3 - q)SF(x_{a'c'}x_{b'd'})$ $(q^4 - 1)SF(x_{a'd'}x_{b'c'})$

Table 6.15. The final three orderings of the left term need to swap b' and c' in order to match up with orderings in the second term. In this case when we swap the positions of b' and c', since they are not in the same relative positions to all the swapping elements the q creation must be tracked manually.

Single Swap Canceling	Left Term	1st	Swapped 2nd
$a \leq a' < b < c \leq c' < d \leq d' = b'$ $(a \leq a' < b < c \leq b' < d \leq d' = c')$	$q^2SF(x_{c'd'}x_{a'b'})$	$SF(x_{c'd'}x_{a'b'})$	$(q^2 - 1)SF(x_{c'd'}x_{a'b'})$

Table 6.16. The pair cancelation version of Table 6.15.

Pair Swap Canceling	α	Remainder Term	2nd
$a \leq a' < b < c \leq c' < d \leq b' < d'$ $a \leq a' < b < c \leq c' < d \leq d' < b'$ $(a \leq a' < b < c \leq b' < d \leq d' < c')$ $(a \leq a' < b < c \leq b' < d \leq c' < d')$	q^2	$(q^2 - 1)SF(x_{c'd'}x_{a'b'})$ $(q^3 - q)SF(x_{a'd'}x_{c'b'})$	$(q^2 - 1)SF(x_{c'd'}x_{a'b'})$ $(q^3 - q)SF(x_{a'd'}x_{c'b'})$

Now we note that we have canceled out all the orderings from the left term but have not yet dealt with all the orderings from the two extra terms on the right. These extra terms will cancel with each other in a very similar way to what we have seen before.

TABLE 6.17. Terms where the last two terms on the right cancel with each other.

Single 2nd 3rd Term Canceling	2nd	3rd
$a < b \leq a' = b' < c \leq c' < d \leq d'$	$(q^2 - 1)SF(x_{a'c'}x_{b'd'})$	$-(q^2 - 1)SF(x_{a'c'}x_{b'd'})$
$a \leq a' < b \leq b' < c < d \leq d' = c'$	$(1 - q^{-2})SF(x_{a'c'}x_{b'd'})$	$-(1 - q^{-2})SF(x_{a'c'}x_{b'd'})$
$a < b \leq a' < b' < c < d \leq d' = c'$	$(1 - q^{-2})SF(x_{a'c'}x_{b'd'})$	$-(1 - q^{-2})SF(x_{a'c'}x_{b'd'})$
$a < b \leq b' < a' < c < d \leq d' = c'$	$(1 - q^{-2})SF(x_{a'c'}x_{b'd'})$	$-(1 - q^{-2})SF(x_{a'c'}x_{b'd'})$
$a < b \leq a' = b' < c < d \leq d' = c'$	$(q - q^{-1})SF(x_{a'c'}x_{b'd'})$	$-(q - q^{-1})SF(x_{a'c'}x_{b'd'})$

TABLE 6.18. Terms where the last two terms on the right cancel with each other in pairs.

Pair 2nd 3rd Term Canceling	2nd	3rd
$a < b \leq a' < b' < c \leq c' < d \leq d'$	$(q - q^{-1})SF(x_{a'c'}x_{b'd'})$	$-(q^2 - 1)SF(x_{a'd'}x_{b'c'})$
$a < b \leq b' < a' < c \leq c' < d \leq d'$	$(1 - q^{-2})SF(x_{a'd'}x_{b'c'})$	$-(q - q^{-1})SF(x_{a'c'}x_{b'd'})$
		$(q^2 - 2 + q^{-2})SF(x_{a'd'}x_{b'c'})$
$a < b \leq a' = b' < c < d \leq d' < c'$	$(q - q^{-1})SF(x_{a'd'}x_{c'b'})$	$-(q^3 - q)SF(x_{a'd'}x_{c'd'})$
$a < b \leq a' = b' < c < d \leq c' < d'$	$(q^3 - q)SF(x_{a'd'}x_{c'b'})$	$-(q - q^{-1})SF(x_{a'd'}x_{c'd'})$
$a \leq a' < b \leq b' < c < d \leq d' < c'$	$(q - q^{-1})SF(x_{a'c'}x_{b'd'})$	$-(q^2 - q)SF(x_{a'd'}x_{b'c'})$
$a \leq a' < b \leq b' < c < d \leq c' < d'$	$(q^2 - q)SF(x_{a'd'}x_{b'c'})$	$-(q - q^{-1})SF(x_{a'c'}x_{b'd'})$
$a < b \leq a' < b' < c < d \leq d' < c'$	$(q^2 - 1)SF(x_{b'c'}x_{a'd'})$	$-(q^3 - q)SF(x_{b'd'}x_{a'c'})$
$a < b \leq a' < b' < c < d \leq c' < d'$	$(q^3 - q)SF(x_{b'd'}x_{a'c'})$	$-(q^2 - 1)SF(x_{b'c'}x_{a'd'})$
$a < b \leq b' < a' < c < d \leq d' < c'$	$(q - q^{-1})SF(x_{a'c'}x_{b'd'})$	$-(q^2 - 1)SF(x_{a'd'}x_{b'c'})$
$a < b \leq b' < a' < c < d \leq c' < d'$	$(q^2 - 1)SF(x_{a'd'}x_{b'c'})$	$-(q - q^{-1})SF(x_{a'c'}x_{b'd'})$

TABLE 6.19. The final group of orderings, in order to match up orderings from both the second and the third term need to switch a' and b'. When we switch these two we want b' in front of a' as it is in the second term. Getting into this positioning will create an extra q^{-1} in the third term.

Single Swap Cancel 2nd and 3rd	2nd	3rd
$a < b \leq a' < c \leq c' = b' < d \leq d'$	$(q - q^{-1})SF(x_{a'c'}x_{b'd'})$	
$(a < b \leq b' < c \leq a' = c' < d \leq d')$		$-(q - q^{-1})SF(x_{a'c'}x_{b'd'})$
$a < b \leq a' < c \leq b' < d \leq d' = c'$	$(1 - q^{-2})SF(x_{a'c'}x_{b'd'})$	
$(a < b \leq b' < c \leq a' < d \leq d' = c')$		$-(1 - q^{-2})SF(x_{a'c'}x_{b'd'})$

TABLE 6.20. The pair version of Table 6.19.

Pair Swap Cancel	2nd	3rd
$a < b \le a' < c \le b' < c' < d \le d'$	$(q - q^{-1})SF(x_{a'c'}x_{b'd'})$	
$a < b \le a' < c \le c' < b' < d \le d'$	$(q^2 - 1)SF(x_{a'b'}x_{c'd'})$	
$(a < b \le b' < c \le a' < c' < d \le d')$		$-(q^3 - q)SF(x_{a'c'}x_{b'd'})$
		$(q^4 - q^{-2} - q^{-2} + 1)SF(x_{a'd'}x_{b'c'})$
$(a < b \le b' < c \le c' < a' < d \le d')$		$(q^3 - q - q + q^{-1})SF(x_{a'c'}x_{b'd'})$
		$-(q^2 - 1)SF(x_{a'b'}x_{c'd'})$
		$-(q^4 - q^{-2} - q^{-2} + 1)SF(x_{a'd'}x_{b'c'})$
$a < b \le a' < c \le b' < d \le c' < d'$	$(q - q^{-1})SF(x_{b'd'}x_{a'c'})$	
	$(q^2 - 2 + q^{-2})SF(x_{a'd'}x_{b'c'})$	
$a < b \le a' < c \le b' < d \le d' < c'$	$(1 - q^{-2})SF(x_{a'd'}x_{b'c'})$	
$(a < b \le b' < c \le a' < d \le d' < c')$		$-(q^2 - 1)SF(x_{a'd'}x_{c'd'})$
$(a < b \le b' < c \le a' < d \le c' < d')$		$-(q - q^{-1})SF(x_{b'd'}x_{a'c'})$

We have now shown that the generators of A_n satisfy all of the relations of $\mathcal{O}_q(Sk_n)$. This shows us that ϕ is well-defined, and therefore is an isomorphism. So now, we have a nice combinatorial model for $\mathcal{O}_q(Sk_n)$. This is useful for many applications, such as studying the spectrum of prime ideals.

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