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Singularities on complete algebraic varieties*

Fedor A. Bogomolov[1†], Paolo Cascini[2‡], Bruno de Oliveira[3§]

[1] *Courant Institute of Mathematical Sciences, New York, NY 10012, USA*

[2] *Department of Mathematics, UC Santa Barbara, Santa Barbara CA 93106, USA*

[3] *Department of mathematics, University of Miami, Coral Gables, FL 33124, USA*

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Abstract: We prove that any finite set of n-dimensional isolated algebraic singularities can be afforded on a simply connected projective variety.
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1 Introduction

It is a classical question in algebraic geometry to understand the constraints imposed on the singularities that can be afforded on a given class of algebraic varieties. A general result in this direction appeared in [4]. There it was shown that for any algebraic family of algebraic varieties there are isolated singular points which can not be afforded on any variety which is birationally equivalent to any member of this family. Our aim is to prove that any set of isolated algebraic n-dimensional singularities can be afforded on a simply connected projective variety.

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† E-mail: bogomolo@cims.nyu.edu
‡ E-mail: cascini@math.ucsb.edu
§ E-mail: bdeolive@math.miami.edu

More precisely we are going to prove the following result:

Theorem A. *Let (Y, y) be an isolated singularity. There exists a simply connected projective variety X having a unique singular point $x \in X$ such that the singularities (X, x) and (Y, y) are isomorphic.*

The variety X we are constructing is of general type and we believe that general type condition is necessary in order to afford arbitrary isolated singularity. We will give an interesting example of the result in [4]. We will describe which sets of rational double points can be afforded on rational surfaces (with the surprising fact that two E_6 can not be afforded).

This paper is also devoted to what we consider to be a useful description of singularities. We describe the germ of a reduced and irreducible analytic space as a finite cover of a polydisc Δ^n branched along smooth divisors of Δ^n. In particular, this gives a new description for the deformations of isolated singularities and provides with a simple proof of the fact that an irreducible and reduced germ of an analytic surface is algebraic.

Another motivation for theorem A was the work of C. Epstein and G. Henkin on the stability of the embeddability property of a strictly pseudo-convex 3-dimensional CR-structure [EH]. More precisely, C Epstein asked the third author if one can always embed an embeddable strictly pseudo-convex 3-dimensional CR-structure inside a regular variety. The methods used in [EH] view the embeddable CR-manifold M as the boundary of a pseudo-concave surface Y which can be attached to the Stein filling S of M to give a projective variety $X = Y \amalg_M S$. The properties of X, and especially the regularity, played an important role in their results.

2 Analytic Singularities

This section introduces a local description of analytic spaces that we think is very useful to the analysis of a spectrum of problems about singularities. We describe the germ of a reduced and irreducible analytic space as a finite cover of a polydisc Δ^n branched along smooth divisors of Δ^n. We give then a new description of the deformation space of an isolated singularity. Another application is a simple proof of the algebraicity of isolated surface singularities.

2.1 Local Parameterization

The following result is a simple modification of the lemma from [3] which extends Belyi's argument to the case of arbitrary field of characteristic zero.

Lemma 2.1. *Let Y be an n-dimensional affine variety. Then there exists a proper map $f : Y \to \mathbb{C}^n$ and a linear projection $p : \mathbb{C}^n \to \mathbb{C}^{n-1}$ such that f is ramified over a finite set of sections S_i of p.*

Proof. Consider an arbitrary finite surjective map $g : Y \to \mathbb{C}^n$. Let D be the ramification divisor of g in $\mathbb{C}^n$ and let $p : \mathbb{C}^n \to \mathbb{C}^{n-1}$ be a linear projection whose restriction to D is proper and surjective. The projection p is defined by a point $x \in \mathbb{P}^{n-1}_\infty$. To guarantee properness take x outside of the intersection of the closure $\bar{D} \in \mathbb{P}^n$ with $\mathbb{P}^{n-1}_\infty$. After a linear change of coordinates, $\phi : \mathbb{C}^n \to \mathbb{C}^n$, the projection p can be seen as the standard projection onto the last coordinate. Hence, we have a linear parameter z_n on all the fibers of p and $p(z_1, ..., z_n) = (z_1, ..., z_{n-1})$. By Noether normalization, the ramification divisor D_0 of $g_0 = \phi \circ g$ is given as the set of zeroes of a monic polynomial $f_0(z_n) = z_n^d + a_{d-1}z_n^{d-1} + ... + a_0 = 0$ with coefficients $a_i \in \mathbb{C}[z_1, ..., z_{n-1}]$, for any $i < n$.

Let $F_0 : \mathbb{C}^n \to \mathbb{C}^n$ be the branch cover of degree d defined by

$$F_0(z_1, ..., z_n) = (z_1, ..., z_{n-1}, f_0(z_n))$$

and denote by g_1 the composition $g_1 = F_0 \circ \phi \circ g_0$. The ramification divisor of g_1 is the union of the divisor $z_n = 0$ (corresponding to $F_0(D_0)$) and the divisor $D_1 = F_0(R_0)$, with $R_0 = \{(f_0)_{z_n}(z_n) = 0\}$ where $(f_0)_{z_n}(z_n) = dz_n^{d-1} + (d-1)a_{n-1}z_n^{d-2} + ... + a_1$. The projection p maps the divisor D_1 properly onto $\mathbb{C}^{n-1}$. The divisor R_0 has degree $d-1$ with respect to z_n and hence it's image $F_0(R_0) = D_1$ is defined by a monic polynomial $f_1(z_n)$ of degree $d_1 \le (d-1)$ in z_n.

Let $F_1 : \mathbb{C}^n \to \mathbb{C}^n$ be the branch cover of degree d_1 defined by

$$F_1(z_1, ..., z_n) = (z_1, ..., z_{n-1}, f_1(z_n))$$

and $g_2 = F_1 \circ g_1$. The ramification divisor is the union of two sections of p, $F_1(\{z_n = 0\})$ and $F_1(D_1) = \{z_n = 0\}$, and the divisor $D_2 = F_1(R_1)$ which is defined by a monic polynomial on z_n of degree $\le (d_1 - 1)$. In conclusion, after i-step we have the map $g_i = F_{i-1} \circ g_{i-1}$ with ramification divisor consisting of the union of i sections of p and a divisor $D_i = F_{i-1}(R_{i-1})$ which is defined by a monic polynomial on z_n of degree $\le (d-i)$. Therefore, we obtain the lemma after $l \le d$ steps. □

Remark 2.2. The proof of lemma 2.1 also works for a pair (X, D), where X is an arbitrary affine variety of dimension n and D is a divisor of X. In this case, the result would be that there is a finite map $f : X \to \mathbb{C}^n$ such that the ramification divisor of f and $f(D)$ are a set sections of a projection $p : \mathbb{C}^n \to \mathbb{C}^{n-1}$.

The previous result can be reformulated in the category of complex analytic spaces to give local results. One such reformulation is a refinement of the Local Parameterization Theorem.

Proposition 2.3 (Local Parameterization). *Let x be a point in a complex analytic space X of dimension n and suppose that X is locally irreducible and reduced at x. Then x has neighborhood $U \subset X$ with a finite map $f : U \to \Delta^n$ onto an n-polydisc $\Delta^n = \Delta^{n-1} \times \Delta$ ramified over a finite collection of sections S_i over Δ^{n-1}.*

Proof. The standard Local Parameterization Theorem states that all $x \in X$ have a neighborhood $U \subset X$ admitting a finite map $g : U \to \Delta^n$ onto a n-polydisc with $g(x) = (0, ..., 0)$. The refinement consists of showing that one can make the ramification divisor of the finite map very well behaved, which provides us with a tool to better understand singularities.

First, we remark that there is nothing to prove if x is not a singular point of X. Let $D \subset \Delta^n$ be the ramification divisor of the previously described finite map $g : U \to \Delta^n$. We can shrink U and choose a decomposition of the n-polydisc $\Delta^n = \Delta^{n-1} \times \Delta$ such that the projection of D onto Δ^{n-1} is a finite mapping. The proof of the standard LP theorem also gives that D is given by a Weierstrass polynomial $f_0(z_n) = z_n^d + a_{d-1}z_n^{d-1} + ... + a_0$ with $a_i \in O(\Delta^{n-1})$ with $a_i = O(|(z_1, ..., z_{n-1})|^{d-i})$.

The previous paragraph provides the setup to apply the method used in the previous lemma. We describe one of the steps to make clear the slight modifications. Using the Weierstrass polynomial $f_0(z_n)$ we construct the map $F_0 : \Delta^{n-1} \times \Delta \to \Delta^{n-1} \times \Delta'$, where Δ' is some disc, given by $F_0(z_1, ..., z_n) = (z_1, ..., z_{n-1}, f_0(z_n))$. The map $F_0 \circ g : U \to \Delta^{n-1} \times \Delta'$ might not be surjective. But by picking a smaller disc $\Delta_1 \subset \Delta'$ and shrinking U to $U = (F_0 \circ g)^{-1}(\Delta^{n-1} \times \Delta_1)$ we get a finite mapping $g_1 = F_0 \circ g : U \to \Delta^{n-1} \times \Delta_1$ ramified at $F_0(D) = \{z_n = 0\}$ and $D_1 = F_0(R)$ where $R = \{(f_0)_{z_n} = 0\}$. The divisor D_1 is given by the zero set of a Weierstrass polynomial $f_1(z_n) = z_n^{d_1} + a'_{d_1-1}z_n^{d_1-1} + ... + a'_0$ of degree $d_1 \leq d - 1$. Use $f_1(z_n)$ to construct F_1 and do the necessary shrinking of U, as before, and obtain a finite map $g_2 = F_1 \circ g_1 : U \to \Delta^{n-1} \times \Delta_2$. The desired finite map f will be the map $g_l : U \to \Delta^{n-1} \times \Delta_l$ obtained after some $l \leq d$ steps. □

2.2 Applications

In this section we show how to apply proposition 2.1 to obtain the algebraicity of the germs of normal 2-dimensional complex spaces and give a description of isolated singularities that might prove to be useful for the description of their deformations.

The Local Parameterization theorem presented in section 2.1 provides a simple proof of the algebraicity of any germ of an analytic surface. For isolated singularities this is a well known result due to Artin [A2] and later extended to a global result by Lempert [Le]. More precisely, Lempert showed that any reduced Stein space S with boundary $\partial S = M$ a smooth CR-manifold can be embedded in an algebraic variety. On the other hand, recall that in [13] (examples 14.1 and 14.2) Whitney shows that analytic singularities in dimensions $n \geq 3$ are, in general, not locally algebraic. Whitney constructs an example of a normal analytic variety V of dimension 3 and with a singular point $p \in V$, such that there exists no algebraic variety that is locally (in an analytic sense) biholomorphic to any open neighboorhood of p in V.

We proceed to show that all the analytic singularities in dimension 2 are locally algebraic. Let p be a point of a complex analytic surface S, and suppose that S is normal at p. By Proposition 2.3, there exists an open neighborhood $U \subseteq S$ of p, admitting a finite map $g : U \to \Delta^2$, where Δ^2 is a polydisc in $\mathbb{C}^2$ and such that $g(p) = (0, 0)$.

If the ramification divisor of g, $D \subset \Delta^2$, was an algebraic curve on Δ^2 (i.e. given by the zero locus of a polynomial), then U would be an open subset of an algebraic surface. But D is possibly reducible to an analytic curve in Δ^2. To deal with this case, we have:

Lemma 2.4. *Let $D \subseteq \Delta^2$ be a reduced analytic divisor, such that $(0,0) \in D$. Then, up to shrinking Δ^2, there exists a biholomorphic map from Δ^2 onto an open neighborhood V of $(0,0)$ in $\mathbb{C}^2$ such that the image D' of D is an algebraic divisor passing through $(0,0)$.*

Proof. Levinson proves a more general result in [10] (see also [13], remark 14.3). But, for the sake of completeness, we show an easy proof of the lemma. Let D be a union of irreducible components D_i, with $i = 1, \ldots, N$, passing through $(0,0)$.

By choosing a suitable system of coordinates z_1, z_2 and after shrinking the polydisc Δ^2, we can suppose that if $p_1 : \Delta^2 \to \Delta$ is the projection with respect to the first coordinate, then each D_i is a section of p_1. In other words, we can write each D_i as the zero set of the function $F_i(z_1, z_2) = z_2 - f_i(z_1)$, with f_i analytic.

We want to prove the lemma by induction on N (the number of irreduible components of D). Suppose that $f_1, \ldots, f_k$ are polynomials, with $k < N$. We want to construct a biholomorphism of the form

$$\Phi(z_1, z_2) = (z_1, z_2 + g(z_1, z_2) \prod_{i=1}^{k} F_i(z_1, z_2))$$

where g is an analytic function such that $\Phi(D_{k+1})$ is algebraic. In fact, by construction it follows that $\Phi(D_i) = D_i$ for any $i \le k$.

In order to reach our aim, we have to choose g such that the analytic function

$$P(z_1) = f_{k+1}(z_1) + g(z_1, f_{k+1}(z_1)) \prod_{i=1}^{k} (f_{k+1}(z_1) - f_i(z_1)) \tag{1}$$

is indeed a polynomial.

By shrinking Δ^2 again, if necessary, we can suppose that

$$\prod_{i=1}^{k} (f_{k+1}(z_1) - f_i(z_1)) = z_1^M \phi(z_1)$$

for some $M > 0$ and ϕ analytic function such that $\phi(0) \neq 0$.

Therefore we can find a holomorphic function g, satisfying (1), for any polynomial P such that $P(z_1) - f_{k+1}(z_1)$ is divisible by z_1^M. □

Our claim follows from the lemma. Choose U' defined by $h^{-1}(\Delta^2)$, where $h = \Phi \circ g : U \to V$ and $(0,0) \in \Delta^2 \subset V$, as the neighborhood of p. The open set U' is a branched covering of Δ^2 branched over an algebraic curve.

The Local Parameterization result described in proposition 2.1 gives directly the following description of isolated singularities.

Proposition 2.5. *Let s be a normal isolated singularity in a n-dimensional complex analytic space Y. Then:*

a) There is an open neighborhood of s, $U \subset Y$, admitting a finite map $f : U \to \Delta^n$, onto an n-polydisc, which is unramified outside of a finite set of smooth subvarieties $S_i \subset \Delta^n$.

b) The germ of the singularity $s \in Y$ is determined by the pair $(\Delta^n - \bigcup_i^k S_i, \Gamma)$, where Γ is the subgroup of finite index of $\pi_1(\Delta^n - \bigcup_i^k S_i)$ defining the covering.

The above picture of a singularity can be quite useful to determine the structure of the deformation space for many isolated singularities. Let (Y, s) be the germ of a normal n-dimensional singularity corresponding to the pair $(\Delta^n \setminus \bigcup_i^k S_i, \Gamma)$. Denote by $s_i \in \pi_1(\Delta^n \setminus \bigcup_i^k S_i)$ the simple loops around the irreducible components S_i. The s_1,...,s_k generate $\pi_1(\Delta^n \setminus \bigcup_i^k S_i) \cong \mathbb{Z}^m$, where m is the multiplicity of the irreducible holomorphic function germ g with $g^{-1}(0) = \bigcup_i^k S_i$. Let Γ' be the maximal normal subgroup of $\pi_1(\Delta^n \setminus \bigcup_i^k S_i)$ contained in Γ. The next short exact sequence holds:

$$1 \to \Gamma' \to \pi_1(\Delta^n \setminus \bigcup_i^k S_i) \to G \to 1$$

where G is the Galois group of the cover induced by Γ'.

Consider a deformation $\bigcup_i^k S_i^t$ of $\bigcup_i^k S_i$. Let T be a tubular neighborhood of $\bigcup_i^k S_i$. The complement $\Delta^n \setminus T$ is homotopically equivalent to $\Delta^n \setminus \bigcup_i^k S_i$ and it is immersed in $\Delta^n \setminus \bigcup_i^k S_i^t$ for $\frac{1}{|t|} \gg 0$. Hence there is a natural homomorphism $j_t : \pi_1(\Delta^n \setminus \bigcup_i^k S_i) \to \pi_1(\Delta^n \setminus \bigcup_i^k S_i^t)$ for $\frac{1}{|t|} \gg 0$.

Assume that a surjection $r_t : \pi_1(\Delta^n \setminus \bigcup_i^k S_i^t) \twoheadrightarrow G$ holds and moreover that $r_t o j_t : \pi_1(\Delta^n \setminus \bigcup_i^k S_i) \twoheadrightarrow G$ is constant for $\frac{1}{|t|} \gg 0$. This implies that a Galois cover, associated with G, of $\Delta^n \setminus \bigcup_i^k S_i^t$ persists for small t and the induced covering of $\Delta^n \setminus T \subset \Delta^n \setminus \bigcup_i^k S_i^t$ is constant along the family. In turn, this implies that an intermediate covering associated with Γ inducing a constant covering of $\Delta^n \setminus T \subset \Delta^n \setminus \bigcup_i^k S_i^t$ also persists for small t.

The end result is that from a family of divisors $\bigcup_i^k S_i^t \subset \Delta^n$ for which $r_t o j_t : \pi_1(\Delta^n \setminus \bigcup_i^k S_i) \twoheadrightarrow G$ is constant for $\frac{1}{|t|} \gg 0$ one obtains a family of singularities Y_t associated with the pairs $(\Delta^n \setminus \bigcup_i^k S_i^t, \Gamma)$. The singularities Y_t all have a finite map $f_t : Y_t \to \Delta^n$ of the same degree branched at $\bigcup_i^k S_i^t$. Moreover, the Y_t have an arbitrarily large open subset $f_t^{-1}(\Delta^n \setminus T) \subset f_t^{-1}(\Delta^n \setminus \bigcup_i^k S_i^t)$ which is biholomorphic to $f_0^{-1}(\Delta^n \setminus T)$ for all sufficiently small t. The conditions to impose on the S_i^t to guarantee the constancy of $r_t o j_t : \pi_1(\Delta^n \setminus \bigcup_i^k S_i) \twoheadrightarrow G$ will be investigated in future work.

3 Singularities inside Projective varieties

Any collection of isolated singularities can be afforded in some projective variety (see paragraph below). On the other hand, a collection of singularities, or even one single singularity, does impose global constraints on the type of the variety that possesses it (see the next subsection). The main goal of this section, theorem A, is to show that the property of being simply connected is not one of the properties which is conditioned by the presence of singularities. Along the same lines we would like to conjecture a stronger result:

Conjecture 3.1. *Let (Y, y) be the germ of a given isolated singularity. There exists a projective variety X containing Y and with $X \setminus \{y\}$ smooth whose resolution $\hat{X}$ is simply connected.*

The following lemma shows that every finite set of isolated singularities, can be afforded in a unique projective variety.

Lemma 3.2. *Let $\Gamma = \{(Y_i, y_i)\}_{i=1,\dots,k}$ be any collection of germs of algebraic isolated singularities of dimension n. There exists a projective variety Y having Γ as its singular locus.*

Proof. Let X_i be a variety with only one singular point and the germ of the singularity is equivalent to (Y_i, y_i). The lemma follows from induction on k. Let us assume that we constructed a projective variety Y'_{k-1} with $\Gamma_{k-1} = \{(Y_i, y_i)\}_{i=1\dots k-1}$ as its singular locus. Let H_n be a general n-codimensional plane in the product variety $Y'_{k-1} \times X_k$ and let us consider

$$Y'_k = (Y'_{k-1} \times X_k) \cap H_n.$$

We can choose H_n so that it intersects transversely the singular locus $\{y_i\} \times X_k$ or $Y'_{k-1} \times \{y_k\}$ and avoids the points $y_i \times y_k$. Therefore Y'_k is a n-dimensional projective variety whose singularities are isomorphic to the singularities $(Y_i, y_i)_{i=1\dots k}$.

In order to have exactly one copy of each singularity, it is enough to resolve the possible extra copies of the singularities $(Y_i, y_i)_{i=1\dots k}$ that might occur. □

3.1 An example of constraints imposed by singularities

In the introduction we recalled a recent result of Ciliberto and Greco stating that for any algebraic family of algebraic varieties there are isolated singular points which can not be afforded on any variety birational to a member of this family. We proceed to give a concrete example of this result. More precisely, we describe all the sets of rational double points, RDP's, that a rational surface can contain (the same result holds for all surfaces of Kodaira dimension $-\infty$).

Notation. Let X and Y be analytic normal complex surfaces and $f : Y \to X$ be a birational morphism with exceptional set $E = \sum E_i$. The negative definiteness of the intersection matrix (E_i, E_j) allow the existence of a unique solution to:

$$K_Y \equiv f^* K_X + \sum a_i E_i$$

The numbers a_i are called the discrepancy of E_i with respect to X, discrep$(E_i, X) = a_i$. The birational morphism will be called totally discrepant if $E \neq \emptyset$ and the $a_i > 0$, for all i.

It was shown in Sakai [S1] that given a normal surface X, there is a sequence $X \to X_1 \to X_2 \to ... \to X_n$ of contractions of exceptional curves of the first kind, i.e $C^2 < 0$ and $K_{X_i}.C < 0$, such that X_n has no such curves. X_n is then called a minimal model of X and the morphism $f : X \to X_n$ is totally discrepant. Let X' be the minimal model of the normal surface X and $f : X \to X'$ be the totally discreptant birational morphism. Let $\pi' : Y' \to X'$ and $\pi : Y \to X$ be respectively the minimal resolutions of X' and X, with $K_{Y'} = \pi'^* K_{X'} + \Delta'$ and $K_Y = \pi^* K_X + \Delta$. Then f induces a birational morphism $g : Y' \to Y$ such that $g_* \Delta \geq \Delta'$ (this result supports the statement that going to the minimal model does not make singularities worse).

A normal surface singularity (X, x_0) is an RDP (rational double point) iff $K_Y.E_i = 0$ for every exceptional curve E_i of the minimal resolution $f : Y \to X$ or equivalently $f^* K_X = K_Y + \Delta$ with $\Delta = 0$. From the definition of an RDP singularity follows that the negative configuration of curves that form the exceptional set of f is composed of smooth rational curves with self intersection -2 in one of formations of the Dinkin diagrams $A_n(n = 1, ...), D_n(n = 4, ...), E_n(n = 6, 7, 8)$. The observation of the previous paragraph implies that if a normal surface X has only RDP singularities then the same is true for its minimal model.

Theorem 3.3. *The collection of rational double points that can be in a rational surface X are the following:*
1. Arbitrary collections of A_n and D_n singularities
2. An E_n singularity and an arbitrary collections of A_n singularities.

Proof. First, we give the positive results. By blowing up over a point, one can get an A_n configuration of negative curves. Hence all birational classes of surfaces can have as many A_n singularities as desired.

A D_n configuration of negative curves can be obtained by blowing up over a smooth rational curve C with $C^2 = 0$. Hence one can get as many D_n singularities as desired in all birational classes of ruled surfaces.

An E_n configuration can not be obtained by blowing up over a smooth rational curve C with $C^2 = 0$. This is the reason behind the asymmetry of the theorem. On the other hand, one E_n configuration of negative curves can be obtained by blowing up over two lines in $\mathbb{P}^2$. Hence one can get one E_n singularity in the birational class of rational surfaces.

The minimal model program for normal singular surfaces developed by Sakai will give the negative results. Let Y be a normal surface with two E_n singularities or an E_n and one D_m singularity. Resolve all the other singularities and still name that surface Y. The minimal model Y_m of Y is a surface with only rational double points, as explained above. Moreover, the singularities E_n and D_m of Y will still exist in the minimal model Y_m [Mo]. Let $f : X \to Y_m$ be the minimal resolution of Y_m (i.e. no (-1)-curves on the exceptional locus). Assume, as in the hypothesis of the theorem, that X is rational, then since the singularities of Y_m are rational, $K_X = f^* K_{Y_m}$, one has $\mathrm{Kod}(Y_m) = -\infty$ and K_{Y_m} is not nef.

Sakai [S2] proved that if W is a minimal normal surface whose canonical bundle K_W is not nef then W is projective, $\mathrm{Kod}(W) = -\infty$ and either:

i) $\rho(W) = 1$ and $-K_W$ is numerically ample, i.e. $K_W^2 > 0$ and $K_W.C < 0$ for all curves $C \subset W$, or

ii) W has a $\mathbb{P}^1$-fibration.

So according to Sakai's result Y_m must be one of the two cases described above. We will show that both cases are not possible.

Suppose Y_m is as in i). The minimal resolution X of Y_m is rational and has $K_X^2 = (f^* K_{Y_m})^2 > 0$. Hence X is $\mathbb{P}^2$ blown up at most 8 times or one of the Hirzebruch surfaces is F_n blown up at most 7 points. In both cases $b_2^-(X) \leq 8$. But on the other hand the minimal resolution of Y_m must have $b_2^-(X) \geq n + m \geq 10$ and we obtain a contradiction. The inequality is just a consequence of the linear independence of homology classes of the curves in the exceptional locus.

Suppose Y_m is as in ii). The $\mathbb{P}^1$-fibration of Y_m induces a ruled-fibration, $\pi : X \to C$, of X. The configuration of (-2)-curves coming from the resolution of the E_n singularity lies in one of the fibers. The surface resulting from contracting the (-1)-curves in the fibers of π is an Hirzebruch surface F_n. But an E_n configuration of (-2)-curves can not be obtained by blowing up over a smooth rational curve C with $C^2 = 0$ and the desired contradiction follows. □

Corollary 3.4. *A surface X which is a resolution of a surface Y containing a E_n and a D_n singularity must have its Kodaira dimension* $\mathrm{Kod}(X) \geq 0$.

Proof. The last theorem states that X is not a rational surface. On the other hand, an E_n configuration of negative curves does not lie entirely in the blow up pre-image of a fiber of a ruled surface. This would force one of the (-2)-curves to surject to the base of the ruled surface imposing that X is rational. □

Corollary 3.5. *There is a singularity that can not be afforded in a projective surface X with* $\mathrm{Kod}(X) = -\infty$.

Proof. Let X' be a smooth projective surface with a E_n and a D_m configurations of -2-curves which are disjoint. Let H be an ample divisor on X', blow up X' at a sufficiently large number of points on H but not on the configurations E_n or D_m. We obtain a new surface X'' with a negative configuration of curves consisting of H' (the strict transform of H) plus the curves coming from E_n and D_m (the negative definiteness is guaranteed by making $H'^2 << 0$). Now contract this negative configuration of curves. By the previous corollary, the singularity that is obtained does not lie in a surface X with $\mathrm{Kod}(X) = -\infty$. □

3.2 Symmetric powers

In this subsection, we show that any germ of an algebraic singularity (X, s) can be realized in a projective variety Y satisfying $Y_{Sing} = s$ and such that its smooth locus has abelian fundamental group. In particular, also Y will have abelian fundamental group.

The construction will be based on the topological properties of symmetric powers of algebraic varieties. In fact Y will be a generic complete intersection of S^2X.

For any CW-complex X we can define an m-th symmetric power S^mX as the quotient of the CW-complex $X^m = X \times ... \times X$ by the symmetric group of m-letters S_m. Hence S^mX is also a CW-complex with a natural morphism $s_m : X^m \to S^mX$. For the sake of the readability of this paper, we recall some key topological properties of symmetric products with a short proof.

Lemma 3.6. *Let X be a CW-complex then the induced CW-complex S^mX has the following properties.*
(1) $\pi_1(S^nX) = H_1(X, \mathbb{Z}), n > 1$
(2) $H^i(S^lX, \mathbb{R}) = H^i(S^mX, \mathbb{R})$ *for* $m, l > i$

Proof.
(1) The fundamental group of S^nX is generated by the fundamental group of X. In particular, given the map $s_n : X^n \to X^n$ with $n > 1$, every two elements in $\pi_1(S^nX)$ can be thought as induced by the first and second factor respectively. Thus, their commutator is trivial in $\pi_1(S^nX)$.

(2) The cochains of S^mX are symmetrizations of the cochains in the product of m copies of X. Thus for $i < n$ symmetric cochains are generated by cochains in the product of $\leq i$ copies of X multiplied by 0-dimensional cochains. It implies that the i-skeletons of S^nX and S^mX are isomorphic for $i < \min(m, n)$.

□

If X is an algebraic or projective variety then S^mX is respectively an algebraic or projective variety. The variety S^mX is singular unless X is a non-singular curve. Let us consider the case where X is an algebraic variety of dimension n with a finite collection of singular points $\Gamma = \{s_1, ..., s_k\}$. Denote $U = X_{reg} = X \setminus X_{Sing}$ and any of the i-diagonals of U^m (the entries of a fixed set of i places of U^m are identical) by Δ_i. We have the following stratification $\mathcal{S}$ of S^mX:
(1) $(S^mX)_{reg}$.

(2) $P_i[m] = s_m(\Delta_{i+1}) \setminus \cup_{j=i+2}^m s_m(\Delta_j)$, $1 \leq i \leq m-1$.
(3) $\Sigma_i = S^i(\Gamma) \times (S^{m-i}U)_{reg}$ $1 \leq i \leq m$.
(4) $\Sigma P_{ij} = S^i(\Gamma) \times P_j[m-i]$, $1 \leq i \leq m$ and $1 \leq j \leq m-i$.

We denote the complement of the union of of all strata of codimension $\geq (i+1)n$ by $(S^mX)_i$. The $(S^mX)_i$ are Zariski open subsets of S^mX. For example, $(S^mX)_0 = (S^mX)_{reg}$ and $(S^mX)_1 = S^mX)_{reg} \cup P_1[m] \cup \Sigma_1$. The following dimensional properties hold for the strata:

(1) $\text{codim} P_i[m] = in$, the singularities along $P_i[m]$ are simple quotient singularities.
(2) $\text{codim}\Sigma_i = in$.
(3) $\text{codim}\Sigma P_{ij} = (i+j)n$.

We are now ready to state the main result of this section:

Theorem 3.7. *Let $\Gamma = \{(Y_i, y_i)\}_{i=1,\ldots,k}$ be any collection of germs of equidimensional isolated singularities. There exists a projective variety X with abelian fundamental group whose collection of singular points coincide with Γ. Moreover if Y is a projective variety with $Y_{Sing} = \Gamma$, then X can be made such that $\pi_1(X \setminus \Gamma) = H_1(Y \setminus \Gamma, \mathbb{Z})$.*

Proof. Let Y be a projective variety whose collection of singular points coincide with Γ (lemma 3.2).

From lemma 3.6, it follows that the fundamental group of the symmetric product of any algebraic variety is abelian, and therefore we would like to take a generic complete intersection Z in S^2Y of the same dimension of Y and that contains the same singularities of Y in such a way that $\pi_1(Z) = \pi_1(S^2Y)$. That would imply that Z has an abelian fundamental group.

Lefschetz theorem on hyperplane sections states that if W is an algebraic variety with $\dim W > 2$ and $H \subset W$ is an hyperplane section such that $W \setminus H$ is smooth then $\pi_1(W) = \pi_1(H)$. For the variety S^2Y that we are considering, Lefshetz theorem cannot be applied directly, because we want to study complete intersection subvarieties that are transverse to the singular locus of S^2Y.

Hence, let us consider n generic hyperplane sections $H_1, \ldots, H_n$ of Y, passing through the singular points of Y.

On the product $Y^2 = Y \times Y$, let $p_i : Y^2 \to Y$ with $i = 1, 2$, be the respective projections and let H be the intersection of the divisors $p_1^{-1}(H_j) \cup p_2^{-1}(H_j)$ with $j = 1, \ldots n$.

Then H is a complete intersection of very ample divisors on Y^2 that is invariant with respect of the natural action of the group $\mathbb{Z}_2$ on Y^2. We denote its quotient by

$$Z = H/\mathbb{Z}_2 \subseteq S^2Y.$$

Let U be the smooth locus of Y, i.e. $U = Y \setminus \Gamma$, and let $H_U = H \cap (U \times U)$.

By applying Lefshetz theorem on H_U. it follows that $\pi_1(H_U) = \pi(U \times U)$ and since the action induced by $\mathbb{Z}_2$ on those groups is the same, we have that if Z_U is the quotient of H_U by $\mathbb{Z}_2$, then Z_U has the same fundamental group of S^2U and in particular this is abelian.

On the other hand Z_U is also an open set of Z such that its complement is a union of a finite number of point and therefore the fundamental group $\pi_1(Z)$ is abelian since surjection $\pi_1(Z_U) \to \pi_1(Z)$ holds.

From the construction of H, it follows easily that the singularities of Z are isolated and decompose into $Z_{Sing} = (Z \cap \Sigma_1) \cup (Z \cap P_1)$. The singularities in $Z \cap \Sigma_1$ are equivalent to the isolated singularities of Y and the singularities in $Z \cap P_1$ are double points.

Let X be the projective variety obtained from Z by resolving the double points. Then X has abelian fundamental group and its singular locus coincides with Γ, as desired. $\square$

Corollary 3.8. *Let Y be a projective variety with a given collection Γ of isolated singular points such that $H_1(Y, \mathbb{Z}) = 0$. Then there exists a projective variety X with $X_{Sing} = \Gamma$ which is simply connected. If additionally Y is such that $H_1(Y \setminus \Gamma, \mathbb{Z}) = 0$ then X can be made so that $X \setminus \Gamma$ is also simply connected.*

4 Reducing the abelian fundamental group

We are now ready to prove theorem A. Let S be a given isolated singolarity. By the results in the previous section, we can suppose that there exists a variety X such that $X_{Sing} \simeq S$, and, if $U = X \setminus X_{Sing}$, then $\pi_1(U)$ is abelian and the imbedding $U \hookrightarrow X$ defines a surjection $\pi_1(U) \to \pi_1(X)$.

Let us consider the infinite part of the fundamental group of X. If it is trivial, then the group is finite and hence there is a finite nonramified covering of X which is simply connected.

Thus, we can suppose that the Albanese map $f : X \to A := Alb(X)$ is not trivial. The torsion subgroup $\pi_1(X)_{tors}$ is a direct summand of $\pi_1(X)$, and therefore if $\pi_1(X)_F$ is the complementary subgroup, the induced morphism

$$f_* : \pi_1(X)_F \to \pi_1(A)$$

is an isomorphism.

For every $n > 0$, we can consider the iteration map $f_n : S^n X \to A$, given by $f_n(x_1 \dots x_n) = \sum f(x_i)$. Since $f(X)$ generates A, there exists n_0 such that if $n \geq n_0$, then f_n is surjective.

Moreover, we have

Lemma 4.1. *There exists a positive integer $m > 0$ such that the map $f_m : S^m X \to A$ admits a topological section $s : A \to S^m X$.*

Proof. Let $g = \dim A$, and let $[\gamma_1], \dots, [\gamma_{2g}]$ be generators of $\pi_1(A)$, given by considering the i-th component of $\Pi_{i=1}^{2g} S^1$, for some homeomorphism $\Pi_{i=1}^{2g} S^1 \simeq A$.

Since the induced map $f_* : \pi_1(X) \to \pi_1(A)$ is surjective, for each $i = 1, \dots, 2g$, there exists an injective map $r_i : S^1 \hookrightarrow X$, so that $f_*(r_i) = \gamma_i$. In particular, these maps induce an isomorphism $\pi_1(A) \to \pi_1(X)_F$.

The map $f_{2g} \circ r : \Pi S_i^1 \to A$, given by $f_{2g} \circ r(z_1, \dots, z_{2g}) = \sum_{i=1}^{2g} \gamma_i(z_i)$ is a homeomorphism and therefore the continuous map

$$r = \Pi r_i : A \simeq \Pi S^1 \to S^{2g} X$$

defines a topological section for $f_{2g} : S^{2g}X \to A$ (and in fact, more generally for $X^{2g} \to A$). □

Remark 4.2. The homotopy class of the section s above, is defined by the homotopy class of the corresponding subgroup $\pi_1(X)_F$, generated by the elements $r_i : S^1 \hookrightarrow X$.

Lemma 4.3. *If* $\dim X = n$*, then for* $N \geq i$*, the natural imbedding* $S^i X \hookrightarrow S^N X$ *is a homotopy equivalence up to dimension* $i - 1$*.*

Proof. By lemma 3.6, the statement is true for the fundamental group of X.

Moreover, it also holds for the homotopy groups $H^i(S^i X)$, since it is generated by products of elements in $H^{i_s}(X)$ with $i_1, \dots, i_k$ so that their sum is equal to i and hence all such product are represented on $S^i X$.

In fact this is true on the level of complexes. Indeed the cells of dimension i in $S^N X$ are obtained from the cells $i_1, ..., i_k$ with sum equal to i in X. Thus each cell is the image of a product of at most i simplices from X and hence it comes from $S^i X$.

In particular the imbedding $S^i X \hookrightarrow S^N X$ is an homotopy equivalence up to dimension $i - 1$. □

By lemma 4.1, there exists $m > 0$ and a topological section $s : A \to S^m X$. In particular, $f_n : S^n X \to A$ is surjective for any $n \geq m$.

Let $R_x^n \subseteq S^n X$ be the fiber of $f_n : S^n X \to A$ over a point $x \in A$. Moreover, let $\tilde{S}^n X \to S^n X$ be the abelian cover induced by the universal cover $\mathbb{C}^g \to A$, and let $\tilde{f}_n : \tilde{S}^n X \to \mathbb{C}^g$.

In particular, $\tilde{f}_m$ admits a topogical section $\tilde{s} : \mathbb{C}^g \to \tilde{S}^m X$, obtained as a cover of $s(A)$. Moreover the natural map $S^n X \times S^k X \to S^{n+k} X$, can be lifted to the map $\tilde{S}^n X \times \tilde{S}^k X \to \tilde{S}^{n+k} X$.

Lemma 4.4. *The natural imbedding* $i_n : R_x^n \hookrightarrow \tilde{S}^n X$ *is an homotopy equivalence up to dimension* $n - m - 1$*.*

Proof. By lemma 4.3, the k-skeleton of $\tilde{S}^m X$ can be contracted to any subvariety

$$\tilde{S}^{k+1} X \times c \subseteq \tilde{S}^m X$$

where $c \in S^{m-k-1} X$ is any cycle.

Consider the map

$$\phi_k : \tilde{S}^{k+1} X \to \tilde{S}^{k+1+m} X$$

which maps $p \in \tilde{S}^{k+1}$ to $p \cdot \tilde{s}(-\tilde{f}_{k+1}(p)) \in \tilde{S}^{k+1+m}$.

Thus, we have

$$\tilde{f}_{k+m+1}(\phi_k(p)) = \tilde{f}_{k+1}(p) + \tilde{f}_m(\tilde{s}(-\tilde{f}_{k+1}(p))) = 0.$$

Therefore ϕ_k maps $S^{k+1}X$ inside $\tilde{R}_0^{k+m+1}$.

Moreover it is homotopy equivalent to the standard imbedding. In fact the map

$$(t,p) \mapsto p \cdot s(-t\tilde{f}_{k+1}(p)) \qquad \text{for } 0 \leq t \leq 1$$

defines the homotopy equivalence.

Thus $\tilde{S}^{k+1}X \subseteq \tilde{S}^{k+m+1}X$ is homotopy equivalent to its image in R_0^{k+m+1}, and therefore any nontrivial homotopy in $\tilde{S}^{k+m+1}$ is the same as in R_0^{k+m+1}, up to dimension k. This implies the lemma. □

Remark 4.5. The same result and proof applies for any continuous map $a : S \to T$ from a topological space S to a torus T, provided that the induced map $a^{\#} : \pi(S) \to \pi(T)$ is surjective and the map $\pi(S)^{ab} \to \pi(T)^{ab}$ is an isomorphism.

Let $U \subseteq X$ be an open smooth subvariety of X, so that the natural map $\pi_1(U) \to \pi_1(X)$ is a surjection. Let $RU_x^n \subseteq R_x^n$ be the fiber for the induced map $S^nU \to A$. In particular, RU_x^n is quasi-smooth, i.e. it has only quotient singularities and, by the same arguments used in lemma 4.4, the fibers RU_x^n are homotopically equivalent up to dimension $n-m-1$.

Thus, if $n > m+1$ then the fundamental group of RU_x^n is abelian and equal to the kernel of the map $\pi_1(U) \to \pi_1(A)$.

Fixed $x \in A$, let M be the union of RU_x^n with the intersection of R_x^n and the image of the map $S^{n-1}U \times X_{Sing} \to S^nX$ then the resulting variety $M \subset R_n(X)$ has the following properties:

(1) $\pi_1(M) = \pi_1(R_x^n)$;
(2) $\operatorname{codim}(R_x^n \setminus M) > \dim X$;
(3) M is quasi-smooth outside a singular subset which is locally isomorphic to $SingX \times D$, where D is a polydisk.

Thus if we take now a complete intersection of M of dimension equal to the dimension of X then the resulting variety X' will have isolated singularities which are the same as X and $\pi_1(X') = \pi_1(M)$ and hence it is a finite abelian group. This finishes the construction and the proof. □

References

[1] M. Artin: "On the solutions of analytic equations", *Invent. Math.*, Vol. 5, (1968), pp. 277–291.

[2] M. Artin: "Algebraic approximation of structures over complete local rings", *Publ. Math. I.H.E.S.*, Vol. 36, (1969), pp. 23–58.

[3] F.A. Bogomolov and T. Pantev: "Weak Hironaka Theorem", *Math. Res. Let.*, Vol. 3, (1996), pp. 299–307.

[4] C. Ciliberto and S. Greco: "On normal surface singularities and a problem of Enriques", *Commun. Algebra*, Vol. 28(12), (2000), pp. 5891–5913.

[5] C. Epstein and G. Henkin: "Stability of embeddings for pseudoconcave surfaces and their boundaries", *Acta Math.*, Vol. 185(2), (2000), pp. 161–237.

[6] M J. Mather: *Notes on Topological Stability*, Mimeographed Notes, Harvard University, 1970.

[7] D. Morrison: "The birational geometry of surfaces with rational double points", *Math. Ann.*, Vol. 271, (1985), pp. 415–438.

[8] R. Hartdt: "Topological Properties of subanalytic sets", *Trans. Amer. Math. Soc.*, Vol. 211, (1975), pp. 193–208.

[9] L. Lempert: "Algebraic approximations in analytic geometry", *Inv. Math.*, Vol. 121(2), (1995), pp. 335–353.

[10] N. Levinson: "A polynomial canonical form for certain analytic functions of two variables at a critical point", *Bull. Am. Math. Soc.*, Vol. 66, (1960), 366–368.

[11] F. Sakai: "Weil divisors on normal surfaces", *Duke Math.*, Vol. 51, (1984), pp. 877–887.

[12] F. Sakai: "The structure of normal surfaces", *Duke Math.*, Vol. 52, (1985), pp. 627–648.

[13] H. Whitney: *Local Properties of Analytic Varieties*, Differential and Combinatorial Topology, Princeton Univ. Press, Princeton N.J., 1965, pp. 205–244.