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Subsheaves of the cotangent bundle

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Abstract: For any smooth projective variety, we study a birational invariant, defined by Campana which depends on the Kodaira dimension of the subsheaves of the cotangent bundle of the variety and its exterior powers.
We provide new bounds for a related invariant in any dimension and in particular we show that it is equal to the Kodaira dimension of the variety, in dimension up to 4, if this is not negative.

Keywords: Kodaira dimension, birational geometry
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1 Introduction

The attempt at classifying algebraic varieties has always been based on the study of the positivity (or negativity) of their cotangent bundles.

One of the most important invariants of a smooth algebraic variety X defined over the complex numbers is its Kodaira dimension, which measures the number of global pluricanonical forms on X (see [12] for more details).

A decade or so ago, Campana defined another important invariant for any smooth projective variety X (see e.g. [5]) :

$$k^+(X) = \max_{\mathcal{F}\subseteq\Omega^p_X} \{\mathrm{kod}(X, \det(\mathcal{F}))\}$$

where the maximum is taken after considering all coherent subsheaves $\mathcal{F} \subseteq \Omega^p_X$ of rank r for any $r, p > 0$.

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Clearly $k^+(X)$ is strictly related to the Kodaira dimension of X and, in particular, we always have $k^+(X) \geq \text{kod}(X)$. The equality does not hold in general (consider, for example, the surface $X = \mathbb{P}^1 \times C$ where C is any curve of genus $g \geq 1$). On the other hand, it is natural to conjecture that for any non-uniruled projective variety, this invariant coincides with the Kodaira dimension of the variety. More precisely:

Conjecture (C_n). Let X be a smooth projective variety of dimension n and let $r : X \dashrightarrow Z$ be the Maximal Rationally Connected (MRC) fibration associated to X. Then $k^+(X) = \text{kod}(Z)$.

(e.g. see [4], for a construction of the MRC fibration)

In [5], Campana showed that (C_n) is a consequence of the minimal model program and the following

Conjecture (R_n). A smooth algebraic variety of dimension n is uniruled if and only if $\text{kod}(X) = -\infty$.

We recall that a smooth variety is said to be uniruled if, for any generic point, there exists a rational curve passing through it. At the moment the conjecture (R_n) (and therefore (C_n)) is known to be true for $n \leq 3$ [19].

A weaker version of the conjecture (C_n) is obtained by considering only the line bundles $L \subseteq \Omega^p_X$ and the invariant

$$k_1^+(X) = \max_{L \subseteq \Omega^p_X} \{\text{kod}(X, L))\}$$

where L is any line bundle and $p > 0$ is arbitrary. Therefore, we have:

Conjecture (C'_n). : Let X, Z be as above then

$$k_1^+(X) = \text{kod}(Z).$$

The main result of this paper is

Theorem 1.1. *Let X be a projective variety of dimension ≤ 4 and with non-negative Kodaira dimension. Then $k_1^+(X) = \text{kod}(X)$.*

Theorem 1.1 will be a consequence of the following results:

Theorem 1.2. *Let X be a smooth and non-uniruled projective variety and let $L \subseteq \Omega^{n-1}_X$ be any invertible subsheaf such that* $\text{kod}(X, K_X + L) \geq \dim(X) - 2$.

Then $\text{kod}(X, L) \leq \max\{0, \text{kod}(X)\}$.

The requirement $\text{kod}(X, K_X + L) \geq \dim(X) - 2$ is justified by the fact that the relative minimal model is known only for relative dimension ≤ 2. The theorem will follow from the generic semi-positivity of the cotangent bundle of X.

On the other hand, we have:

Theorem 1.3. *Let X be a smooth projective variety with $\mathrm{kod}(X) = 0$ and let L be an invertible subsheaf of Ω^i_X for some $i > 0$.*

If $\mathrm{kod}(X, L + K_X) = \mathrm{kod}(X, L)$, then $\mathrm{kod}(X, L) \leq \dim X - 4$.

The last result is a generalization of Campana's theorem on the "speciality" of varieties of zero Kodaira dimension (see [6]):

Theorem 1.4. *Let X be a smooth projective variety of dimension 4 and zero Kodaira dimension, and let $L \subseteq \Omega^i_X$. Then $\mathrm{kod}(X, L) \leq i - 2$.*

The above bounds immediately imply the conjecture (C'_4) for varieties with zero Kodaira dimension. In fact, if X is a a 4-fold with $\mathrm{kod}(X) = 0$ and $k^+_1(X) > 0$, then theorem 1.4 rules out all the cases, except a possible sub-line bundle $L \subseteq \Omega^3_X$ of Kodaira dimension 1. By theorem 1.2, it follows that $\mathrm{kod}(X, K_X + L) = 1$, but that contradicts theorem 1.3. In section 6, we will see that it suffices to consider varieties with zero Kodaira dimension in order to prove (C'_n) for varieties with positive Kodaira dimension (see also [5]).

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I would also like to point out that many of the results of this paper were obtained independently by Campana and Peternell [7].

2 A Positivity Result

The aim of this section is to prove:

Theorem 2.1. *Let X be a non-uniruled variety and let L be a line bundle that is a quotient of Ω^1_X. Then L is pseudo-effective.*

We recall that a line bundle is said to be *pseudo-effective* if it is a limit of line bundles with non-zero sections.

In order to prove the theorem, we will use a recent characterization of pseudo-effective line bundles [2]:

Theorem 2.2. *[Boucksom - Demailly - Paun - Peternell]*
A line bundle L is pseudo-effective if and only if $L \cdot C \geq 0$, for every moving curve C, i.e. for every curve C that is a member of a family C_t that covers X.

Theorem 2.1 can be seen as a variation of Miyaoka's generic semi-positivity theorem [19]:

Theorem 2.3. *Let X be a normal projective variety of dimension n that is not uniruled, and let $H_1, \ldots, H_{n-2}$ be ample Cartier divisors on X. If $m_1, \ldots, m_{n-1}$ are any sufficiently large integers, then the restriction of Ω^1_X to any general complete intersection curve C cut out by the linear systems $|m_i H_i|$, is a semi-positive vector bundle.*

Miyaoka's theorem implies that on a non-uniruled projective variety X, every line bundle that is a quotient of Ω^1_X is non-negative on a generic complete intersection curve. Unfortunately, this alone does not imply the pseudo-effectiveness of the line bundle. In fact, Demailly, Peternell and Schneider [8] constructed an example of a non-pseudo-effective line bundle that has non-negative degree on any generic complete intersection curve, by considering $X = \mathbb{P}(T_K)$ to be the projectivization of the tangent bundle of a generic quartic surface $K \subseteq \mathbb{P}^3$ and $L = \mathcal{O}(1)$ be the associated line bundle on X.

On the other hand, the proof of theorem 2.1 closely follows the proof of Miyaoka's theorem. In fact, we will reduce X modulo p so that the line bundle L in theorem 2.1, defines a foliation on X, and thereafter apply Ekedahl's theory of foliations over a field in positive characteristic [9].

In order to do that, we recall some basic definitions. Let X be a normal projective variety defined over a field k. A saturated subsheaf of the tangent bundle of X, $\mathcal{F} \subseteq T_X$, defines a *foliation*, denoted by $(X, \mathcal{F})$, if it is closed under Lie brackets. For any foliation, there exists an immersion:

$$j : (X_0, \mathcal{F}_0) \hookrightarrow (X, \mathcal{F})$$

such that $X \setminus X_0$ has codimension 2, X_0 is non-singular, and $\mathcal{F}_0 = \mathcal{F}_{|X_0}$ is locally free. Moreover we define $K_{\mathcal{F}} = j_*(c_1(\mathcal{F}_0^*))$

If k is a field of characteristic $p > 0$, there exists a natural map:

$$\mathcal{F}_0^p \to T_{X_0}/\mathcal{F}_0.$$

We say that $(X, \mathcal{F})$ is p-closed if such a map is zero.

Theorem 2.4 (Ekedahl). *Given a normal variety X in characteristic $p > 0$, there is a one-to-one correspondence between:*
(1) p-closed foliations with rank r
(2) Factorizations

$$X \xrightarrow{\rho} Y \to X^{(1)} \tag{1}$$

of the geometric Frobenius map $F : X \to X^{(1)}$ *with* $\deg \rho = p^r$*.*

In particular, if $\mathcal{F} \subseteq T_X$ is a saturated subsheaf such that

$$\mathrm{Hom}_{\mathcal{O}_{X_0}}(\wedge^2 \mathcal{F}, T_X/\mathcal{F}) = 0 \quad \text{and} \quad \mathrm{Hom}_{\mathcal{O}_{X_0}}(F^*\mathcal{F}, T_X/\mathcal{F}) = 0$$

then $(X, \mathcal{F})$ defines a p-closed foliation and therefore a factorization as in (1).

We will write the quotient Y as $X/\mathcal{F}$ and we have:

$$\rho^* K_{X/\mathcal{F}} = (p-1)K_{\mathcal{F}} + K_X. \tag{2}$$

To proceed with the proof of 2.1, let us reduce X modulo p. Any line bundle L that is a quotient of Ω^1_X, defines a foliation given by $\mathcal{F} = L^* \subsetneq T_X$.

Suppose that there exists a moving curve C such that $LC < 0$. Then

$$\mathrm{Hom}_{\mathcal{O}_C}(F^*\mathcal{F}, T_X/\mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_C}(L^{-p}, T_X/\mathcal{F}) = \mathrm{H}^0(C, T_X/\mathcal{F} \otimes L^p \otimes \mathcal{O}_C) = 0$$

provided that $p >> 0$.

Since C is a moving curve, we have: $\mathrm{Hom}(F^*\mathcal{F}, T_X/\mathcal{F}) = 0$ and therefore $\mathcal{F}$ is a $p-$closed foliation for any sufficiently large p that defines a map $\rho : X \to Y = X/\mathcal{F}$ as in (1). By (2), it follows that $\rho^* K_Y C = (p-1)LC + K_X C$. Since $LC < 0$, by bend-and-break we have that for any p sufficiently large, there exists a rational curve L through a general point of $\rho(C)$, and since ρ is purely inseparable, L pulls back to a rational curve passing through a generic point of X.

Therefore, X must be uniruled and this completes the proof of theorem 2.1.

3 Proof of theorem 1.2

Let X be a smooth projective variety of dimension n that is non-uniruled and let L be an invertible subsheaf of Ω^{n-1}_X.

From the isomorphism $\Omega^{n-1}_X \simeq T_X \otimes \omega_X$, it follows that $L \otimes \omega^*_X \subseteq T_X$ defines a foliation on X and therefore, by theorem 2.1, $K_X - L$ is a pseudo-effective divisor, i.e. $L \cdot C \leq K_X \cdot C$, for any moving curve C (theorem 2.2).

Lemma 3.1. *Suppose that* X *is a non-uniruled smooth variety that is not of general type and let* $L \subset \Omega^{n-1}_X$*. Then* $L + K_X$ *is not big.*

Proof. If $L + K_X$ is big, then by the pseudo-effectiveness of $K_X - L$, it follows that $2K_X = (K_X - L) + (K_X + L)$ is big, for the cone of big divisors is the interior of the cone of pseudo-effective divisors on X. Thus, X is of general type. □

Let us suppose now, as in the hypothesis of theorem 1.2, that

$$\mathrm{kod}(X, K_X + L) \geq \dim(X) - 2.$$

We need to show that $\text{kod}(X, L) \le \max\{0, \text{kod}(X)\}$.

Let $\phi : X \to Z$ be the Iitaka fibration of $K_X + L$. Without loss of generality, by taking a suitable modification of X, we may suppose that ϕ is a regular morphism such that for any generic fiber X_z of ϕ, we have

$$\text{kod}(X_z, L_z + K_{X_z}) = 0.$$

where L_z is the restriction of L to the generic fiber X_z. We may also suppose that X is not of general type and that $\text{kod}(X, L) \ge 0$, otherwise there would be nothing to prove.

By lemma 3.1, the fibration ϕ is not an isomorphism. Moreover, by the assumption with regard to the Kodaira dimension of $K_X + L$ (i.e. $\text{kod}(X, K_X + L) \ge \dim(X) - 2$), we have that the generic fiber of ϕ is either a non-rational curve or a non-uniruled surface, otherwise X would be uniruled. Moreover, $\text{kod}(L) \ge 0$ implies that $\text{kod}(X_z, L_z) \ge 0$ for the generic point $z \in Z$, and therefore $\text{kod}(X_z) \le \text{kod}(X_z, L_z + K_{X_z}) = 0$ and by the classification of curves and surfaces, it follows immediately that the generic fiber of ϕ has zero Kodaira dimension.
Thus, since $\text{kod}(X_z, L_z + K_{X_z}) = \text{kod}(X_z) = 0$, it follows that $\text{kod}(X_z, L_z) \le \text{kod}(X_z, L_z + K_{X_z}) = 0$, and since $\text{kod}(X, L) \ge 0$, it follows that $\text{kod}(X_z, L_z)$ must vanish for the generic point $z \in Z$. This implies that the Iitaka fibration associated to L must factor through ϕ. Hence, without loss of generality we may suppose that $L = \phi^* L_1$ where L_1 is a $\mathbb{Q}-$divisor on Z.

Let us first assume that X_z is a curve, then $\phi : X \to Z$ is an elliptic fibration. Consequently, the canonical bundle of X is contained in the fibers of ϕ, i.e. there exists a divisor D on Z such that $\phi^* D - K_X$ is effective and $\text{kod}(X) = \text{kod}(Z, D)$. We wish to show that D is a big divisor on Z.

Theorem 1.2 will follow immediately. In fact,

$$\text{kod}(X, L) = \text{kod}(Z, L_1) \le \dim Z = \text{kod}(Z, D) = \text{kod}(X).$$

To prove the claim, we first notice that, since

$$\phi^*(D - L_1) = (\phi^* D - K_X) + (K_X - L)$$

is a sum of an effective divisor with a pseudo-effective one, $\phi^*(D - L_1)$ and therefore, $D - L_1$, must be pseudo-effective.

Moreover, since $\text{kod}(X, K_X + L) = \dim Z$, we have that $D + L_1$ is big on Z, and therefore

$$2D = (D - L_1) + (D + L_1)$$

is a sum of a pseudo-effective divisor with a big divisor. Thus D is big.

Now, let X_z be a surface. The situation is very similar. In fact, we have the following relative version of the minimal model program for algebraic surfaces (e.g. see [18], pag. 161):

Lemma 3.2. *Let $\phi : S \to B$ a family of algebraic surfaces such that a generic fiber S_b is a surface with zero Kodaira dimension.*

Then, there exists a rational map $S \to S'$ to a smooth variety $S' \xrightarrow{\phi'} B$ over B, such that the generic fiber of ϕ' is a surface with numerically trivial canonical bundle.

From the lemma it follows that we may suppose, without loss of generality, that the generic fiber of f has numerically trivial canonical line bundle. In fact, let $\pi : X \to X'$, be a rational map over Z as in the lemma, and let $\phi' : X' \to Z$, be the associated map. As in the case (?) of the elliptic fibration, there exists a divisor D on Z and a positive integer m, such that $\phi'^*(D) - mK_{X'}$ is effective, and $\mathrm{kod}(X) = \mathrm{kod}(X') = \mathrm{kod}(Z, D)$.

If $L' = \phi'^*(L_1)$, as in the previous claim, it follows that D is big and therefore $\mathrm{kod}(X, L) = \mathrm{kod}(X, L') \leq \mathrm{kod}(X)$.

4 Proof of theorem 1.3

Let X be a smooth variety with zero Kodaira dimension. We may assume, without loss of generality, that $P_g(X) = \dim \mathrm{H}^0(X, K_X) = 1$, in fact for any smooth variety X with $\mathrm{kod}(X) \geq 0$, there exists a cyclic cover $\pi : X' \to X$, such that $P_g(X') > 0$, and such that $K_{X'}$ is a multiple of $\pi^*(?)K_X$. Therefore, $\mathrm{kod}(X') = \mathrm{kod}(X)$ (see [13], pag.263).

Before proceeding with the proof of theorem 1.3, we need some preliminary results.

The following is a theorem of Griffiths [11] (see also [25]):

Theorem 4.1. *Let $f : Z \to C$ be an algebraic fibration from a smooth projective variety Z to a smooth curve C. Let $B \subset C$ be such that f is smooth over $C \setminus B$ and $D = f^{-1}(B)$ is a reduced normal crossing divisor on Z. If m is the dimension of the generic fiber of f, then, for any $q = 1, \ldots, m$, any invertible sub-sheaf of $\mathrm{R}^m f_* \Omega^q_{Z|C}(\log D)$ has non-positive degree on C.*

Let Z be a smooth projective variety of dimension n. We will say that Z admits a *perfect complex Poincaré pairing*, if the natural map

$$\mathrm{H}^0(Z, \Omega^i_Z) \times \mathrm{H}^0(Z, \Omega^{n-i}_Z) \to \mathrm{H}^0(Z, \omega_Z) \quad (3)$$

is a perfect pairing, i.e. for any non-zero $\eta \in \mathrm{H}^0(Z, \Omega^i_Z)$, the linear map $\cdot \wedge \eta : \mathrm{H}^0(Z, \Omega^{n-i}_Z) \to \mathrm{H}^0(Z, \omega_Z)$ is not identically zero.

Lemma 4.2. *[Complex Poincaré Pairing] Any smooth variety Z of dimension ≤ 3, with zero Kodaira dimension and with $P_g(X) = 1$, admits a perfect complex Poincaré pairing.*

This lemma was proved by Bogomolov in the special case $K_Z = 0$, and it should hold, at least conjecturally, for any variety of zero Kodaira dimension, as a consequence of the minimal model program. On the other hand, by taking a generic hyper-surface of $K \times K$, where K is a $K3$ surface, it can be easily shown that such a statement does not hold for varieties of higher Kodaira dimension, even in the case $P_g(X) > 0$.

Proof. The only interesting case is when X is a three-fold.

Let us first review the proof in the case $K_Z = 0$ (the same proof would hold in general for any dimension). Let $\omega \in \mathrm{H}^0(Z, \omega_Z)$ be a non-zero canonical form, that is, by assumption, nowhere zero. Any non-zero $(i,0)$-form $s \in \mathrm{H}^0(Z, \Omega^i_Z)$ corresponds, by the Hodge theorem, to a non-zero $(3-i,i)$-form $s' \in \mathrm{H}^3(Z, \Omega^{3-i}_Z)$. On dividing s' by $\bar{\omega}$, it can be easily shown that $s'/\bar{\omega}$ is a $(3-i)$-holomorphic form that is not homologous to zero. Therefore, we have defined an isomorphism $\mathrm{H}^0(Z, \Omega^i_Z) \xrightarrow{\sim} \mathrm{H}^0(Z, \Omega^{3-i}_Z)$ which implies the exactness of the pairing (3).

Let us consider now the more general case of a three-fold Z with $\mathrm{kod}(Z) = 0$ and with $P_g(Z) = 1$. The minimal model program implies that Z is birational to a product of an abelian variety with a simply-connected variety Z' with only terminal singularities and with numerically trivial canonical bundle (see e.g. [17]). Let us suppose that Z' is a three-fold. By Namikawa's theorem [21], Z' is smoothable, i.e. there exists a family $\mathcal{Z}$ over a disc Δ such that the central fiber is isomorphic to Z' and with $\mathcal{Z}$ smooth. Since terminal singularities are rational, by Steenbrink's theorem [22] it follows that $\mathrm{H}^i(\mathcal{O}_Z)$ is naturally isomorphic to $\mathrm{H}^i(\mathcal{O}_{\mathcal{Z}_\eta})$, where $\mathcal{Z}_\eta$ is a generic member of the family $\mathcal{Z}$. Since $\mathcal{Z}_\eta$ is smooth, the claim follows from the previous case. □

We are now ready to proceed with the proof of theorem 1.3.

Let us consider a line bundle $L \subseteq \Omega^i_X$ of Kodaira dimension $k > 0$ such that $\mathrm{kod}(X, L + K_X) = \mathrm{kod}(L)$ as per the hypothesis of the theorem.

Let $\phi : X \to Y$ be the Iitaka map associated to L, with $\dim Y = k$. We may assume once again that ϕ is a regular morphism and that there exists a big $\mathbb{Q}$-divisor H on Y such that $\phi^*(H) = L$. In fact, let $\pi : \hat{X} \to X$ be a modification, for which the Iitaka fibration $\hat{\phi} : \hat{X} \to Y$ of $\pi^* L$ is a regular morphism. Then we can find an effective divisor A on $\hat{X}$ and a big divisor H on Y, such that $L' := \pi^* L - A = \hat{\phi}^*(H)$. We want to show that $\mathrm{kod}(\hat{X}, L' + K_{\hat{X}}) = \mathrm{kod}(L')$. Obviously we have that $\mathrm{kod}(\hat{X}, L' + K_{\hat{X}}) \geq \mathrm{kod}(L')$. On the other hand, if E is the exceptional divisor for $\pi : \hat{X} \to X$, then

$$\begin{aligned}\mathrm{kod}(\hat{X}, L') = \mathrm{kod}(\hat{X}, \pi^* L) = \mathrm{kod}(\hat{X}, \pi^*(L + K_X)) = \\ = \mathrm{kod}(\hat{X}, \pi^*(L + K_X) + E) \geq \mathrm{kod}(\hat{X}, L' + K_{\hat{X}}).\end{aligned}$$

Thus, we can suppose that $\hat{X} = X$ and $L' = L$.

Lemma 4.3. *Under the assumptions of theorem 1.3, the generic fiber X_y of ϕ has zero Kodaira dimension.*

Proof. Let $\eta : X' \to Y'$ be the Iitaka fibration associated to the divisor $K_X + L$ defined on a smooth modification X' of X.

We have, by assumption, $\dim(Y') = \mathrm{kod}(X, K_X + L) = \mathrm{kod}(X, L) = \dim(Y)$.

The restriction of $K_X + L$ to the generic fiber has zero Kodaira dimension and since that restriction of K_X to such a fiber is $\mathbb{Q}$-effective, it follows that the restriction of L to the generic fiber of η has zero Kodaira dimension.

Thus, by theorem 10.6 in [12], it follows that η coincides with the fibration ϕ. In particular, the generic fiber of ϕ will have zero Kodaira dimension. □

By lemma 4.2, Theorem 1.3 will be a consequence of the following:

Lemma 4.4. *Let $f : X \to Y$ be an algebraic fibration such that its generic fiber X_y has zero Kodaira dimension and $P_g(X_y) = 1$ and let $H \in \operatorname{Pic} Y$ be a big divisor, such that if $L = f^*H$, there exists an embedding $L \subseteq \Omega^i_X$.*

Then $\dim(Y) \leq \operatorname{kod}(X)$.

Proof. Let $k = \dim Y$ and let C_0 be the generic member of a family of moving curves on Y (if Y is a curve, we just take $C_0 = Y$). In particular, we may assume that C_0 is not contained in the singular locus of f. Moreover, by theorem 1.5 in [2], we may suppose that the curve C_0 is *strongly movable*, i.e. there exists a modification $\mu : \tilde{Y} \to Y$, and very ample divisors $\tilde{A}_1, \dots, \tilde{A}_{k-1}$ such that $\mu_* C = C_0$, where

$$C = \tilde{A}_1 \cap \cdots \cap \tilde{A}_{k-1}.$$

Let $g : Z \to C$ be the fibration over C, obtained as the pull-back of the fibration f on C. By taking a semi-stable reduction of g, we can construct a ramified cover $v : C' \to C$, such that if Z' is a desingularization of $Z \times_C C'$, then the induced map $g' : Z' \to C'$ does not have multiple fibers [14].

Moreover, if $u : Z' \to Z$ is the induced cover, then it is possible to check with a local computation, that there exists an isomorphism in codimension 1

$$u^* \Omega^q_{Z|C}(\log D) \simeq \Omega^q_{Z'|C'}(\log D').$$

where C_0 is the locus on which g is smooth, D is the inverse image of $C \setminus C_0$ and $D' = u^{-1}(D)$.

Let $\psi : Z' \to X$ be the map induced by $\mu \circ v$. If $M = \omega_X \otimes L^*$ on X, and $M' = \psi^* M$ on Z', we want to show

Claim: There exists an embedding

$$M'^* \hookrightarrow \Omega^{i-k}_{Z'|C'}(\log D') \otimes \omega^*_{Z'|C'}.$$

and therefore M' is a quotient of $\Omega^{i-k}_{Z'|C'}(\log D)^* \otimes \omega_{Z'|C'}$.

Let B be a divisor on Y such that f is smooth on $Y \setminus B$, and let $S = f^{-1}(B) = \sum a_i S_i$. By Hironaka's theorem, we can suppose that B and S are normal crossing(?).

Let $S' = \sum (a_i - 1) S_i$. We will first show that there exists an embedding

$$L \hookrightarrow \Omega^{i-k}_{X|Y}(\log S) \otimes f^* \omega_Y \otimes \mathcal{O}(S'), \tag{4}$$

(see also lemma 5.1).

Over $X \setminus S$, we choose local coordinates $z_1, \ldots, z_n$, and so we are enabled to write $f(z_1, \ldots, z_n) = (z_1, \ldots, z_k)$. Moreover, by assumption on L, we can suppose that there exist $k+1$ analytically independent local sections of L, $s_0, \ldots s_k$ such that $s_i = z_i s_0$. Since Z is a Kähler manifold, by the "covering trick" (see [1]) the holomorphic forms s_i are closed and therefore we have:

$$0 = ds_i = dz_i \wedge s_0 + z_i ds_0 = dz_i \wedge s_0,$$

i.e. dz_i divides s_0 and locally we have $s_0 = dz_1 \wedge \ldots dz_k \wedge w$, where w is a $i-k$ form, that is a local section of $\Omega^{i-k}_{X|Y}(\log S)$. The same local computation shows, more in general, that the embedding in (4) holds in an open set of codimension 1 in X. Thus, we have

$$M^* \hookrightarrow \Omega^{i-k}_{X|Y}(\log S) \otimes \omega^*_{X|Y} \otimes \mathcal{O}(S').$$

From the exact sequence

$$0 \to f^*\Omega^1_Y(\log B) \to \Omega^1_X(\log S) \to \Omega^1_{X|Y}(\log S) \to 0,$$

it follows

$$\det \Omega^1_{X|Y}(\log S) \simeq \omega_{X|Y} \otimes \mathcal{O}(-S').$$

Hence, the isomorphism in codimension 1

$$\psi^*(\Omega^1_{X|Y}(\log S)) \simeq \Omega^1_{Z'|C'}(\log D')$$

implies the claim.

The generic fiber F of g' has zero Kodaira dimension and $P_g(F) = 1$. Therefore by lemma 4.2, it admits a perfect complex Poincaré pairing. Thus, by considering the restriction to the generic fiber of g', it follows that the map

$$g'_*(\Omega^{i-k}_{Z'|C'}(\log D')^* \otimes \omega_{Z'|C'}) \to g'_*M'$$

induced by the quotient map, given by the claim above, is not trivial.

By Steenbrink's theorem [22], the sheaves $\mathrm{R}^q\, g'_*\Omega^{i-k}_{Z'|C'}(\log D')$ are locally free, and therefore, by Grothendieck-Serre duality, there exists an isomorphism

$$g'_*(\Omega^{i-k}_{Z'|C'}(\log D')^* \otimes \omega_{Z'|C'}) \simeq (\mathrm{R}^{n-k}\, g'_*\Omega^{i-k}_{Z'|C'}(\log D'))^*$$

Thus, the map

$$(\mathrm{R}^{n-k}\, g'_*\Omega^{i-k}_{Z'|C'}(\log D'))^* \to g'_*M'$$

is generically surjective.

Since $n-k$ is the dimension of the generic fiber of g', it follows, by theorem 4.1, that $\mathrm{R}^{n-k}\, g'_*\Omega^q_{Z'|C'}(\log(D'))^*$ is a semi-positive vector bundle, and in particular g'_*M' has positive degree on C'. Thus, by theorem 2.2, M is a pseudo-effective divisor on X. Since H is big on Y, it follows that if $E = M + \frac{1}{2}L$, then

$$\mathrm{kod}(X, E) = \mathrm{kod}(X, M + \frac{1}{2}L) = \mathrm{kod}(Y, f_*M + \frac{1}{2}H) \geq 0,$$

and therefore $\mathrm{kod}(X) = \mathrm{kod}(X, E + \frac{1}{2}L) \geq \mathrm{kod}(X, L) = \dim Y$. □

In the proof of lemma 4.4, we used only the fact that the generic fiber of ϕ admits a perfect complex Poincaré parity. Therefore, lemma 4.4 can be stated in a more general way:

Proposition 4.5. *Let $\phi : X \to Y$ be an algebraic fibration such that its generic fiber admits a perfect complex Poincaré pairing and let $H \in \mathrm{Pic}\, Y$ be a big divisor, such that if $L = \phi^* H$, there exists an embedding $L \subseteq \Omega^i_X$.*

Then $\dim(Y) \leq \mathrm{kod}(X)$.

We will use this statement again later on.

5 Proof of theorem 1.4

In [1], Bogomolov proved that in a smooth projective variety X, any sub-line bundle $L \subseteq \Omega^i_X$ has Kodaira dimension less than i (this statement is known to be false in characteristic $p > 0$, see [16]). The idea of the proof was to consider the Iitaka map $\phi : X \to Y$ associated to L and show that any global form of L is monomial (see definition below) and induced by a form in Y.

Campana [6] improved the inequality for a variety X with zero Kodaira dimension (or, in greater generality, when X is special), by showing that for any line bundle $L \subseteq \Omega^i_X$, we have $\mathrm{kod}(X, L) < i$.

Inspired by the above results, we are going to prove that under the same assumption on the Kodaira dimension of X, we have $\mathrm{kod}(X, L) \leq i - 2$ (Campana's conjecture would imply $\mathrm{kod}(X, L) \leq 0$).

For the sake of completeness, we also sketch the proof of the above results.

A line bundle $L \subseteq \Omega^i_X$ is said to be *monomial* if any global section of $L^{\otimes k}$ can be written in a neighborhood of a generic point $p \in X$, as $\phi(z)(dz_1 \wedge \ldots dz_i)^{\otimes k}$, where ϕ is a holomorphic function.

Lemma 5.1. *Let $L \subseteq \Omega^i_X$ such that* $\mathrm{kod}(X, L) \geq i - 1$. *Then L is monomial.*

Proof. Let $\phi : X \to Y$ be the Iitaka fibration associated to L. We can suppose, without loss of generality, that ϕ is a regular map. Moreover, for the covering trick [1], we need only consider the global sections of L; in fact for any section $s \in \mathrm{H}^0(X, L^{\otimes k})$, there exists a cyclic covering $\pi : X' \to X$ and a section $t \in \mathrm{H}^0(X', \pi^* L)$ such that $\pi^* s = t^{\otimes k}$. For the same reason, we can suppose that ϕ is the map defined by the global sections of L.

On an analytic neighborhood of a generic point $p \in X$, we choose local coordinates so that the map ϕ has the form $\phi(x_1, \ldots, x_n) = (x_1, \ldots, x_p)$, where $p = \dim Y = \mathrm{kod}(X, L)$. Thus, for any section s_0 of L and for any $j = 1, \ldots, p$, $s_j = x_j s_0$ is also a section of L. Since any holomorphic $p-$form on X is closed, we have that $0 = ds_j = dx_j \wedge s_0 + x_j\, ds_0 = dx_j \wedge s_0$, i.e. dx_j divides s_0 and since $p \geq i-1$ we can write

$$s_0 = dx_0 \wedge dx_1 \wedge \cdots \wedge dx_{i-1} \wedge \omega,$$

for some holomorphic $1-$form ω on X. Thus L is monomial. □

From the proof of the previous lemma, it follows immediately that for any line bundle $L \subseteq \Omega^i_X$, we have $\mathrm{kod}(X, L) \leq i$. Let us suppose now that the equality holds and let $\phi : X \to Y$ be its Iitaka fibration. By Hironaka's theorem, we can suppose that ϕ is a *prepared* morphism, i.e. it is a regular map and, the locus S where ϕ is not smooth and its inverse image $\phi^{-1}(S)$ are contained inside simple normal crossing divisors in Y and X respectively. By lemma 5.1 (see also [6], thm 2.25), $L = \phi^*(K_Y)$ at a generic point of Y.

By [13] and [24] (see also prop. 4.15 in [6]), there exists a finite and flat covering $v : Y' \to Y$, with Y' smooth and such that if $\phi' : X' \to Y'$ is the map obtained by v, smoothing the base change of ϕ, then there exists an injection

$$\phi'_*(K_{X'/Y'}) \hookrightarrow v^*(\phi_*(K_X - L)).$$

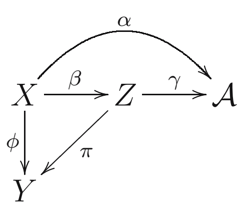
It follows that $\phi_*(K_X - L)$ is semi-positive, and in particular (lemma 4.10 in [6]) we have $\mathrm{kod}(X) \geq \mathrm{kod}(X, L) = i > 0$.

Let us suppose now that $\mathrm{kod}(L) = i-1$ and let us still denote by $\phi : X \to Y$ the fibration associated to L. As in the previous case, we can suppose that ϕ is a prepared morphism.

Lemma 5.1 implies that any section s of L can be locally written as $s = \phi^*\theta \wedge \omega$, where θ is an $(i-1)$-form of Y, i.e. a section of ω_Y and ω is a 1-form on X that defines a global $1-$form on a generic fiber of ϕ. In particular, it follows that $L \subseteq \phi^*\omega_Y \otimes \Omega^1_{X/Y}$, at the generic point of Y.

Every generic fiber of ϕ admits a global 1-form, and therefore it admits a non-trivial relative Albanese map $\alpha : X \to \mathcal{A}$ (see [3]). Let $\alpha : X \overset{\beta}{\to} Z \overset{\gamma}{\to} \mathcal{A}$ be its Stein Factorization, where β is an algebraic fibration and γ is a finite map onto its image.

We have the following diagram:



We will distinguish two different cases:

First Case: $\dim Z < \dim X$, i.e. β is not an isomorphism.

Since $L \subseteq \phi^*\Omega^{i-1}_Y \otimes \Omega^1_{X/Y}$, from the exact sequence of sheaves:

$$0 \to \beta^*\Omega^1_{Z/Y} \to \Omega^1_{X|Y} \to \Omega^1_{X/Z} \to 0$$

and by the definition of Albanese map, we have that

$$L \subseteq \phi^*\omega_Y \otimes \beta^*\Omega^1_{Z/Y} = \beta^*(\pi^*\omega_Y \otimes \Omega^1_{Z/Y}). \tag{5}$$

Since ϕ is the Iitaka fibration associated to L, we can suppose that $L = \phi^* L_0$, where L_0 is a big $\mathbb{Q}$-divisor on Y. Therefore, if $L_1 = \pi^* L_0$, then $L = \beta^* L_1$ and, by (5), it follows that

$$L_1 \subseteq \pi^*\omega_Y \otimes \Omega^1_{Z/Y} \hookrightarrow \Omega^i_Z.$$

Moreover, $\operatorname{kod}(Z, L_1) = \operatorname{kod}(X, L) = i - 1$.

Let us consider now the maximal rationally connected fibration $\eta : Z \dashrightarrow Z'$, such that its generic fiber is rationally connected.

By the results in [10], it follows that Z' is non-uniruled. Since $m = \dim Z < \dim X$, it follows that (R_m) holds. Moreover, Iitaka's conjecture $(C_{n,m})$ holds true for any $n \leq 4$ (e.g. see [20]). Therefore, since there exists a fibration from X onto Z', we have:

$$0 = \operatorname{kod}(X) \geq \operatorname{kod}(Z') \geq 0,$$

i.e. Z' has zero Kodaira dimension. If we can show that there exists a line bundle $L_2 \subseteq \Omega^j_{Z'}$ for some $j > 0$ such that $\operatorname{kod}(Z', L_2) = \operatorname{kod}(X, L)$, then we can conclude by induction that $\operatorname{kod}(X, L) = \operatorname{kod}(Z', L_1) = 0$.

The claim follows immediately by considering the exact sequence

$$0 \to \mathcal{N}^*_{Z_t|Z} \to i^*\Omega^1_Z \to \Omega^1_{Z_t} \to 0$$

where $\mathcal{N}_{Z_t|Z} \simeq \mathcal{O}^{\oplus d}_{Z_t}$ is the normal bundle of $i : Z_t \hookrightarrow Z$, and by the fact that the generic fiber of η is rationally connected and therefore $\mathrm{H}^0(Z_t, (\Omega^i_{Z_t})^k) = 0$ for any $i, m > 0$ (e.g. see [17]).

Second Case: $\dim Z = \dim X$, i.e. α is a finite map,

We want to show that this implies that a generic fiber of ϕ admits a perfect complex Poincaré pairing as in lemma 4.2.

Lemma 5.2. *Let W be a smooth variety such that its Albanese map $\alpha : W \to \operatorname{Alb}(W)$ is finite onto its image $\alpha(W)$.*

Then W admits a perfect complex Poincaré pairing.

Proof. Let $0 \neq \eta \in \mathrm{H}^0(\Omega^i_W)$. We want to show that the linear map

$$\cdot \wedge \eta : \mathrm{H}^0(W, \Omega^{m-i}_W) \to \mathrm{H}^0(W, \omega_W)$$

is not zero, where m is the dimension of W.

Since the Albanese map α is finite onto its image, we have that Ω^1_W is generically globally generated, i.e. the natural map

$$\mathrm{H}^0(W, \Omega^1_W) \otimes O_W \longrightarrow \Omega^1_W$$

is surjective on an open set of W. In fact, if $W' = \alpha(W)$ is the image of the Albanese map of W and $A = \operatorname{Alb} W$, then the map

$$\mathcal{O}_{W'} \otimes \mathrm{H}^0(W', \Omega^1_{W'}) = \Omega^1_{A|_{W'}} \longrightarrow \Omega^1_{W'}$$

is surjective and therefore the sheaf $\Omega^1_{W'}$ is globally generated. Since α is by assumption a finite map, the claim follows.

In particular, it follows that the map

$$\wedge^j \mathrm{H}^0(W, \Omega^1_W) \otimes O_W \longrightarrow \Omega^j_W$$

is surjective for any $i = 1, \ldots, m$.

Therefore, if $p \in W$ is a generic point, we can suppose that η is locally monomial, i.e. there exist local coordinates $z = (z_1, \ldots, z_m)$ and an holomorphic function ϕ, around a point $p \in W$, such that $\eta = \phi(z) dz_1 \wedge \cdots \wedge dz_i$.

Thus there exist linearly independent global 1-forms $\omega_1, \ldots, \omega_m \in \mathrm{H}^0(\Omega^1_W)$, such that, in a neighborhood of p, $\omega_j = dz_j$, for any $j = 1, \ldots, m$, and such that the m−form $\omega_1 \wedge \cdots \wedge \omega_m$ is not zero. In particular, there exists $\eta' \in \mathrm{H}^0(\Omega^{m-i})$, defined by $\eta' = \omega_{i+1} \wedge \cdots \wedge \omega_m$, such that $\eta \wedge \eta'$ is not zero. □

The proof of theorem 1.4 in this second case, follows therefore from prop. 4.5.

6 Conclusions

As explained in the introduction, theorems 1.2, 1.3 and 1.4 imply

Lemma 6.1. *Let X be a projective variety of dimension 4 and with zero Kodaira dimension. Then $k_1^+(X) = 0$, i.e. any invertible subsheaf $L \subseteq \Omega^i_X$ has non-positive Kodaira dimension.*

We can now proceed with the proof of the main result of this paper.

Proof (Proof of theorem 1.1).

It follows immediately from the definition that $k_1^+(X) \geq \operatorname{kod}(X)$.

Let $\phi : X \to \mathcal{I}_X$ be the Iitaka fibration of X. Then $\dim \mathcal{I}_X = \operatorname{kod}(X) > 0$. Let X_z be a generic fiber of ϕ, then X_z is connected and $\operatorname{kod}(X_z) = 0$. By theorem 6.1, we have $k_+(X_z) = 0$. Let $L \subseteq \Omega^i_X$ (for some $i > 0$) be a line bundle with positive Kodaira dimension. We want to show

Claim: $\operatorname{kod}(X_z, L_{|X_z}) \leq 0$.

The theorem will be a consequence of the claim. In fact, it implies that $h^0(X_z, L^n_{|X_z}) \leq 1$ for any n and therefore $\phi_* L^n$ is a sheaf of rank at most 1. Thus, $h^0(X, L^n) = h^0(\mathcal{I}_X, \phi_* L^n) \leq a n^k + b$, with a, b positive integers, and $k = \dim(\mathcal{I}_X) = \operatorname{kod}(X)$. Thus $\operatorname{kod}(X, L) \leq \operatorname{kod}(X)$ that concludes the proof.

In order to prove the claim, we consider the exact sequence

$$0 \to \mathcal{N}^*_{X_z|X} \to i^*\Omega^1_X \to \Omega^1_{X_z} \to 0.$$

where $\mathcal{N}_{X_z|X} \simeq \mathcal{O}^{\oplus d}_{X_z}$ is the normal bundle of $i : X_z \hookrightarrow X$, with $d = \mathrm{kod}(X)$.

By taking the $i-$th exterior power, we get a long exact sequence:

$$0 \to S^i\mathcal{O}^{\oplus d}_{X_z} \to S^{i-1}\mathcal{O}^{\oplus d}_{X_z} \otimes i^*\Omega^1_X \to \cdots \to i^*\Omega^i_X \to \Omega^i_{X_z} \to 0,$$

where $S^\bullet$ denotes the symmetric power.

Since L is contained in Ω^i_X, it follows that $L_{|X_z}$ must be contained in $\Omega^j_{X_z}$ for some $j \le i$, and therefore, by lemma 6.1, $\mathrm{kod}(X_z, L_{|X_z}) \le k_1^+(X_z) = 0$. □

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