Final Exam
Math 145, UCSB, Fall '07

Exercise 1. Let $\{x_n\}$ and $\{y_n\}$ be two Cauchy sequences in a complete metric space X, with distance d. Prove that they have the same limit if and only if

$$\lim_{n \to +\infty} d(x_n, y_n) = 0.$$

Exercise 2.

Determine which of the following functions are continuous (justify your answer):

1. Let X and Y be topological spaces, let $y_0 \in Y$ be a fixed point and let $f : X \to Y$ defined as $f(x) = y_0$ for any $x \in X$.
2. Let $X = \mathbb{R}$ with the cofinite topology and let $f : X \to X$ defined as $f(x) = \sin(x)$, for any $x \in X$.
3. Let $X = \mathbb{R}$ with the cofinite topology and let $Y = \mathbb{R}$ with the usual topology. Let $f : X \to Y$ defined as $f(x) = 2 \cdot x$, for any $x \in X$.

Exercise 3.

Determine which of the following topological spaces is connected (justify your answer):

1. The set of real numbers $\mathbb{R}$ with the confinite topology;
2. The subspace S of $\mathbb{R}^2$ with the usual topology, defined as
$$S = \{(x, y) | x \neq 0\};$$
3. The set of rational numbers $\mathbb{Q}$ with the usual topology;
4. **(Extra)** The cylinder defined by the equations $x^2 + y^2 = 1$ inside $\mathbb{R}^3$ with the usual topology.

Exercise 4.

Let X and Y be topological space.

1. Assume that $X \times Y$ is compact. Show that X is also compact.
2. Assume that $X \times Y$ is connected. Show that X is also connected.
3. Assume that $X \times Y$ is Hausdorff. Show that X is also Hausdorff.

Exercise 5.

Let $\{x_n\}$ be a sequence in a topological space X which converges to $x \in X$ and let $\{y_n\}$ be a sequence in a topological space Y which converges to $y \in Y$. Show that $z_n = (x_n, y_n)$ converges to (x, y) in $X \times Y$.

Exercise 6.

Determine if the following pairs of topological spaces are homeomorphic within each other (justify your answer):

1. $[0, 1]$ with the usual topology and $\mathbb{R}$ with the usual topology;
2. $\mathbb{Q}$ with the usual topology and $\mathbb{R}$ with the usual topology;
3. $\mathbb{R}^2$ with the usual topology and the circle $\{(x, y)|x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$ with the usual topology;
4. **(Extra)** $[0, 1]$ with the discrete topology and $\mathbb{R}$ with the discrete topology.