SESHADRI CONSTANTS ON SMOOTH THREEFOLDS

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ABSTRACT. We prove that the Seshadri constant of an ample line bundle at a very general point of a smooth projective threefold is at least $2/3$. While falling short of the conjectured lower bound of one, this improves significantly on known results. More importantly, we systematically exploit new results and ideas related to the variation and complexity of base loci.

1. INTRODUCTION

Originally introduced by Demailly [D] in studying local analytic positivity of ample line bundles, Seshadri constants have since been studied in many contexts and now constitute an important tool for understanding many geometric problems: the reader can consult [L1] Chapter V for a survey of results and techniques.

In general, there tend to be two approaches to studying Seshadri constants. On the one hand, explicit computations can be made for certain special varieties: the work of Bauer, Szemberg, and Garcia [B, BS, G] gives a good sense of this side of the story. On the other hand, lower bounds valid on any variety can be found [EL, EKL, N] but only at a very general point. We mention also work of Hwang and Keum [HK] which shows how Seshadri constants can govern the global geometry of a projective variety. Our interest in Seshadri constants is thus motivated by the surprisingly rich source of geometric problems and properties hidden inside these seemingly simple numbers.

In this paper we address the problem of bounding from below the Seshadri constant of an ample line bundle at a very general point of a smooth threefold, improving the bounds of [EKL, C, N] while falling short of the conjectured best bound [L1] Conjecture 5.2.4. Interestingly, our method of proof combines general ideas of positivity, also used in [EKL], with a close analysis of special geometric situations which is similar in flavor to [B, BS].

There are many equivalent ways to define Seshadri constants:

Definition 1.1. *Suppose X is a smooth projective variety, A an ample line bundle on X, and $x \in X$ a point. Let $\pi : Y \to X$ be the blow–up* *of X at x with exceptional divisor E. Then the Seshadri constant of A at x is defined by*

$$\epsilon(x, A) = \sup_{\alpha \in \mathbf{Q}} \{\pi^*(A)(-\alpha E) \text{ is nef}\}.$$

Alternatively, the Seshadri constant can be defined in terms of curves on X:

$$\epsilon(x, A) = \inf_{C \ni x} \frac{\deg_A(C)}{\mathrm{mult}_x(C)}$$

where the infimum runs over all integral curves $C \subset X$ passing through x. The equivalence of the two definitions is found in [L1] Proposition 5.1.5. Finally, in [L1] Theorem 5.1.17 it is established that both definitions are equivalent to measuring the jets separated by the linear series $|kA|$ at x, asymptotically with respect to k. An irreducible subvariety $V \subset X$, of positive dimension, satisfying

$$\left(\frac{\deg_A(V)}{\mathrm{mult}_x(V)}\right)^{\frac{1}{\dim(V)}} = \epsilon(x, A)$$

is called *Seshadri exceptional* at x relative to A provided V is not properly contained in a larger subvariety satisfying the same equality.

We can now state our main theorem:

Theorem 1.2. *Suppose X is a smooth projective threefold and A an ample line bundle on X. If η is a very general point of X then*

$$\epsilon(\eta, A) \geq \frac{2}{3}.$$

Theorem 1.2 improves upon prior lower bounds of one third [EKL], $\frac{\sqrt{3}}{4}$ [C], and one half [N].

Before giving a detailed strategy for proving Theorem 1.2, we would like to make two basic but important general remarks. First, although, from the purely numerical point of view, having a Seshadri exceptional curve C_η with, say, $\deg_A(C_\eta) = 2$ and $\mathrm{mult}_\eta(C_\eta) = 3$ is the same as having a curve C'_η with $\deg_A(C'_\eta) = 4$ and $\mathrm{mult}_\eta(C'_\eta) = 6$, these two scenarios are not at all the same from the point of view of counting jets generated by the linear series $|kA|$. Indeed, from this latter point of view C'_η is identical to having *two* Seshadri exceptional curves C_η, each of degree 2 relative to A and each with multiplicity 3 at η. In other words, if $\epsilon(\eta, A) < 1$ one can expect that the fraction $\deg_A(C_\eta)/\mathrm{mult}_\eta(C_\eta)$ will be very close to being a reduced fraction. Thus it is not surprising that the most difficult case to deal with, within the range from 0 to 2/3 treated here, is the case of a curve C_η with $\deg_A(C_\eta) = 1$ and $\mathrm{mult}_\eta(C_\eta) = 2$. Secondly, the structure of the argument in [EKL] is inductive: in other words, in order to obtain lower bounds for the Seshadri constant of a smooth fourfold at a very general point one uses the bound for threefolds. In particular, Theorem 1.2 implies an improvement in all dimensions.

The fundamental idea behind the proof of Theorem 1.2, namely exploiting the fact that as $\epsilon(\eta, A)$ becomes smaller the linear series $|kA|$ separates fewer and fewer jets of higher order, comes from [N]. More specifically, let

$$m(A) = \sup_{D \equiv A} \{\mathrm{mult}_\eta(D) |\ D \in \mathrm{Div}(X) \otimes \mathbf{Q} \text{ effective}\}$$

where $\equiv$ denotes numerical equivalence. Counting jets of each order shows that $m(A)$ becomes larger as $\epsilon(\eta, A)$ becomes smaller. In [EKL] the lower bound $m(A) \geq 1$ is used in order to prove $\epsilon(\eta, A) \geq 1/3$. In [N] the fact that a small Seshadri constant means a larger value for $m(A)$ was used in order to prove $\epsilon(\eta, A) \geq 1/2$. Here we use the same basic idea but have a much more sophistocated method of counting jets which takes advantage of some new abstract ideas developed in [ELMNP2].

The strategy behind the proof of Theorem 1.2 is as follows. We assume that $\epsilon(\eta, A) < \frac{2}{3}$. If there is a Seshadri exceptional surface S_η at η then its multiplicity at η has to be at least 2 and a contradiction is quickly established: namely, $m(A)$ would be infinite as the restrictions on jets in each order are so great that the vector space $|kA|$ is never exhausted when raising the order of vanishing at η. Suppose then that C_η is a Seshadri exceptional curve at η. Let $p = \deg_A(C_\eta)$ and $q = \mathrm{mult}_\eta(C_\eta)$. A basic result of [EKL] says that if $D \in \left|kA \otimes m_\eta^{\otimes k\alpha}\right|$ and $\alpha > p/q$ then D vanishes to large order, roughly $k(\alpha - p/q)$ at least, along C_η. This allows one to bound

$$H^0\left(X, kA \otimes m_\eta^{\otimes k\alpha} / m_\eta^{\otimes k\alpha - 1}\right)$$

from above for different values of α. Allowing $k\alpha$ to vary and adding up the dimensions of these cohomology groups gives a lower bound for $h^0\left(X, kA \otimes m_\eta^{k\alpha}\right)$ for different values of α and hence gives a lower bound for $m(A)$. The techniques of [ELMNP2] are used in a crucial way in the counting as the dimension of the above cohomology group is equal to a restricted volume which is then identified, asymptotically, with an intersection number which is relatively easy to compute. The counting is slightly involved as we also have to pay close attention to when the base locus of $H^0\left(X, kA \otimes m_\eta^{k\alpha}\right)$ is two dimensional at η. The contradiction we reach, under the assumption that $\epsilon(\eta, A) < \frac{2}{3}$, is that $m(A)$ becomes infinite.

The contents of the paper are as follows. In §2 we introduce the language and basic formalism governing the base loci studied here. Many of the results here can be found in [EKL, ELMNP1, ELMNP2]. In §3, the technical heart of the paper, we prove that $m(A)$ can be bounded below by integrals involving the restricted volume function and we establish some basic bounds for these restricted volumes. As a first example of the strength of these counting techniques, we recover the result of [EL] that the Seshadri constant of an ample line bundle on a surface is at least one at a very general point. In §4, the counting techniques of §3 are put to use, eliminating all possible Seshadri constants less than 2/3 except for 1/2. The final case is then treated separately in §5.

2. Preliminary Results

Inspired by [ELMNP1] and [ELMNP2], in this section we introduce some technical language and results which will be useful to describe the base loci of linear series and how they vary with respect to the line bundle in question.

2.1. **Base Loci.** Let X be a smooth projective variety of dimension n and let D be an effective $\mathbb{Q}-$divisor on X. We denote by $\mathbf{B}(D)$ the *stable base locus* of D, that is the intersection of the base loci of the linear systems $|kD|$ as k varies over all positive integers. More generally, given an ideal sheaf $\mathcal{I}$ and a rational number $c > 0$, we define the stable base locus of $D \otimes \mathcal{I}^c$, as

$$\mathbf{B}(D \otimes \mathcal{I}^c) = \bigcap_k \mathrm{Bs}\left|kD \otimes \mathcal{I}^{kc}\right|,$$

where, again, k varies over all sufficiently divisible positive integers and $\mathrm{Bs}|L \otimes \mathcal{J}|$ denotes the base locus of $H^0(X, L \otimes \mathcal{J})$. Note that by noetherian induction, for any k sufficiently divisible one has

$$\mathbf{B}(D \otimes \mathcal{I}^c) = \mathrm{Bs}\left|kD \otimes \mathcal{I}^{kc}\right|.$$

The *augmented base locus* $\mathbf{B}_+(D)$ of D, studied in detail in [ELMNP1], is the stable base locus of any perturbation of D by a sufficiently small ample divisor. More precisely,

$$\mathbf{B}_+(D) = \bigcap_{A \text{ ample}} \mathbf{B}(D - A)$$

where $D - A$ is a $\mathbf{Q}$–divisor.

Moreover, if F is a subvariety of $\mathbf{B}(D)$, the *multiplicity* of D along F is defined by

$$\operatorname{mult}_F(D) = \liminf_{k\to\infty} \frac{\operatorname{mult}_F D_k}{k}.$$

where $D_k \in |kD|$ is a general representative for any sufficiently large and divisible integer k. Nakayama [Na], in the divisorial case and [ELMNP2] in general, established that $\operatorname{mult}_F(D)$ varies continuously with respect to D and thus extends uniquely to a function $\operatorname{mult}_F(D)$ defined for all real effective divisors D.

Given a subscheme S of X passing through a smooth point η of X, the projectivization of the tangent cone $T_\eta(S)$ of S at η defines a subscheme

$$\mathbb{P}(T_\eta(S)) \subseteq \mathbb{P}(T_\eta(X)) \simeq \mathbb{P}^{n-1}.$$

Lemma 2.1. *Given an effective $\mathbb{Q}$-divisor D on X, an ideal sheaf $\mathcal{I}$, a rational number $c > 0$, and a smooth point $\eta \in X$, there exist a sufficiently large and divisible integer k and $s_1, \ldots, s_m \in H^0\left(X, kD \otimes \mathcal{I}^{kc}\right)$ such that*

$$\operatorname{supp}\left(\cap_{i=1}^m T_\eta(Z(s_i))\right) = \operatorname{supp}\left(T_\eta\left(\mathbf{B}(D \otimes \mathcal{I}^c)\right)\right),$$

where $Z(s_i)$ is the zero set of the section s_i.

Proof. Given any system of local parameters $(z_1, \ldots, z_n)$ at $\eta \in X$, the tangent cone $T_\eta(S)$ of a scheme S at η is obtained by taking the leading terms of the elements of the ideal $\mathcal{I}_S$ associated to S with respect to the coordinates $(z_1, \ldots, z_n)$. Therefore it follows immediately that $\operatorname{supp}(T_\eta\left(\mathbf{B}(D \otimes \mathcal{I}^c)\right)) \subseteq \operatorname{supp}(T_\eta(Z(s)))$, for any section $s \in H^0\left(X, kD \otimes \mathcal{I}^{kc}\right)$, with k sufficiently divisible. In particular, one inclusion is clear:

$$\operatorname{supp}\left(\cap_{i=1}^m T_\eta(Z(s_i))\right) \supset \operatorname{supp}\left(T_\eta\left(\mathbf{B}(D \otimes \mathcal{I}^c)\right)\right).$$

For the opposite inclusion, choose $k > 0$ so that kc is an integer and so that

$$\operatorname{supp}(\mathcal{I}\left(\mathbf{B}\left(kD \otimes \mathcal{I}^{kc}\right)\right)) = \operatorname{supp}(\mathcal{I}\left(\mathbf{B}(D \otimes \mathcal{I}^c)\right)).$$

We claim that there exists a positive integer r so that

$$\left(\mathcal{I}\left(\mathbf{B}(D \otimes \mathcal{I}^c)\right)\right)^r \subset \mathcal{I}\left(\mathbf{B}\left(kD \otimes \mathcal{I}^{kc}\right)\right). \tag{1}$$

This can be seen using [L2] 9.6.28. Indeed, since the zero sets of the ideals $\mathcal{I}(\mathbf{B}(D \otimes \mathcal{I}^c))$ and $\mathcal{I}\left(\mathbf{B}\left(kD \otimes \mathcal{I}^{kc}\right)\right)$ have the same support, taking a common resolution and pushing forward the exceptional divisors gives an integer r_1 so that

$$\mathcal{I}\left(\mathbf{B}(D \otimes \mathcal{I}^c)\right)^{r_1} \subset \overline{\mathcal{I}(\mathbf{B}\left(kD \otimes \mathcal{I}^{kc}\right))}$$

where the bar denotes integral closure. Taking a further power, as noted in [L2] 9.6.28, establishes (1). The support of the zeroes of the leading terms of each element of $\mathcal{I}(\mathbf{B}(D\otimes\mathcal{I}^c))^r$ cut out $T_\eta\left(\mathbf{B}(D\otimes\mathcal{I}^c)\right)$ set theoretically and this concludes the proof of Lemma 2.1. □

Central to our technique, as is the case in [EKL], is the notion of differentiation. The following is a consequence of the fundamental "differentiation" result in [EKL]:

Lemma 2.2. *Let $\eta \in X$ be a very general point, let D be an effective integral divisor on X, and let $W \subseteq X$ be an irreducible subvariety. Let $\pi\colon Y \to X$ be the blow-up of X at η with exceptional divisor E and let $\tilde{W}$ be the strict transform of W in Y. Write*

$$\alpha(W) = \inf_{\beta\in\mathbb{Q}}\{\tilde{W} \subseteq \mathbf{B}(\pi^*D - \beta E)\}.$$

Then

$$\operatorname{mult}_W(\pi^*D - \beta E) \geq \beta - \alpha(W)$$

for any $\beta \geq \alpha(W)$.

Proof. See Lemma 1.3 in [N]. □

In particular, we will apply Lemma 2.2 in the situation where W is a Seshadri exceptional subvariety relative to A at η. In this setting, $\alpha(W) = \epsilon(\eta, A)$ and Lemma 2.2 states that W enters the base locus of $|kA|$ once the multiplicity at η excedes $k\epsilon(\eta, A)$: moreover, as the multiplicity at η is increased the multiplicity along W grows at least linearly.

2.2. **Volume and Restricted Volume.** Let X be a smooth projective variety of dimension n and let D be an effective $\mathbb{Q}$-divisor on X. The *volume* of D is defined by

$$\operatorname{vol}_X(D) = \limsup_{k\to\infty} \frac{h^0(X, kD)}{k^n/n!},$$

where k is taken sufficiently divisible. In particular, $\operatorname{vol}_X(D) > 0$ if and only if D is big. For the basic properties of the volume, as a function of D, we refer the reader to [L1] §2.2C.

If D is integral and V is a subvariety of X of dimension $d > 0$, the *restricted volume* of D along V is

$$\operatorname{vol}_{X|V}(D) = \limsup_{k\to\infty} \frac{h^0(X|V, kD)}{k^d/d!},$$

where

$$h^0(X|V, kD) = \dim\left(\operatorname{Im}\left(H^0(X, kD) \to H^0(V, kD|_V)\right)\right).$$

By [ELMNP2] Corollary 2.15, the $\limsup_{m\to\infty}$ can be replaced with $\lim_{m\to\infty}$ and D can be replaced by any $\mathbb{Q}$-divisor.

Restricted volume functions are studied in detail in [ELMNP2]. We recall here some of their fundamental properties:

Theorem 2.3. *Let D be a $\mathbb{Q}$-divisor on a smooth projective variety X and let V be a subvariety of dimension $d > 0$ such that V is not contained in $\mathbf{B}_+(D)$. We have the following:*

1. $\mathrm{vol}_{X|V}(D) > 0$,
2. $\mathrm{vol}_{X|V}(D)$ *depends only on the numerical equivalence class of D,*
3. $\mathrm{vol}_{X|V}$ *extends uniquely to a continuous function*

   $$\mathrm{vol}_{X|V}\colon \mathrm{Big}^V(X)^+_{\mathbb{R}} \to \mathbb{R}$$

   where $\mathrm{Big}^V(X)^+_{\mathbb{R}}$ denotes the set of all real divisor ξ such that V is not contained in $\mathbf{B}_+(\xi)$,
4. $\mathrm{vol}_{X|V}$ *is homogeneous of degree d and satisfies the log-concavity property:*

   $$\mathrm{vol}_{X|V}(\xi_1+\xi_2)^{1/d} \geq \mathrm{vol}_{X|V}(\xi_1)^{1/d} + \mathrm{vol}_{X|V}(\xi_2)^{1/d},$$

   for any $\xi_1, \xi_2 \in \mathrm{Big}^V(X)^+_{\mathbb{R}}$,
5. *For any sufficiently divisible integer k and for any general divisors $D_{k,1}, \ldots, D_{k,d} \in |kD|$,*

   $$\mathrm{vol}_{X|V}(D) = \lim_{k\to\infty} \frac{\#(V \cap D_{k,1} \cap \cdots \cap D_{k,d} \setminus \mathbf{B}(D))}{k^d}.$$

Proof. See Theorem A and B in [ELMNP2]. □

Lemma 2.4. *Let D be an effective $\mathbb{R}$-divisor on a smooth projective variety X and let V be a subvariety of dimension $d > 0$ such that V is not contained in the support of D. If $F_1, \ldots, F_l$ are irreducible divisors contained inside $\mathbf{B}(D)$ and $\gamma_i = \mathrm{mult}_{F_i}(D)$, then*

$$\mathrm{vol}_{X|V}(D) = \mathrm{vol}_{X|V}\left(D - \sum t_i F\right)$$

for any $t_1, \ldots, t_l$ such that $0 \leq t_i \leq \gamma_i$.

Proof. By induction on the number of irreducible components in the fixed part of D, we can assume $l = 1$. Let $F = F_1$ and $t = t_1$. By continuity of the restricted volume function, (3) of Theorem 2.3, we can assume $t \in \mathbb{Q}$. For any sufficiently large and divisible integer k, we have the following commutative diagram:

$$\begin{tikzcd}
H^0(X, k(D - tF)) \arrow[r, "u"] \arrow[d] & H^0(X, kD)) \arrow[d] \\
H^0(V, k(D - tF)|_V) \arrow[r] & H^0(V, kD|_V).
\end{tikzcd}$$

By definition of F, the map u is an isomorphism and this concludes the proof of Lemma 2.4. ☐

Proposition 2.5. *Let D be an effective $\mathbb{R}$-divisor on a smooth projective variety X and let V be a subvariety of dimension $d > 0$ such that V is not contained in the support of D. Then*

$$\mathrm{vol}_{X|V}(D) = \limsup_{k\to\infty} \frac{h^0(X|V, \llcorner kD \lrcorner)}{k^d/d!}$$

where $\llcorner kD \lrcorner$ denotes the round-down of kD.

Proof. The claim follows from the definition when D is a $\mathbb{Q}$-divisor. For any effective $\mathbb{R}$–divisor D, we define

$$\phi_k(D) = \frac{h^0(X|V, \llcorner kD \lrcorner)}{k^d/d!}.$$

Moreover, for every positive integer m, we define the divisors

$$D_m = \frac{\llcorner mD \lrcorner}{m} \qquad \text{and} \qquad D_m^+ = \frac{\ulcorner mD \urcorner}{m},$$

where $\ulcorner mD \urcorner$ denotes the round-up of mD.

Since V is not contained in the support of D and hence it is not contained in the support of D_m and D_m^+, it follows that

$$\phi_k(D_m) \leq \phi_k(D) \leq \phi_k(D_m^+).$$

Hence

$$\mathrm{vol}_{X|V}(D_m) \leq \limsup_{k\to\infty} \phi_k(D) \leq \mathrm{vol}_{X|V}(D_m^+) \tag{2}$$

for any $m > 0$. Since

$$\lim_{m\to\infty} D_m = \lim_{m\to\infty} D_m^+ = D,$$

the continuity of the restricted volume (cf. (3) of Thm. 2.3) implies the claim after taking the limit as $m \to \infty$ in (2). ☐

Suppose $D_1, \ldots, D_n$ are effective Cartier divsiors on X, where $n = \dim(X)$. Suppose Z is a union of irreducible components of $D_1 \cap \ldots \cap D_n$, including all positive dimensional components. We denote, somewhat abusively, by $i_Z(D_1 \cdot \ldots \cdot D_n)$ the part of the intersection class $D_1 \cdot \ldots \cdot D_n$ supported on Z (cf. [F], 7.1.10). Note that for the positive dimensional components Z_i of Z the contribution of Z_i to $i_Z(D_1 \cdot \ldots \cdot D_n)$ is a rational equivalence class supported on Z_i while if P is a zero dimensional component of Z then $i_P(D_1 \cdot \ldots \cdot D_n)$ is a positive multiple of $[P]$ (see [F] Definition 6.1.2). If D is an effective $\mathbb{Q}$-divisor on X and V is a smooth divisor not contained in $\mathbf{B}(D)$, we define

$$i_{X|V}(D) = \lim_{k\to\infty} \frac{i_{\mathbf{B}(D)\cap V}(V \cdot D_{k,1} \ldots \cdot D_{k,n-1})}{k^{n-1}} \tag{3}$$

where $D_{k,1}, \ldots, D_{k,n-1}$ are general divisors in the linear series $|kD|$: note that the right hand side of (3) is well–defined as each component of $\mathbf{B}(D) \cap V$ will be an irreducible component of $D_{k,1} \cap \ldots \cap D_{k,n-1} \cap V$. Since the total degree of the class $V \cdot D_{k,1} \cdot \ldots \cdot D_{k,n-1}$ is $k^{n-1}D^{n-1} \cdot V$, and since the intersection $V \cap D_{k,1} \cap \ldots \cap D_{k,n-1}$ is proper away from $\mathbf{B}(D) \cap V$, (5) of Theorem 2.3 implies

Proposition 2.6. *With the notation introduced above,*

$$\mathrm{vol}_{X|V}(D) = D^{n-1} \cdot V - i_{X|V}(D).$$

Warning. In general Proposition 2.6 is not useful because $i_{X|V}(D)$ can be negative. We will only use Proposition 2.6, however, in the case where $V = \mathbf{P}^{n-1}$.

3. Counting Jets

Let X be a smooth projective variety of dimension n and for fixed $\eta \in X$ let $\pi\colon Y \to X$ be the blow-up of X at η with exceptional divisor E. Given a $\mathbb{Q}$-divisor D on X and $\alpha \geq 0$, we will denote

$$D_\alpha = \pi^*D - \alpha E.$$

In this section our goal is to compute the volume of D_α for different values of α, reducing the computation to one of restricted volumes where we will restrict to the exceptional divisor E. In particular, we begin with the following fundamental connection between the volume function on X and the restricted volume function on E.

Lemma 3.1. *With the notation introduced above, if $\alpha \in \mathbb{Q}_{\geq 0}$ then*

$$\mathrm{vol}_Y(D_\alpha) = \mathrm{vol}_X(D) - n \int_0^\alpha \mathrm{vol}_{Y|E}(D_\gamma)\, d\gamma. \tag{4}$$

Note that $\mathrm{vol}_{Y|E}(D_\gamma)$, on the right hand side of (4), is defined for real γ by continuity using (3) of Theorem of 2.3.

Proof. We will assume, after rescaling if necessary, that α is a positive integer. For any integers $k > 0$ and $m \geq 0$, we have the exact sequence:

$$0 \to \mathcal{O}_Y(k\pi^*D - (m+1)E) \to \mathcal{O}_Y(k\pi^*D - mE) \to \mathcal{O}_E(-mE) \to 0.$$

Thus,

$$h^0(Y, k\pi^*D - k\alpha E) = h^0(X, kD) - \sum_{l=0}^{k\alpha-1} h^0(Y|E, k\pi^*D - lE). \tag{5}$$

For any $x \in (0, \alpha)$ we define

$$f_k(x) = \frac{h^0(Y|E, \llcorner kD_x \lrcorner)}{k^{n-1}/(n-1)!}.$$

Proposition 2.5 implies

$$\limsup_{k\to\infty} f_k(x) = \operatorname{vol}_{Y|E}(D_x). \tag{6}$$

Since $f_k(x)$ is a simple function for each k, that is it assumes only a finite numbers of values, we have

$$\int_0^\alpha f_k(x)\, dx = \frac{1}{k} \cdot \sum_{l=0}^{k\alpha-1} f_k\left(\frac{l}{k}\right) = \frac{1}{k^n/(n-1)!} \sum_{l=0}^{k\alpha-1} h^0(Y|E, k\pi^*D - lE).$$

Combining this with (5) shows that

$$\operatorname{vol}_Y(D_\alpha) = \operatorname{vol}_X(D) - \limsup_{k\to\infty} \int_0^\alpha f_k(x)dx.$$

Since the functions $|f_k(x)|$ are uniformly bounded, it follows that

$$\limsup_{k\to\infty} \int_0^\alpha f_k = \int_0^\alpha \limsup_{k\to\infty} f_k.$$

Thus (6) implies the claim. $\square$

We will apply Proposition 2.6 and Lemma 3.1 to obtain a lower bound for the volume of D_α for different values of α. To this end, it is important to divide into cases according to the dimension of the base locus of D_α. Thus, we define for $1 \le r \le d$,

$$\epsilon_r = \epsilon_r(\eta, D) = \inf \left\{ \alpha \in \mathbb{Q}_{\ge 0} | \dim \mathbf{B}(D_\alpha) \ge r \right\}.$$

Moreover, suppose at a very general point η all Seshadri exceptional subvarieties are curves. Then we let

$$\sigma = \inf\{\alpha \in \mathbb{Q}_{\ge 0} | \exists\ D_\eta \subseteq Y \text{ such that } \tilde{C}_\eta \subsetneq D_\eta \subseteq \mathbf{B}(A_\alpha)\}; \tag{7}$$

here C_η is a Seshadri exceptional curve at η and $\tilde{C}_\eta$ is its proper transform in Y. By definition, $\sigma \ge \epsilon_2$.

Proposition 3.2. *Suppose σ is well–defined and in particular that all Seshadri exceptional subvarieties for A at η are curves. Let $\alpha \in (0, \sigma)$*

be a rational number and let C_η *be a Seshadri exceptional curve for* A *at* η*. Let*

$$q = \operatorname{mult}_\eta(C_\eta) \qquad \textit{and} \qquad \beta = \operatorname{mult}_{\tilde{C}_\eta}(D_\alpha)$$

where $\tilde{C}_\eta$ *is the proper transform of* C_η *in* Y*. Then*

$$\operatorname{vol}_{Y|E}(D_\alpha) \leq \max\left\{\alpha^{n-1} - q\beta^{n-1}, 0\right\}. \tag{8}$$

Proof. The goal is to apply Proposition 2.6 with $V = E$. If $E \subseteq \mathbf{B}_+(D_\alpha)$, then by Theorem C in [ELMNP2] we have $\operatorname{vol}_{Y|E}(D_\alpha) = 0$ and (8) follows immediately. Therefore, we may assume $E \not\subseteq \mathbf{B}_+(D_\alpha)$. Let $D_{k,1} \ldots D_{k,n-1}$ be general divisors of the linear system $|kD_\alpha|$, for a sufficiently large and divisible integer k. Using Proposition 2.6 it is sufficient to show that

$$i_{Y|E}(D_\alpha) \geq q\beta^{n-1}. \tag{9}$$

Assume $\beta > 0$ since otherwise $\alpha \in (0, \epsilon(\eta, A)]$ and (9) is trivial. If $\dim(\mathbf{B}(D_\alpha)) = 1$ then by Lemma 2.1 we may assume that $D_{k_1} \cap \ldots \cap D_{k_{n-1}} \cap E$ is proper. We have $E \cdot \tilde{C}_\eta = q$ and $\operatorname{mult}_{\tilde{C}_\eta}(D_{k,i}) \geq k\beta$ for each $i = 1, \ldots, n-1$ by Lemma 2.2. Since $\deg(T_\eta(\tilde{C}_\eta)) = q$, it follows that at least $qk^{n-1}\beta^{n-1}$ of the points of $D_{k_1} \cap \ldots \cap D_{k_{n-1}} \cap E$, counted with multiplicity, are contained inside $\mathbf{B}(D_\alpha) \cap E$ and in particular

$$i_{Y|E}(D_\alpha) \geq q\beta^{n-1}.$$

Thus, Proposition 2.6 implies (8) when $\dim(\mathbf{B}(D_\alpha)) = 1$.

Suppose next that $\dim(\mathbf{B}(D_\alpha)) \geq 2$. By definition of σ, $\tilde{C}_\eta$ is an irreducible component of $D_{k_1} \cap \ldots \cap D_{k_{n-1}}$, with multiplicity at least β^{n-1}. The argument of the previous paragraph, using Lemma 2.1 again, gives a contribution of at least $q\beta^{n-1}$ to the part of

$$D_{k_1} \cap \ldots \cap D_{k_{n-1}} \cap E$$

supported on $\mathbf{B}(D_\alpha) \cap E$. Thus it is sufficient to verify that the part of $D_{k_1} \cap \ldots \cap D_{k_{n-1}} \cap E$ which is a rational equivalence class, supported on $\mathbf{B}(D_\alpha) \cap E$, has non–negative degree. Since $E \simeq \mathbf{P}^{n-1}$ and since all cycles being intersected are effective, this follows immediately, establishing (9) and thus Proposition 3.2. □

Lemma 3.3. *With the notation introduced above, if* $\operatorname{vol}_{Y|E}(D_\alpha) = 0$ *for some* $\alpha > 0$*, then* $\operatorname{vol}_Y(D_\alpha) = 0$.

Proof. Suppose $\operatorname{vol}_Y(D_\alpha) > 0$. Since $\operatorname{vol}_{Y|E}(D_\alpha) = 0$, (1) of Theorem 2.3 implies that $E \subset \mathbf{B}_+(D_\alpha)$. In particular, $\operatorname{vol}_{Y|E}(D_\alpha) = 0$ for every $\beta > \alpha$ and by Lemma 3.1 it follows $m(D) = \infty$. This is a contradiction establishing Lemma 3.3. □

As a first application of the techniques used above, we examine the Seshadri constant of a smooth surface at a very general point:

Proposition 3.4. *Let X be a smooth surface and let A be an ample line bundle on X. If C_η is an exceptional curve relative to A at a general point η in X and $q = \operatorname{mult}_\eta(C_\eta)$, then*

$$\frac{\epsilon(\eta, A)}{\sqrt{A^2}} \geq \sqrt{1 - \frac{1}{q}}. \tag{10}$$

Note that Proposition 3.4 establishes that $\epsilon(\eta, A) \geq 1$ for a very general point of a smooth surface, thus recovering the main result of [EL].

Proof. We can assume $q > 1$ as otherwise (10) is trivial. Let $p = A \cdot C_\eta$, so that $\epsilon(\eta, A) = p/q$. Let Y be the blow-up of X at η, with exceptional divisor E and let $\tilde{C}_\eta$ be the proper transform of C_η in Y . Then by assumption $\tilde{C}_\eta \subseteq \mathbf{B}_+(A_{p/q})$ and Lemma 2.2 implies, for $\alpha > \frac{p}{q}$,

$$\operatorname{mult}_{\tilde{C}_\eta}(A_\alpha) \geq \alpha - \frac{p}{q}.$$

By Proposition 3.2, it follows that for any $\alpha > \frac{p}{q}$,

$$\operatorname{vol}_{Y|E}(A_\alpha) \leq \max\{p - (q-1)\ \alpha, 0\}. \tag{11}$$

From (11) we see that $\operatorname{vol}_{Y|E}\left(A_{\frac{p}{q-1}}\right) = 0$. On the other hand, Lemma 3.1 and (11) give

$$\begin{aligned} \operatorname{vol}_Y\left(A_{\frac{p}{q-1}}\right) &= \operatorname{vol}_Y\left(A_{\frac{p}{q}}\right) - \int_{\frac{p}{q}}^{\frac{p}{q-1}} \operatorname{vol}_{Y|E}(A_\gamma)\ d\gamma \\ &\geq A^2 - \left(\frac{p}{q}\right)^2 \frac{1}{1 - 1/q} \end{aligned} \tag{12}$$

By Lemma 3.3, the right hand side of (12) must be ≤ 0, which yields (10). □

4. Proof of the Main Theorem: the general case

The goal of this section is to prove Theorem 1.2 for all but two cases which will be handled in §5. Let X be a smooth threefold and A an ample line bundle on X. Let $\pi\colon Y \to X$ be the blow-up of X at a very general point η with exceptional divisor E and given $\alpha > 0$, let $A_\alpha = \pi^* A - \alpha E$. We will first reduce to the case where all

Seshadri exceptional subvarieties are curves. Suppose C_η is a Seshadri exceptional curve for A at η. By Lemma 3.1

$$\mathrm{vol}_X(A) = 3\int_0^{m(A)} \mathrm{vol}_{Y|E}(A_\gamma)d\gamma. \tag{13}$$

In general, the more singular C_η is at η the smaller the integrand $\mathrm{vol}_{Y|E}(A_\gamma)$ will be. On the other hand, Lemma 2.2 and the techniques of [EKL] imply that $m(A) \leq 3\epsilon(\eta, A)$. Thus when $\epsilon(\eta, A)$ is small $m(A)$ and $\mathrm{vol}_{Y|E}(A_\gamma)$ are also small so that Lemma 3.1 gives a contradiction since $\mathrm{vol}_X(A) \geq 1$. The goal of this section to is to make this argument rigorous.

Proposition 4.1. *With the notation introduced above, assume* $\epsilon(\eta, A) < 1$*. Then there exists a Seshadri exceptional curve* C_η *for* A *at* η*. Thus, setting* $p = A \cdot C_\eta$ *and* $q = \mathrm{mult}_\eta(C_\eta)$*, we have* $\epsilon(\eta, A) = p/q$*.*

Proof. We first note that if X itself is Seshadri exceptional at η relative to A, then $\epsilon(\eta, A) = \sqrt[3]{A^3} \geq 1$. Next suppose that there is a Seshadri exceptional surface S_η at η. Since $\epsilon(\eta, A) < 1$, we have $m = \mathrm{mult}_\eta(S_\eta) \geq 2$. Let $\tilde{S}_\eta$ be the proper transform of S_η in Y. By Lemma 2.4, for any $\alpha > \epsilon(\eta, A)$ we can write

$$A_\alpha = M + \gamma\tilde{S}_\eta$$

for some $\gamma > 0$ where M is an effective $\mathbb{Q}$–divisor such that $\mathrm{vol}_{Y|E}(A_\alpha) = \mathrm{vol}_{Y|E}(M)$. In particular, if $\tilde{M}$ is the proper transform of M in X, then $\mathrm{mult}_\eta(\tilde{M}) < \alpha$. It follows that $\mathrm{vol}_{Y|E}(A_\alpha) < \alpha^3$ for any $\alpha > \epsilon(\eta, A)$. Since $\mathrm{vol}_X(A) = A^3 \geq 1$, (13) implies that $m(A) > 1$. Thus, there exists $D \in |kA|$ with

$$\mathrm{mult}_\eta(D) = k(1+\delta), \;\; \delta > 0.$$

for $k \gg 0$. By Lemma 2.2

$$\mathrm{mult}_{S_\eta}(D) \geq k(1 + \delta - \epsilon(\eta, A)).$$

Thus

$$\begin{aligned} \mathrm{mult}_\eta(D) &\geq mk(1+\delta-\epsilon(\eta, A)) \\ &\geq mk\left(\delta + \frac{1}{m}\right) \\ &\geq k + 2k\delta. \end{aligned}$$

This is impossible, however, because by hypothesis $\mathrm{mult}_\eta(D) = k + k\delta$. Another way of stating this contradiction is that $m(A)$ becomes infinite as, once one passes multiplicity one, it becomes "free" to continue raising the multiplicity. The proof shows, in fact, that *all* Seshadri exceptional subvarieties for A at η are curves. $\square$

By of Proposition 4.1, for the rest of this section, we will assume that $\epsilon(A,\eta) = p/q$ is rational and that there is a curve C_η with $A \cdot C_\eta = p$ and $\operatorname{mult}_\eta(C_\eta) = q$. Moreover, the number σ defined in (7) is well defined. We have the following upper bound on σ:

Lemma 4.2. *With the notation introduced above,*

$$\sigma \leq \frac{p}{q - \sqrt{q}}. \tag{14}$$

Proof. Proposition 3.2 implies that $\operatorname{vol}_{Y|E}\left(A_{\frac{p}{q-\sqrt{q}}}\right) = 0$ and hence, by Lemma 3.3

$$\operatorname{vol}_Y\left(A_{\frac{p}{q-\sqrt{q}}}\right) = 0.$$

In particular, $X = \mathbf{B}(A_\alpha)$ for all $\alpha > \frac{p}{q-\sqrt{q}}$ and (14) follows. $\square$

Lemma 4.3. *Let $\tilde{D}_\eta$ be an irreducible surface inside $\mathbf{B}_+(A_\alpha)$, for some $\alpha > \sigma$ such that $\mathbf{B}(A_\alpha) \neq Y$ and such that $\tilde{C}_\eta \subseteq \tilde{D}_\eta$. Let $D_\eta = p(\tilde{D}_\eta)$ and let $a = \operatorname{mult}_\eta(D_\eta)$ and $b = \operatorname{mult}_{C_\eta}(D_\eta)$ Then*

$$\sigma \leq \frac{p}{q - \frac{a}{b}}. \tag{15}$$

Proof. Let $\alpha < \sigma$ be a rational number and let B be a general element of $|kA_\alpha|$ for some sufficiently divisible integer $k > 0$. Then, by Lemma 2.2, we have $\operatorname{mult}_{\tilde{C}_\eta}(B) \geq k(\alpha - p/q)$. Since $\alpha < \sigma$, B and $\tilde{D}_\eta$ intersect properly and

$$B \cdot \tilde{D}_\eta = bk(\alpha - p/q) \cdot [\tilde{C}_\eta] + R$$

for some effective cycle R. In particular,

$$ak\alpha = B \cdot \tilde{D}_\eta \cdot E \geq qbk(\alpha - p/q) + E \cdot R.$$

Since E is isomorphic to $\mathbf{P}^2$ and since $\tilde{D}_\eta$ and B both meet E properly, the intersection $B \cdot \tilde{D}_\eta \cdot E$ can be computed on E and thus $\deg(R) \geq 0$ since any intersection of effective divisors on $\mathbf{P}^2$ has non–negative degree. The inequality (15) follows by taking the limit as $\alpha \to \sigma$ from below. $\square$

Lemma 4.4. *Assume $\epsilon(\eta, A) < 1$ and let $\alpha > \sigma$ be a rational number, such that $\mathbf{B}(A_\alpha) \neq Y$. Let $\tilde{D}_\eta$ be an irreducible surface inside $\mathbf{B}_+(A_\alpha)$ such that $\tilde{C}_\eta \subseteq \tilde{D}_\eta$, and let $D_\eta = p(\tilde{D}_\eta)$. Then*

$$\operatorname{mult}_\eta(D_\eta) > \operatorname{mult}_{C_\eta}(D_\eta).$$

In particular, D_η is singular at η.

Proof. Suppose not. Let $\xi \in X$ be a general point such that $\eta \in C_\xi$. By the genericity assumption on η, there exists a surface D_ξ inside the stable base locus of $A \otimes m_\xi^\alpha$, numerically equivalent to D_η, and such that $C_\xi \subseteq D_\xi$. Let k be a sufficiently divisible integer and let D_k be a general element in $\left|kA \otimes m_\eta^{k\alpha}\right|$ with $\gamma = \operatorname{mult}_{D_\eta}(D_k)$.

Let

$$D' = D + \gamma(D_\xi - D_\eta).$$

Since, by assumption, $\operatorname{mult}_{C_\xi}(D_\xi) = \operatorname{mult}_{C_\eta}(D_\eta) = \operatorname{mult}_\eta(D_\eta)$, it follows that D' is an effective $\mathbb{Q}$-divisor such that $kD' \in \left|kA' \otimes m_\eta^{k\alpha}\right|$ where

$$A' = A + \gamma(D_\xi - D_\eta).$$

Since A' is numerically equivalent to A, Theorem 2.3 (2) implies that $\mathbf{B}_+(A'_\alpha) = \mathbf{B}_+(A_\alpha)$. Thus, since $\xi \in X$ is a general point with $\eta \in C_\xi$ and since $\mathbf{B}(A_\alpha) \neq Y$, it follows that $D_\eta = D_\xi$.

In particular, we conclude that D_η contains all curves C_ξ such that $\eta \in C_\xi$. The main result of [EL] implies that D_η must be singular along any one parameter family of points $t \in D_\eta$ such that $C_t \subset D_\eta$. This is impossible, however, as D_ξ is singular along C_ξ while D_η can only be singular along a finite number of curves, each of which can be Seshadri exceptional for at most finitely many points $x \in X$. □

Lemma 4.5. *For any $\alpha > \sigma$, let $\tilde{D}_1, \ldots, \tilde{D}_r$ be surfaces contained inside $\mathbf{B}(A_\alpha)$. Let $0 < a_i \leq \operatorname{mult}_\eta(D_i)$, where $D_i = p(\tilde{D}_i)$ and let*

$$\alpha(D_i) = \inf_{\beta \in \mathbb{Q}} \{D_i \subseteq \mathbf{B}(A_\beta)\}$$

Then

$$\operatorname{vol}_{Y|E}(A_\alpha) \leq \operatorname{vol}_{Y|E}(A_{\alpha - \sum a_i \gamma_i}),$$

where $\gamma_i = \alpha - \alpha(D_i)$.

In particular,

$$\operatorname{vol}_{Y|E}(A_{\sigma+\gamma}) \leq \operatorname{vol}_{Y|E}(A_{\sigma-\gamma})$$

for any $\gamma > 0$.

Proof. It is sufficient, using the continuity of the restricted volume functions, to establish the claim for $\alpha \in \mathbb{Q}$. Consider a $\mathbf{Q}$–divisior $F \equiv A$ with multiplicity $\alpha = \sigma + \gamma$ at η. Then by Lemma 2.2, we can write

$$F = \sum \gamma'_i D_i + M.$$

where $\gamma'_i = \frac{a_i}{\operatorname{mult}_\eta D_i}\gamma$ and M is an effective $\mathbb{Q}$-divisor. In particular

$$\operatorname{mult}_\eta(M) = \alpha - \sum a_i \gamma_i. \tag{16}$$

If $\tilde{M}$ is the proper transform of M in Y, then $\tilde{M} \equiv A_\alpha - \sum \gamma_i' \tilde{D}_i$ and, by Lemma 2.4, it follows that

$$\mathrm{vol}_{Y|E}(A_\alpha) = \mathrm{vol}_{Y|E}(\tilde{M}).$$

Since η is a general point and $\mathrm{mult}_\eta(D_i) > 0$, each divisor D_i moves in an algebraic family and η is not a fixed point of this family, for any $i = 1, \dots r$. In particular we can choose D_i' algebraically equivalent to D_i with η not contained in D_i'. Thus, if $\tilde{D}_i'$ is the proper transform of D_i' in Y, then (16) implies

$$\tilde{M} + \sum \gamma_i' \tilde{D}_i \equiv A_{\alpha - \sum a_i \gamma_i}.$$

Therefore, (2) of Theorem 2.3 implies

$$\mathrm{vol}_{Y|E}(\tilde{M}) \leq \mathrm{vol}_{Y|E}\left(\tilde{M} + \sum \gamma_i' D_i'\right) = \mathrm{vol}_{Y|E}\left(A_{\alpha - \sum a_i \gamma_i}\right).$$

Finally, by Lemma 4.4, there exists a surface $\tilde{D} \subseteq \mathbf{B}_+(A_\sigma)$, such that $\mathrm{mult}_\eta(D) \geq 2$ where $D = p(\tilde{D})$ and therefore $\mathrm{vol}_{Y|E}(A_{\sigma+\gamma}) \leq \mathrm{vol}_{Y|E}(A_{\sigma-\gamma})$, for any $\gamma > 0$. □

By [EKL] 3.4 there is an affine variety T, a quasi–finite dominant map $\phi : T \to X$ and a subvariety $\mathcal{C} \subset X \times T$ satisfying the following. Suppose $\pi_1 : \mathcal{C} \to X$ and $\pi_2 : \mathcal{C} \to T$ are the two projections. Then for each $t \in T$, $C_t = \pi_2^{-1}(t)$ is a Seshadri exceptional curve for A at $g(t)$. Let

$$R_\eta = \overline{\bigcup_{g(t) \in g(T) \cap C_\eta} C_t} :$$

here the bar denotes the Zariski closure. Let S_η be an irreducible component of R_η passing through η. Let $\tilde{S}_\eta$ be the proper transform of S_η in Y. By definition, S_η is swept out by Seshadri exceptional curves based at point of C_η. As in [EKL] §3.7, Lemma 2.2 implies

Lemma 4.6. *With the notation introduced above, the surface* $\tilde{S}_\eta$ *is contained inside* $\mathbf{B}_+(A_{2p/q})$.

We want to study carefully the singularities of S_η to improve the estimate on the volume of A_α in some special cases, for which $p/q < 2/3$.

Lemma 4.7. *We have* $\mathrm{mult}_{C_\eta}(S_\eta) \geq 2$. *Moreover, if* $\mathrm{vol}(A_{2p/q}) > 0$, *then* $\mathrm{mult}_\eta(S_\eta) > \mathrm{mult}_{C_\eta}(S_\eta)$.

Proof. If S_η were smooth at a general point $\xi \in C_\eta$ then we would have $\epsilon(\xi, A|_{S_\eta}) = \frac{p}{q} < 1$ and this is impossible since the result of Ein and Lazarsfeld [EL] indicates that on a surface the Seshadri constant can only be smaller than 1 at a countable set of points. Suppose next that $\mathrm{vol}_Y(A_{2p/q}) > 0$. Then $\mathbf{B}(A_{2p/q}) \neq Y$, while Lemma 4.6 implies that $\tilde{S}_\eta$ is contained inside $\mathbf{B}_+(A_{2p/q})$. Since $\tilde{C}_\eta \subseteq \tilde{S}_\eta$, Lemma 4.4 implies the claim. $\square$

Proposition 4.8. *With the notation introduced above, if $q \geq 5$, then $\epsilon(\eta, A) \geq 2/3$.*

Proof. Suppose not. We want to show that if $q \geq 5$ and $p/q < 2/3$, then $\mathrm{vol}_Y(A_{m(\alpha)}) > 0$, which gives a contraddiction.

Let D_η be a (possibly reducible) surface containing $\tilde{C}_\eta$ and contained inside $\mathbf{B}_+(A_\sigma)$ and let $\overline{D}_\eta = p(D_\eta)$. By Lemma 3.1

$$\begin{aligned}\mathrm{vol}(A_{m(A)}) &\geq A^3 - 3\int_0^\sigma \mathrm{vol}_{Y|E}(A_\gamma)d\gamma - 3\int_\sigma^{m(A)} \mathrm{vol}_{Y|E}(A_\gamma)d\gamma \\ &\geq A^3 - 6\int_0^\sigma \mathrm{vol}_{Y|E}(A_\gamma)d\gamma \qquad (17)\\ &\geq A^3 - 2\sigma^3 + 2q\left(\sigma - \frac{p}{q}\right)^3.\end{aligned}$$

where the second inequality follows from Lemma 4.5 and the third inequality follows from Proposition 3.2.

Elementary calculus shows that the function $x^3 - q(x-p/q)^3$ is an increasing function of x for $p/q < x < p/(q-\sqrt{q})$. Thus Lemma 4.2 implies

$$\begin{aligned}\mathrm{vol}_Y(A_{m(A)}) &\geq A^3 - 2\left(\frac{p}{q-\sqrt{q}}\right)^3 + 2q\left(\frac{p}{q-\sqrt{q}} - \frac{p}{q}\right)^3 \\ &= A^3 - 2\frac{\left(\frac{p}{q}\right)^3}{\left(1-\frac{1}{\sqrt{q}}\right)^2}.\end{aligned}$$

Since $A^3 \geq 1$ and $p/q < 2/3$, it follows that if $q > 18$ then $\mathrm{vol}_Y(A_{m(A)}) > 0$. If $q \leq 18$ and $p/q < 2/3$, it is an easy calculation to show that we get the same conclusion except for the following cases:

(1) $p = 1, \quad q = 2$;
(2) $p = 2, \quad q = 4$;
(3) $p = 3, \quad q = 5$;
(4) $p = 5, \quad q = 8$;
(5) $p = 7, \quad q = 11$.

The first two are ruled out by hypothesis. Let us consider first the case (5). We distinguish two cases. If $\mathrm{mult}_\eta(\overline{D}_\eta) > 2$, then by Lemma

4.4, the inequality (17) becomes

$$\operatorname{vol}(A_{m(A)}) \geq A^3 - \frac{3}{2}\sigma^3 + \frac{3}{2}q\left(\sigma - \frac{p}{q}\right)^3. \tag{18}$$

As above, the right hand side of the inequality is decreasing with respect to σ and by the bound on σ provided by Lemma 4.2, it follows that $\operatorname{vol}(A_{m(A)}) > 0.2$. Thus, we can assume that $\operatorname{mult}_\eta(\overline{D}_\eta) = 2$. By Lemma 4.4, we have $\operatorname{mult}_{\tilde{C}_\eta}(D_\eta) = 1$ and Lemma 4.3 implies $\sigma \leq 7/9$. Thus (17) implies $\operatorname{vol}_Y(A_{m(A)}) > 0.12$.

Let us consider now the case (4). As in the previous case, by (18) we can assume $\operatorname{mult}_\eta(\overline{D}_\eta) = 2$. On the other hand, in this case, the bound $\sigma \leq 5/6$, guaranteed by Lemma 4.3, is not enough. By Lemma 4.6, the surface $\tilde{S}_\eta$ is contained inside the stable base locus of A_α, for each $\alpha > 5/4$ and by Lemma 4.7, we have $\operatorname{mult}_\eta(S_\eta) \geq 3$. In particular $S_\eta \neq D_\eta$. Thus, by Lemma 4.5, the inequality (17) becomes

$$\operatorname{vol}(A_{m(\alpha)}) \geq A^3 - \frac{15}{4}\int_0^{\alpha_0} \operatorname{vol}_{Y|E}(A_\gamma)\, d\gamma - 6\int_{\alpha_0}^{\sigma} \operatorname{vol}_{Y|E}(A_\gamma)\, d\gamma$$

where $\alpha_0 = 2\sigma - \frac{2p}{q}$. Assuming $p = 5$ and $q = 8$ (as in (4)), it is an easy computation to show that the right hand side of the inequality is decreasing with respect of σ for $\sigma \leq 5/6$. Thus, it follows that $\operatorname{vol}(A_{m(A)}) > 0.04$.

It remains to consider the case (3). Using the same methods as above, we can assume $\operatorname{mult}_\eta D_\eta \leq 3$. We will distinguish 3 cases. First, if $\operatorname{mult}_\eta(D_\eta) = 3$ and $\operatorname{mult}_{C_\eta}(D_\eta) = 2$, then by Lemma 4.3, we have $\sigma < 6/7$ and (18) implies that $\operatorname{vol}_Y(A_\alpha) > 0.18$. On the other hand, if $\operatorname{mult}_\eta(D_\eta) = 3$ and $\operatorname{mult}_{C_\eta}(D_\eta) = 1$, then Lemma 4.7 implies $D_\eta \neq S_\eta$ and a similar argument as the one used for (4), implies $\operatorname{vol}(A_{m(A)}) > 0$. Thus, we can assume $\operatorname{mult}_\eta(D_\eta) = 2$. In this case, the argument used above does not guarantee a lower bound for $\operatorname{vol}(A_{m(A)})$ directly. On the other hand, if $\operatorname{mult}_\eta(S_\eta) = 3$ then Lemma 4.7 implies $\operatorname{mult}_{C_\eta}(S_\eta) = 2$ and Lemma 4.3 implies $\sigma < 6/7$. It follows that $\operatorname{vol}_Y(A_{m(A)}) > 0.01$. Otherwise $\operatorname{mult}_\eta(S_\eta) \geq 4$ and also in this case the argument used in (4) implies that $\operatorname{vol}_Y(A_{m(A)}) > 0$. □

5. Proof of the Main Theorem: Case $\frac{p}{q} = \frac{1}{2}$

In this section we will deal with the cases which could not be eliminated by the general counting techniques of §4, namely the cases where $q < 5$ and $p/q < 2/3$ or, in other words, $p = 1$, $q = 2$ and $p = 2$, $q = 4$. As mentioned in the introduction the case where $p = 2$ and $q = 4$ is in fact much easier to eliminate than $p = 1$ and $q = 2$.

First note that when $p/q = 1/2$, Lemma 4.6 implies that

$$\sigma \leq 1$$

as $S_\eta \subset \mathbf{B}(A_\alpha)$ for any $\alpha > 1$ where S_η is the surface defined before Lemma 4.6. We next claim that $m(A) \leq 3/2$ if $\epsilon(\eta, A) = 1/2$. By Lemma 2.2 if $D \in |kA|$ and $m = \mathrm{mult}_\eta(D) > \frac{3k}{2}$ then $\mathrm{mult}_{S_\eta}(D) \geq m - k$. By Lemma 4.7, $\mathrm{mult}_\eta(S_\eta) \geq 3$ and thus $m \geq 3(m-k)$ which is not possible given the hypothesis that $m > 3k/2$.

For the case $p = 2$ and $q = 4$, Proposition 3.2 shows that

$$\mathrm{vol}_Y(A_\sigma) \geq 1 - \sigma^3 + 4\left(\sigma - \frac{2}{4}\right)^3 \geq \frac{1}{2}$$

with equality if and only if $\sigma = 1$. If $\mathrm{vol}_Y(A_\sigma) > 1/2$ then (17) will show that $\mathrm{vol}(A_{m(A)}) > 0$, a contradiction. Thus we must have $\sigma = 1$. Then Lemma 4.7, and the argument used to establish (18), shows that

$$\mathrm{vol}_Y\left(A_{\frac{3}{2}}\right) \geq \frac{1}{4}$$

and this is a contradiction as well since we have already established that $m(A) \leq 3/2$.

We now consider the case $p = 1$ and $q = 2$. In this case there is an open subset $U \subset X$ at each point η of which the Seshadri exceptional curve C_η is unique for if there were a curve C'_η with $\deg_A(C'_\eta) = 1$ and $\mathrm{mult}_\eta(C'_\eta) = 2$ then one would be in the case $p = 2$ and $q = 4$. Thus the scheme T introduced before Lemma 4.6 can be identified with an open set $U \subset X$.

Lemma 5.1. *If $\epsilon(\eta, A) = \frac{1}{2}$ then $S_\eta = \mathbf{B}_+(A_\sigma)$.*

Proof. Suppose $E_\eta \subset \mathbf{B}_+(A_\sigma)$ and $E_\eta \neq S_\eta$. Then both E_η and S_η are in $\mathbf{B}_+(A_1)$ and, by Lemmas 4.4, 4.6, and 4.7, $\mathrm{mult}(E_\eta + S_\eta) \geq 5$. The argument of (18) shows that

$$\mathrm{vol}_Y(A_{m(A)}) \geq A^3 - \frac{15}{4}\int_0^1 \mathrm{vol}_{Y|E}(A_\gamma)\ d\gamma$$

and in particular $\int_0^1 \mathrm{vol}_{Y|E}(A_\gamma) d\gamma \geq \frac{4}{5}$. However, this is impossible, by Proposition 3.2. □

Lemma 5.2. *Let $p = 1$ and $q = 2$. Then $\mathrm{mult}_\eta(S_\eta) = 3$. Moreover $A - \frac{1}{2}S_\eta$ is in the closure of the effective cone and $A^3 = 1$.*

Proof. We begin by showing that $\mathrm{mult}_\eta(S_\eta) = 3$. We know by Lemma 4.7 that $\mathrm{mult}_\eta(S_\eta) \geq 3$. Suppose $\mathrm{mult}_\eta(S_\eta) \geq 5$. The proof of Lemma

5.1 shows that

$$A^3 - \frac{5}{4}\int_0^1 \mathrm{vol}_{Y|E}(A_\gamma)d\gamma \leq 0$$

and this is impossible since $\int_0^1 \mathrm{vol}_{Y|E}(A_\gamma)d\gamma \leq 3/4$ by Proposition 3.2. Suppose next that $\mathrm{mult}_\eta(S_\eta) = 4$. Then, by Lemma 4.7, there are two cases to deal with, $\mathrm{mult}_{C_\eta}(S_\eta) = 2$ and $\mathrm{mult}_{C_\eta}(S_\eta) = 3$. In both cases σ must be 1 and $m(A) = \frac{4}{3}$ or else the analogue of (18), using $\mathrm{mult}_\eta(S_\eta) = 4$, would give a contradiction.

We first treat the case where $\mathrm{mult}_{C_\eta}(S_\eta) = 2$. In this case

$$\begin{aligned} 3\int_1^{\frac{4}{3}} \mathrm{vol}_{Y|E}(A_\gamma)d\gamma &\leq 3\int_1^{\frac{4}{3}} \left((4-3\gamma)^2 - 2\left(2-\gamma-\frac{1}{2}\right)^2\right)d\gamma \\ &\leq \frac{2}{216} + \frac{1}{12}. \end{aligned}$$

Since $\mathrm{vol}(A_1) \geq \frac{1}{4}$ this shows that $\mathrm{vol}(A_{4/3}) > 0$ and this is not possible since $m(A) = 4/3$. When $\mathrm{mult}_{C_\eta}(S_\eta) = 3$ there is a different type of contradiction. In particular, for $\epsilon \ll 1$ choose $k \gg 0$ and an effective representative $D \in \left|kA_{\frac{4}{3}-\epsilon}\right|$. Then, by Lemma 2.2 $D = \left(\frac{1}{3} - \epsilon\right)kS_\eta + E$ where E is an effective divisor. By Lemma 2.2, we know that $\mathrm{mult}_{C_\eta}(D) \geq (\frac{5}{6} - \epsilon)k$. Since $\mathrm{mult}_{C_\eta}(S_\eta) = 3$ this implies that $\mathrm{mult}_{C_\eta}(E) \geq \frac{1}{6}k$ and this is impossible as the multiplicity of D at η would be larger than $km(A)$.

Having established that $\mathrm{mult}_\eta(S_\eta) = 3$ we now show that $m(A) = \frac{3}{2}$. By Lemma 2.2, this implies that $A - \frac{1}{2}S_\eta$ is in the closure of the effective cone. We have

$$\begin{aligned} A^3 &= 3\int_0^{m(A)} \mathrm{vol}_{Y|E}(A_\gamma)d\gamma \\ &= 3\int_0^{\sigma} \mathrm{vol}_{Y|E}(A_\gamma)d\gamma + 3\int_\sigma^{m(A)} \mathrm{vol}_{Y|E}(A_\gamma)d\gamma. \end{aligned}$$

For the first integral, using Proposition 3.2, we have

$$3\int_0^{\sigma} \mathrm{vol}_{Y|E}(A_\gamma)d\gamma \leq \sigma^3 - \left(\sigma - \frac{p}{q}\right)^3 \leq \frac{3}{4}$$

with equality holding only if $\sigma = 1$. By Lemma 5.1, $\tilde{S}_\eta \subset \mathbf{B}_+(A_\sigma)$ and, using Lemma 2.2 we see that $m(A) \leq \frac{3}{2}\sigma$. Moreover, using the facts that $\mathrm{mult}_\eta(S_\eta) = 3$ and, by Lemma 4.7, $\mathrm{mult}_{C_\eta}(S_\eta) = 2$, we find

$$\begin{aligned} 3\int_\sigma^{m(A)} \mathrm{vol}_{Y|E}(A_\gamma)d\gamma &= 3\int_\sigma^{m(A)} \mathrm{vol}_{Y|E}(A_\gamma - (\gamma-\sigma)S_\eta)d\gamma \\ &\leq 3\int_\sigma^{m(A)} \left((3\sigma - 2\gamma)^2 - 2\left(\frac{1}{2}+\sigma-\gamma\right)^2\right)d\gamma. \end{aligned}$$

This last integral is at most $\frac{1}{4}$ with equality holding if and only if $\sigma = 1$ and $m(A) = \frac{3}{2}$. Thus if $A^3 = \int_0^{m(A)} \mathrm{vol}_{Y|E}(A_\gamma)d\gamma$ then $A^3 = 1$ as claimed and $m(A) = \frac{3}{2}$. □

We now define, using the notation prior to Lemma 4.6,

$$T_\eta = \{x \in U | C_x \ni \eta\}.$$

Let D_η be the Zariski closure of T_η in X.

Lemma 5.3. *With the notation introduced above, we have $C_x \subseteq S_\eta$ for any $x \in T_\eta$.*

Proof. Suppose not. Then, there exists $t \in T_\eta$ such that C_t is not contained in S_η. Thus, Lemma 5.2 implies that $S_\eta \cdot C_t \geq 3$. Moreover, $(A - \frac{1}{2}S_\eta)\cdot C_t \geq 0$ since the numerical equivalence class of C_t moves with only finitely many base points. This is impossible, however, since $A \cdot C_t = 1$ by hypothesis.

□

If $D_\eta \neq C_\eta$, we have the following

Lemma 5.4. *With the notation introduced above, assume $D_\eta \neq C_\eta$. For a general point $\xi \in D_\eta$ we have*

$$S_\eta \cdot S_\xi > 2C_\eta + 2C_\xi.$$

Proof. First of all the intersection is proper because the curves C_t cover an open subset of S_η and thus if the intersection were not proper we would have $S_\eta = S_\alpha$ for a general point $\alpha \in S_\eta$ and this is impossible as it would imply that S_η is singular at a general point, using the same method as in Lemma 4.4. Next, Lemma 5.3 implies that C_η and C_ξ are each in the intersection. Moreover, S_η is singular along C_η and S_ξ is singular along C_ξ so we have

$$S_\eta \cdot S_\xi \geq 2C_\eta + 2C_\xi.$$

We claim that

$$\mathrm{mult}_\xi(S_\eta) \geq 2.$$

If not, then $\epsilon(\xi, A|_{S_\eta}) \leq \frac{1}{2}$ since $C_\xi \subset S_\eta$ by hypothesis. The main result of [EL], however, indicates that $\epsilon(x, A|_{S_\eta})$ can only be smaller than one at a countable number of points.

We have, by Lemma 4.7, $\operatorname{mult}_\xi(S_\xi) \geq 3$. Hence

$$\operatorname{mult}_\xi(S_\eta \cdot S_\xi) \geq 6.$$

This is only possible, however, if either

$$S_\eta \cdot S_\xi > 2C_\eta + 2C_\xi,$$

with other curves in the intersection passing through ξ, or

$$\xi \in C_\eta$$

so that $2C_\eta$ will contribute at least 2 toward the multiplicity at ξ of the intersection. The second case has been excluded by hypothesis and so this concludes the proof. $\square$

The case where $p = 1, q = 2$ can now be subdivided. We first consider the case where $D_\eta \neq C_\eta$ so that Lemma 5.4 applies, and

$$S_\xi \cdot S_\eta > 2C_\eta + 2C_\xi.$$

By Lemma 5.2 we have, writing $A \equiv \frac{1}{2}S_\eta + E_\eta$ and $A \equiv \frac{1}{2}S_\xi + E_\xi$ where E_η and E_ξ are in the closure of the effective cone,

$$A^2 \equiv \frac{1}{4}(S_\eta \cdot S_\xi) + \frac{1}{2}S_\eta \cdot E_\xi + A \cdot E_\xi.$$

But $A^2 \cdot E_\xi \geq 0$ since A is ample and E_ξ is in the closure of the effective cone and similarly $A \cdot S_\eta \cdot E_\xi \geq 0$ since $A \cdot S_\eta$ can be represented by a curve which moves with at most finitely many base points. Thus, Lemma 5.2 and Lemma 5.4 imply

$$1 = A^3 \geq \frac{1}{4}S_\eta \cdot S_\xi \cdot A > \frac{1}{4}(2C_\eta + 2C_\xi) \cdot A$$

and this is impossible since $A \cdot C_\eta = A \cdot C_\xi = 2$

Next assume that $D_\eta = C_\eta$. Suppose ξ is a general point of S_η and consider $C_\xi \cdot S_\eta$. We have

$$i_\xi(C_\xi \cdot S_\eta) \geq 2$$

since $\operatorname{mult}_\xi(C_\xi) = 2$. On the other hand, $\xi \in C_t$ for some $t \in C_\eta$ by definition of S_η. But since we have assumed that $D_\eta = C_\eta$ we have that C_ξ passes through t. In particular

$$C_\xi \cdot S_\eta \geq 3$$

and this is impossible since, according to Lemma 5.2, we would have

$$1 = A \cdot C_\xi \geq \frac{1}{2}S_\eta \cdot C_\xi \geq \frac{3}{2}.$$

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