Some existence results for the Navier-Stokes equation driven by heat conduction

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Abstract

The Navier-Stokes equation driven by heat conduction is studied. As a prototype we consider Rayleigh-Bénard convection, in the Boussinesq approximation. Under a large aspect ratio assumption, which is the case in Rayleigh-Bénard experiments with Prandtl numer close to one, we prove the existence of a global strong solution to the 3D Navier-Stokes equation coupled with a heat equation, and the existence of a maximal B-attractor.

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1 Introduction

In this paper we study the Navier-Stokes equation driven by heat conduction. Under a large aspect ratio assumption (the spatial domain being a thin layer) we prove the existence of a global strong solution. The aspect ratio Γ is the width/height ratio of the experimental apparatus and $\Gamma \geq 40$ is considered large, see Hu et al. [16, 17]. For existence results for the Navier-Stokes equation in thin domains we particularly refer to Raugel [36] and the references therein. Our contribution is the addition of a heat equation where the heat conduction is driving the fluid. The prototype we have in mind is the classical Bénard problem and this work is motivated by the instabilities of roll-patterns observed in experiments and simulations. Rayleigh-Bénard convection is a model for pattern formation and has been extensively studied, Busse and Clever [8, 11] established the stability of straight parallel convection rolls and used them to explain many experimental observations, for large Prandtl numbers $\sigma = \frac{\nu}{\kappa}$ (ν is the kinematic viscosity and κ is the thermal diffusivity). For low Prandtl numbers, the situation is much more complicated and it has long been recognized [12, 14, 32, 15] that mean flows are crucial in understanding the complex pattern dynamics observed in experiments. In this paper we establish that this complexity is not caused by singularity formation but our ultimate goal is a theoretical understanding and a quantitative capture of the mean flow. It is known that wave-number distortion, roll curvature and the mean flow make straight convection rolls become unstable [29, 34, 1, 19]. These effects have been succesfully modelled by Decker and Pesch [15] and simulated. However, the resulting equations contain non-local terms due to the mean-flow and are theoretically intractable. It is our hope that our results will put the study of the contribution of the mean flow on a rigorous mathematical footing, both simplifying and reaching a better theoretical understanding in the process. The experimental difficulties of measuring weak global flow in the presence of dominant local roll circulations are formidable, [17], and a theoretical insight may be crucial in the case of dynamical patterns.

As a preparation, we prove the classical, see Leray [25], existence of a global strong solution and a global attractor for the three-dimensional Navier-Stokes equation under smallness assumptions on the data, see Ladyshenskaya [22]. Then we prove the existence of a global solutions of the Rayleigh-Bénard problem with a large aspect ratio. The existence is proven after an initial time-interval, corresponding to a settling-down period in experiments. In experiments in gases (CO_2) this settling-down time is a few hours for experiments that take a few days, [17].

We will also prove that the Rayleigh-Bénard problem has a global attractor. Then theorems of Milnor [30], Birnir and Grauer [5], and Birnir [3] are used to prove that the Rayleigh-Bénard problem has a unique maximal B-attractor. This attractor has the property that every point attracts a set (of functions) that is not shy [20], or of positive infinite-dimensional measure. The discovery of a spiral-defect chaotic attractor [31, 19, 18], in a parameter region were previously only straight rolls were known to be stable [35, 13], was one of the more startling results in recent pattern formation theory. The experimentally observed attractors are B-attractors, and the spiral-defect chaotic attractor and the straight roll attractor may be two minimal B-attractors of the unique maximal B-attractor whose existence we prove.

The existence of the mean-flow will be proven in another paper [7] based on this one. We introduce here a scaling that will produce the mean flow and give an existence theorem for the scaled equations. The existence of the mean flow will then be rigorously established in the sequel to this paper [7] using the technique of homogenization.

2 The Navier-Stokes equation

We will denote by $|\cdot|$ and $\|\cdot\|$ the usual norms in $L^2(\Omega)$ and $H^1(\Omega)$. We denote by $\|\cdot\|_2$ the $H^2(\Omega)$-norm and by $|\cdot|_\infty$ the $L^\infty(\Omega)$-norm. Further, $|u|_{2,\infty} = \sup|u(t)|$, where supremum is taken over all $t \geq 0$.

Let us consider the Navier-Stokes equation for incompressible fluids

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f, & \\ & x \in \Omega,\ t \in \mathbf{R}^+, \\ \operatorname{div} u = 0, & \end{cases} \tag{2.1}$$

where u is the velocity, p is the pressure and ν is the viscosity, with initial condition

$$u(x,0) = u_0(x)$$

and vanishing boundary conditions on $\partial\Omega$ (periodic boundary conditions with mean zero, if $\Omega = T^3$). f denotes the forcing and λ_1 is the smallest eigenvalue of $-\Delta$ on Ω, with vanishing boundary conditions on $\partial\Omega$. We start by the following estimate which goes back to Leray [25]:

Lemma 2.1 *Every weak solution u to the Navier-Stokes equation (2.1) satisfies the estimate*

$$|u(t)| \leq |u_0| e^{-\lambda_1 \nu t} + \frac{|f|_{2,\infty}}{\lambda_1 \nu}(1 - e^{-\lambda_1 \nu t}).$$

Moreover, there exists a sequence $t_j \to \infty$ such that

$$\|u(t_j)\|^2 \leq 3 \frac{|f|_{2,\infty}^2}{\lambda_1^2 \nu^3}.$$

Proof: We take the inner product of (2.1) with u and integrate over Ω. By the divergence theorem and the incompressibility, we are left with

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \nu |\nabla u(t)|^2 = \int_\Omega f(t) \cdot u(t)\, dx.$$

The Schwarz and Poincaré inequalities give

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \lambda_1 \nu |u(t)|^2 \leq |f(t)||u(t)|,$$

so by cancellation of $|u(t)|$,

$$\frac{d}{dt}|u(t)| + \lambda_1 \nu |u(t)| \leq |f(t)|.$$

An integration over $(0,t)$, taking sup in t of f, gives

$$|u(t)| \leq |u_0|e^{-\lambda_1\nu t} + \frac{|f|_{2,\infty}}{\lambda_1\nu}(1-e^{-\lambda_1\nu t}). \tag{2.2}$$

Next we integrate the inequality

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \nu|\nabla u(t)|^2 \leq |f(t)||u(t)|$$

over the interval $[t_1, t_2]$,

$$\nu\int_{t_1}^{t_2}|\nabla u(t)|^2\,dt \leq \frac{1}{2}(|u(t_1)|^2 - |u(t_2)|^2) + \int_{t_1}^{t_2}|f(t)||u(t)|\,dt.$$

By (2.2) we get

$$\nu\int_{t_1}^{t_2}|\nabla u(t)|^2\,dt \leq |u_0|(\frac{|u_0|}{2} + |f|_{2,\infty})(e^{-\lambda_1\nu t_1} + e^{-\lambda_1\nu t_2}) + \frac{|f|_{2,\infty}^2}{(\lambda_1\nu)^2}(1+(t_2-t_1))$$

Finally, we let $t_2 = t_1 + 1$, and choose t sufficiently large to get

$$\int_{t_1}^{t_1+1}\|u(t)\|^2\,dt \leq \frac{3|f|_{2,\infty}^2}{\lambda_1^2\nu^3}.$$

This implies that there is a set of positive Lebesgue measure in every interval $[t_1, t_1+1]$ such that

$$\|u(t)\|^2 \leq \frac{3|f|_{2,\infty}^2}{\lambda_1^2\nu^3},$$

for t in this set. $\square$

We continue by stating a local existence theorem.

Theorem 2.1 *Suppose that the pressure is normalized, i.e.,*

$$\frac{1}{\Omega}\int_\Omega p\,dx = 0,$$

then there exists a unique local solution $(u,\,p)$ *of (2.1) in* $C([0,t];(H^1(\Omega))^{n+1})$, $n = 2,\ 3$, *with initial data* $u_0 = u(x,0)$ *in* $(H^1(\Omega))^n$. *The local existence time* t *depends only on the* L^2*-norms of* $\operatorname{curl} u_0$.

Kreiss and Lorentz [21] can be consulted for details of the proof.

It is well-known that global solutions of the Navier-Stokes equations exist, $u \in C(\mathbf{R}^+; (H^1(\Omega))^2)$, in two dimensions. In three dimensions that analogus statement is open. However, if transcients are allowed to settle for a sufficiently long time, then global solutions exist in three dimensions, after this settling of the initial velocity.

We now state a result saying that every weak global solution to the three-dimensional Navier-Stokes equation becomes a strong solution after some finite time $t_0 > 0$. This also goes back to Leray [25]. The existence of a global attractor is due to Ladyshenskaya [22, 23].

Theorem 2.2 *Consider the three-dimensional Navier-Stokes equation (2.1). For every weak solution $u_w \in C(\mathbf{R}^+; (L^2(\Omega))^3)$ to (2.1) there exists a $t_0 < \infty$, such that if the forcing $|f|_{2,\infty}$ is sufficiently small there exists a unique strong solution $u \in C([t_0, \infty); (H^1(\Omega))^3)$ to (2.1), with data $u(t_0) = u_w(t_0)$. Moreover, (2.1) possesses a global attractor. If the initial data $|u_0|$ is small, then the solutions have global existence, i.e. $t_0 = 0$.*

Proof: We take the inner product of (2.1) with Δu and integrate over Ω. Using Schwarz's inequality we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\nabla u(t)|^2 + \nu|\Delta u(t)|^2 &= \int_\Omega f(t)\Delta u(t)\,dx - \int_\Omega((u(t)\cdot\nabla)u(t))\cdot\Delta u(t)\,dx \\ &\leq |f(t)||\Delta u(t)| + |u(t)\cdot\nabla u(t)||\Delta u(t)|.\end{aligned}$$

Now

$$|u(t)\cdot\nabla u(t)| \leq |u(t)|_\infty|\nabla u(t)| \leq K|u(t)|^{1/4}\|u(t)\|_2^{3/4}|\nabla u(t)|$$

by the Gagliardo-Nirenberg inequalities, where K is a constant. Thus,

$$\frac{1}{2}\frac{d}{dt}|\nabla u(t)|^2 + \nu|\Delta u(t)|^2 \leq (|f(t)| + K|u(t)|^{1/4}\|u(t)\|_2^{3/4}|\nabla u(t)|)|\Delta u(t)|.$$

An application of the inequality $ab \le a^2/2\nu + \nu b^2/2$, on the right hand side, gives

$$\frac{1}{2}\frac{d}{dt}|\nabla u(t)|^2 + \frac{\nu}{2}|\Delta u(t)|^2 \le \frac{1}{\nu}(|f(t)| + K|u(t)|^{1/4}\|u(t)\|_2^{3/4}|\nabla u(t)|)^2$$

$$\le \frac{1}{\nu}(|f(t)|^2 + K^2|u(t)|^{1/2}\|u(t)\|_2^{3/2}|\nabla u(t)|^2)$$

by another application of the inequality above with $\nu = 1$. From Lemma 2.1 we get, again by the same inequality,

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \nu|\nabla u(t)|^2 \le \frac{1}{2}(|f(t)|^2 + |u(t)|^2).$$

Adding these inequalities, applying the Sobolev inequality and repeating the use of the inequality above results in

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \frac{\nu}{4}\|u(t)\|_2^2 \le C_1|u(t)|^2\|u(t)\|^8 + C_2(|f(t)|^2 + |u(t)|^2),$$

where C_1 and C_2 are constants. Using Poincaré's inequality and Lemma 2.1 we find that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \frac{\lambda_1\nu}{4}\|u(t)\|^2 - 2C_1(|f|_{2,\infty}^2 + |u_0|^2e^{-2\lambda_1\nu t})\|u(t)\|^8$$

$$\le 3C_2(|f|_{2,\infty}^2 + |u_0|^2e^{-2\lambda_1\nu t}),$$

since

$$(|f|_{2,\infty} + |u_0|e^{-\lambda_1\nu t})^2 \le 2(|f|_{2,\infty}^2 + |u_0|^2e^{-2\lambda_1\nu t}).$$

Now the point is that if the coefficient $|f|_{2,\infty}^2 + |u_0|^2e^{-2\lambda_1\nu t}$ is small, or the forcing is small and we have waited a sufficiently long time, to let the initial data $|u_0|^2e^{-2\lambda_1\nu t}$ decay, then the inequality gives us a bound on $\|u(t)\|$. The argument is as follows. We integrate the inequality above over $[t_0, t]$ to get

$$\|u(t)\|^2 - 4C_1\int_{t_0}^{t}(|f|_{2,\infty}^2 + |u_0|^2e^{-2\lambda_1\nu t_0})e^{-\beta(t-s)}\|u(s)\|^8\,ds \le$$

$$\frac{6C_2}{\beta}(|f|_{2,\infty}^2 + |u_0|^2e^{-2\lambda_1\nu t_0})(1 - e^{-\beta(t-t_0)}),$$

where $\beta = \lambda_1\nu/2$. Now assume that $\|u(s)\|$ assumes its maximum in the interval $[t_0, t]$ at $s = t$. Then

$$\|u(t)\|^2 - \frac{4C_1}{\beta}(|f|_{2,\infty}^2 + |u_0|^2e^{-2\lambda_1\nu t_0})\|u(t)\|^8\,ds \le$$

$$\frac{6C_2}{\beta}(|f|_{2,\infty}^2 + |u_0|^2 e^{-2\lambda_1 \nu t_0})(1 - e^{-\beta(t-t_0)}),$$

Now put

$$v(t) = \|u(t)\|^2, \quad a = \frac{4C_1}{\beta}(|f|_{2,\infty}^2 + |u_0|^2 e^{-2\lambda_1 \nu t_0})$$

and

$$M = \frac{6C_2}{\beta}(|f|_{2,\infty}^2 + |u_0|^2 e^{-2\lambda_1 \nu t_0}).$$

Then the inequality above can be written as

$$v(t) - av^4(t) \leq M.$$

The graph of the function $F(v) = v - av^4$ is shown in Figure 1. It is concave and attains

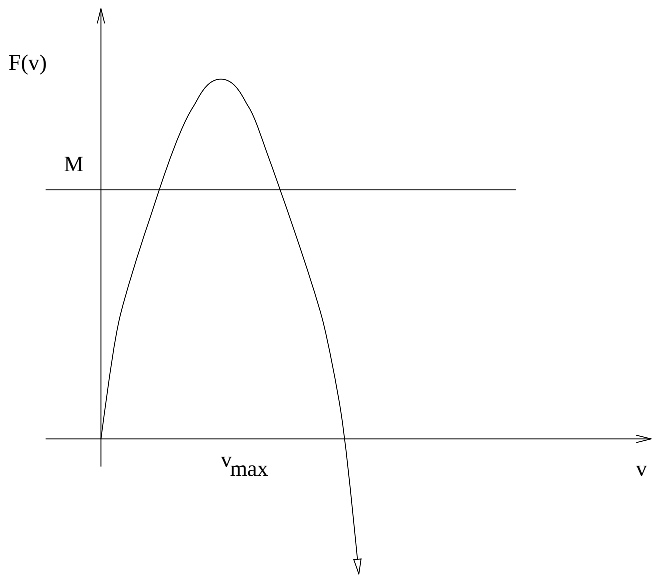

Figure 1: The bound on the norm

its maximum at $v_{max} = 1/(4a)^{1/3}$. By Lemma 2.1 there exists a $t_0 > 0$ such that

$$v(t_0) \leq \|u(t_0)\|^2 \leq \frac{3|f|_{2,\infty}}{\lambda_1^2 \nu^3}$$

and we can choose $|f|_{2,\infty}$ so small that $v(t_0)$ lies between zero and v_{max}, see Figure 1. Moreover, $v(t)$ can never reach v_{max}, because

$$F(v(t)) = v(t) - av^4(t) \leq M < \frac{3}{4}\frac{1}{(4a)^{1/3}} = F(v_{max}).$$

It is clear that we can choose t_0 so large that

$$M = \frac{6C_2}{\beta}(|f|_{2,\infty}^2 + |u_0|^2 e^{-2\lambda_1 \nu t_0}) \leq$$

$$\frac{3}{4}\frac{1}{(4a)^{1/3}} = \frac{3}{4}\frac{1}{(\frac{16C_1}{\beta}|f|_{2,\infty}^2 + |u_0|^2 e^{-2\lambda_1 \nu t_0})^{1/3}}.$$

Moreover, notice that the derivative of F in Figure 1 is positive at the point, where F first reaches M, i.e.,

$$F'(v) = 1 - 4av^3 = 1 + \frac{4(v - av^4)}{v} - 4 = \frac{4M}{v} - 3 \geq 0,$$

which gives the bound

$$v(t) \leq \frac{4M}{3} + \delta$$

where δ is arbitrarily small, and this defines an absorbing set in $(H^1(\Omega))^3$. The only question left is whether the initial data makes sense as a $H^1(\Omega)$ function. But we have already used above that, by Lemma 2.1, there exists a sequence of values $t_j \to \infty$ so that $\|u(t_j)\| < \infty$. We pick the initial time to be the smallest $t_j \geq t_0$. Then we let this t_j be the new t_0. Now we apply the local existence theorem (2.1) and the above bound to get global existence. Moreover, since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ it follows that the Navier-Stokes equation has a global attractor in $(L^2(\Omega))^3$. It is also clear that if $|u_0|$ is small then $v(t)$ satisfies the above bound and the solutions have global existence for $t_0 = 0$. □

Remark 1 The statement and proof of Theorem 2.2 does not seem to be available in the literature although all the estimates are. It is really just a restatement of Leray's classical result of global existence for small initial data. The duality in the smallness condition is that one can either take small initial data or, with arbitrary initial data, just wait for a sufficiently long time. This statement is however what most physicist and and engineers are interested in, at least for low Reynolds numbers. Namely, transients die out and the flow settles down to the flow on the attractor in at most a few hours, in experiments.

Remark 2 The existence of the pressure term follows from a standard orthogonality argument. Consider the Navier-Stokes equation (2.1) and the solution given by Theorem 2.2. We take the inner product by u, integrate over Ω, apply the divergence theorem, use the incompressibility and collect all terms on one side. This gives

$$\int_\Omega (f - \frac{\partial u}{\partial t} - u \cdot \nabla u + \nu \Delta u) \cdot u \, dx = 0.$$

Now we recall that the orthogonal to the divergence free elements in L^2 are gradients in L^2. Therefore, there exists a unique gradient ∇p in $L^2(\Omega)$ given by

$$\nabla p = f - \frac{\partial u}{\partial t} - u \cdot \nabla u + \nu \Delta u.$$

We say that a set M is *invariant* under the flow, defined by a nonlinear semi-group $S(t)$, if $S(t)M = M$, i.e. M is both positively and negatively invariant. An *attractor* $\mathcal{A}$ is an invariant set which attracts a neighborhood U or for all $x_o \in U$, $S(t)x_o$ converges to $\mathcal{A}$ as $t \to \infty$. The largest such set U is called the *basin of attraction* of the attractor $\mathcal{A}$. If the basin of attraction of an attractor $\mathcal{A}$ contains all bounded sets of X and $\mathcal{A}$ is compact, then we say that $\mathcal{A}$ is the *global attractor.*

We have shown above that the Navier-Stokes equations have a global attractor. Morover, by standard results, see Kreiss and Lorentz [21] this attractor consists of spatially smooth solutions and has finite Hausdorff and fractal dimensions, see [28, 37, 38, 26]. The dimension estimates are however large, see [40], and not necessarily a good indication of the size of the attractor.

We focus on the core of the attractor $\mathcal{A}$ which is called the basic attractor $\mathcal{B}$. For now, let us assume that the Banach space X is finite dimensional. An attractor $\mathcal{B}$ is called a *basic attractor* if

- The basin of attraction of $\mathcal{B}$ has positive measure.

- There exists no strictly smaller $\mathcal{B}' \subset \mathcal{B}$, such that up to sets of measure zero, $basin(\mathcal{B}) \subset basin(\mathcal{B}')$.

A *global basic attractor* is a basic attractor whose basin of attraction contains all bounded sets of X up to sets of measure zero.

A theorem by Milnor [30] states that in finite dimensions there exists a unique decomposition of the attractor $\mathcal{A}$,

$$\mathcal{A} = \mathcal{B} \cup \mathcal{C}$$

where $\mathcal{B}$ is a basic attractor and $\mathcal{C}$ is a remainder such that $m(basin(\mathcal{C}) \backslash basin(\mathcal{B})) = 0$ where m is the Lebesque measure.

The notion of a basic attractor can be extended to infinite dimensions and Milnor's Theorem was first proven in an infinite-dimensional setting by Birnir and Grauer [5]. They used cumbersome projections to a finite dimensional space where the $\mathcal{A}$-attractor resides. A more elegant notion of a $\mathcal{B}$-attractor requires an extension of the concepts of measure zero and almost every. Their counterparts in infinite dimensions are *shy* and *prevalent* sets respectively, see Hunt *et al.* [20]. These are defined in the following way. Let X denote a Banach space. We denote by $S + v$ the translate of the set $S \subset X$ by a vector $v \in X$. A measure μ is said to be *transverse* to a Borel set $S \subset X$ if the following two conditions hold:

- There exists a compact set $U \subset X$ for which $0 < \mu(U) < \infty$.
- $\mu(S + v) = 0$ for every $v \in X$.

A Borel set $S \subset X$ is called *shy* if there exists a compactly supported measure transverse to S. More generally, a subset of X is called shy if it is contained in a shy Borel set. The complement of a shy set is called a *prevalent* set.

The infinite-dimensional analog of Milnor's Theorem can now be stated,

Theorem 2.3 *Let $\mathcal{A}$ be the compact attractor of a continuous map $S(t)$ on a separable Banach space X. Then $\mathcal{A}$ can be decomposed into a maximal basic attractor $\mathcal{B}$ and a remainder $\mathcal{C}$,*

$$\mathcal{A} = \mathcal{B} \cup \mathcal{C}$$

such that the realm of attraction of $\mathcal{B}$ is prevalent but the realm of attraction of $\mathcal{C}$, excluding points that are attracted to $\mathcal{B}$, is shy.

For a proof of this Theorem see Birnir [3] and [4]. It implies that the Navier-Stokes equation has a maximal $\mathcal{B}$-attractor in two and three dimensions. If $\mathcal{B}$ can be decomposed into finitely many (disjoint) minimal $\mathcal{B}$-attractors, then the union of the realms of attraction of these minimal $\mathcal{B}$-attractors is the whole space $(L^2(\Omega \subset \mathbf{R}))^n, n = 2, 3$. The realm of attraction is a slight generalization the basin of attraction, as the basin is an open set, but the realm can be either open or closed.

For systems that are simple enough, the basic attractor contains only the stable trajectories of the global attractor. When transients are ignored, one will only see the basic attractor in physical experiments and numerical simulations, not the remainder $\mathcal{C} = \mathcal{A} \backslash \mathcal{B}$ of the attractor. All the relevant dynamics are therefore contained in the basic attractor. Ladyshenskaya [24] gives more examples of $\mathcal{B}$-attractors for the Navier-Stokes equation with nonlinear viscosity.

3 The Boussinesq equations

We start with the Boussinesq equations which are two coupled equations for the fluid velocity u, the pressure p and the temperature T,

$$
(3.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u - \nu \Delta u + \nabla p = g\alpha(T - T_2), \\ \\ \dfrac{\partial T}{\partial t} + (u \cdot \nabla) T - \kappa \Delta T = 0, \qquad\qquad x \in \Omega,\ t \in \mathbf{R}^+ \\ \\ \operatorname{div} u = 0 \end{cases}
$$

Here g is the gravitational acceleration and α is the volume expansion coefficient of the fluid. Moreover ν and κ are the viscosity and conductivity coefficients which determine the dimensionless Rayleigh $R = \frac{\alpha g (T_1 - T_2) h^3}{\nu \kappa}$ and Prandtl $P = \nu / \kappa$ numbers. We assume that Ω is a rectangular box and fix the temperatures at the bottom T_1 and top T_2 of the box. The box is heated from below so $T_1 > T_2$. We can impose periodic boundary conditions for u on the lateral sides of the box. However, as the temperature T_1 increases these have to be relaxed due to the presence of a boundary layer. The velocity is assumed to vanish on the horizontal surfaces of the box. Finally, we must supply the appropriate initial data.

3.1 Problem Setting

We let Ω be a rectangular box, of thickness h, in $\mathbf{R}^n$, $n = 2$ or 3. By (e_i), $i = 1, 2$ or $i = 1, 2, 3$, we denote the canonical basis in $\mathbf{R}^2$ or $\mathbf{R}^3$, respectively. The system (1.2) is equipped with the following initial data:

$$u(x, 0) = u_0(x) \;\text{ and }\; T(x, 0) = T_0(x),$$

which are assumed to belong to $L^2(\Omega)$, and boundary data:

$$u = 0 \;\text{ at }\; x_n = 0 \;\text{ and at }\; x_n = h.$$

As above

$$T = T_1 \;\text{ at }\; x_n = 0 \;\text{ and }\; T = T_2 \;\text{ at }\; x_n = h.$$

Moreover we assume that

$$u, \;\; \nabla u, \;\; T, \;\; \nabla T \;\text{ and }\; p$$

are periodic with period l in the horizontal x_1-direction, in the two-dimensional case and periodic with period l in the horizontal x_1-direction and period L in the x_2-direction in the three-dimensional case. In fact we will without loss of generality assume that $l = L$ throughout the paper. This determines the pressure p up to a constant that can be fixed by normalization, see Remark 2.

Since we are interested in the fluctuations in the temperature we follow [10] and put

$$\theta = (T - T_1 - \frac{x_n}{h}(T_2 - T_1))$$

and

$$p = p - g\alpha(x_n + \frac{x_n^2}{2h})(T_1 - T_2).$$

We get the following system which is equivalent to (3.1).

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u + \nabla p = g\alpha e_n \theta, \\ \\ \dfrac{\partial \theta}{\partial t} + (u \cdot \nabla)\theta - \kappa\Delta\theta = \dfrac{(T_1 - T_2)}{h}(u)_n, \quad x \in \Omega, \ t \in \mathbf{R}^+. \\ \\ \operatorname{div} u = 0. \end{cases} \tag{3.2}$$

For the temperature θ we get initial data $\theta(x, 0) = \theta_0(x)$ and the new boundary data $\theta = 0$ at $x_n = 0$ and at $x_n = h$. The initial and boundary data for u remain unchanged. Moreover, u and θ and their gradients and p are periodic as above. We will find it useful to work with the system in the form (3.1) in some situations, whereas the form (3.2) is more suitable in other situations.

3.2 Existence results

Now consider the Boussinesq system (3.1) equipped with initial data

$$u(x, 0) = u_0(x) \text{ and } T(x, 0) = T_0(x).$$

The boundary conditions are periodic on the vertical surfaces $x_k = 0, l$ for $k < n$. Furthermore, on the horizontal surfaces $x_n = 0$ and $x_n = h$, the velocity u vanishes (no-slip condition) and the temperature T is kept constant, i.e.,

$$T = T_1 \text{ on } x_n = 0, \ T = T_2 \text{ on } x_n = h.$$

We want to state a global existence theorem for the solution to the system (3.1). For that purpose we start with the following maximum principle, c.f. [10]

Lemma 3.1 *Suppose that u and T solve (3.1). If*

$$T_2 \leq T(x, 0) \leq T_1,$$

for a.e. $x \in \Omega$*, then*

$$T_2 \leq T(x,t) \leq T_1,$$

for a.e. $x \in \Omega$ *and all* $t \geq 0$*.*

Proof: We consider

$$(T - T_1)_+(x,t) = \text{ess} \sup_{x \in \Omega} (T - T_1)(x,t).$$

The second equation in (3.1) gives

$$\frac{\partial}{\partial t}(T - T_1)_+ + (u \cdot \nabla)(T - T_1)_+ - \kappa \Delta (T - T_1)_+ = 0.$$

We multiply this equation by $(T - T_1)_+$, integrate over Ω and use the divergence theorem, to get

$$\frac{1}{2}\frac{d}{dt}|(T - T_1)_+(t)|^2 + \kappa |\nabla (T - T_1)_+(t)|^2 = 0.$$

Thus, by Poincaré's inequality

$$\frac{1}{2}\frac{d}{dt}|(T - T_1)_+(t)|^2 + \lambda_1 \kappa |(T - T_1)_+(t)|^2 \leq 0,$$

where again λ_1 is the first eigenvalue of the negative Laplacian with vanishing boundary conditions on Ω. Consequently

$$|(T - T_1)_+(t)| \leq |(T - T_1)_+(0)| e^{-\lambda_1 \kappa t},$$

which shows that

$$(T - T_1)_+(\cdot, t) = 0,$$

for all $t \geq 0$, if

$$(T - T_1)_+(\cdot, 0) = 0.$$

Similarly we get that

$$(T - T_2)_-(x,t) = \text{ess} \sup_{x \in \Omega} (-(T - T_2)(x,t)) = 0,$$

if $(T - T_2)_-(x, 0) = 0$ and we conclude that

$$T_2 \leq T(x,t) \leq T_1.$$

$\square$

Remark 3 Lemma 3.1 actually yields a uniform bound on T in $L^2([0,s]\times\Omega)$ for any $s>0$. This also gives a bound for θ, see below.

Lemma 3.2 *Every weak solution u to the Boussinesq equations (3.2) satisfies the estimate*

$$|u(t)| \le |u_0|e^{-\lambda_1\nu t} + \frac{Kh^{1/2}}{\lambda_1\nu}(1-e^{-\lambda_1\nu t}).$$

$$|\theta(t)| \le \frac{Kh^{1/2}}{g\alpha},$$

where $K = g\alpha(T_1-T_2)L/3^{1/2}$. The equations possess an absorbing set in $(L^2(\Omega))^4$, defined by

$$|u(t)|+|\theta(t)| \le \left(\frac{1}{g\alpha}+\frac{1}{\lambda_1\nu}\right)Kh^{1/2}+\delta,$$

where δ is arbitrarily small. Moreover, there exists a sequence $t_j\to\infty$ such that

$$\|u(t_j)\|^2+\|\theta(t_j)\|^2 \le K_1,$$

where

$$K_1 = 3\frac{K^2}{\lambda_1^2\nu^2}\left(\frac{h}{\nu}+\frac{(T_1-T_2)^2}{\lambda_1^2\kappa^3 h}\right).$$

Proof: We recall the relationship between T and θ

$$T = \theta + T_1 - \frac{x_n}{h}(T_1-T_2).$$

The maximum principle in Lemma 3.1, for T, $T_2 \le T \le T_1$, implies that

$$-(1-\frac{x_n}{h})(T_1-T_2) \le \theta \le \frac{x_n}{h}(T_1-T_2).$$

Thus

$$|\theta|_2^2 \le (T_1-T_2)^2L^2h/3.$$

Now consider the system (3.2). We multiply the first equation by u and integrate over Ω. Integration by parts and Schwarz's inequality give

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \lambda_1 \nu |u(t)|^2 \leq g\alpha|\theta(t)||u_n(t)|.$$

Thus by the use of the bound for θ above, and Poincaré's inequality, we get

$$\frac{d}{dt}|u(t)| + \lambda_1 \nu |u(t)| \leq g\alpha|\theta(t)| \leq g\alpha(T_1 - T_2)L(h/3)^{1/2}.$$

Integration in t then gives

$$|u(t)| \leq |u_0|e^{-\lambda_1 \nu t} + \frac{Kh^{1/2}}{\lambda_1 \nu}(1 - e^{-\lambda_1 \nu t}),$$

where $K = g\alpha(T_1 - T_2)^{1/2}L/(3)^{1/2}$.

We combine the bounds for θ and u to get the absorbing set

$$|\theta| + |u| \leq Kh^{1/2}\left(\frac{1}{g\alpha} + \frac{1}{\lambda_1 \nu}\right) + \delta$$

in $(L^2(\Omega))^4$, where

$$K = g\alpha\frac{(T_1 - T_2)}{3^{1/2}}L$$

and δ is arbitrarily small.

The last statment of the lemma is proven by a straigthforward application of Lemma 2.1, to the equations (3.2), if we recall that the nonlinear term did not play a role in the proof of Lemma 2.1. Namely, for the first equation in (3.2),

$$|f|^2_{2,\infty} = g^2\alpha^2|\theta|^2 = K^2 h,$$

and for the second equation

$$|f|^2_{2,\infty} = \frac{(T_1 - T_2)^2}{h^2}|u|^2 = \frac{(T_1 - T_2)^2 K^2}{\lambda_1^2 \nu h}.$$

We divide these by the decay coefficients $\lambda_1^2\nu^3$ and $\lambda_1^2\kappa^3$, respectively and add them. This produces the bound on the H^1 norm for the sequence t_j. $\square$

Theorem 3.1 *Suppose that the pressure is normalized, i.e.,*

$$\frac{1}{\Omega}\int_\Omega p\,dx = 0,$$

then there exists a unique local solution (u, θ, p) *of (3.2) in* $C([0,t];(H^1(\Omega))^{n+2})$, $n = 2,\ 3$, *with initial data* $(u_0, \theta_0)(x)$ *in* $(H^1(\Omega))^{n+1}$.

Proof: The maximum principle in Lemma 3.1, for T, $T_2 \leq T \leq T_1$, implies a maximum principle for θ,

$$-(1-\frac{x_n}{h})(T_1 - T_2) \leq \theta \leq \frac{x_n}{h}(T_1 - T_2),$$

by the relationship between T and θ

$$T = \theta + T_1 - \frac{x_n}{h}(T_1 - T_2).$$

This implies that

$$|\theta|_2^2 \leq (T_1 - T_2)^2 L^2 h/3.$$

The rest of the proof, using this bound on θ, is similar to the proof of the local existence of the Navier Stokes equation. Kreiss and Lorentz [21] can be consulted for details. Then given the local solution $(u, \theta)(x, t)$ the pressure is recovered as in Remark 2. $\square$

In two dimensions we have the following global existence result.

Theorem 3.2 *The Boussinesq system (3.2) has a unique global solution* $(u,\ \theta)$ *in* $C(\mathbf{R}^+;(H^1(\Omega))^3)$, *where* Ω *is a bounded open set in* $\mathbf{R}^2$. *Moreover, the system (3.2) possesses a global attractor in* $(L^2(\Omega))^3$.

Proof: A proof of Theorem 3.2 can be found in Foias et. al. [10].

In three dimensions the following global existence result holds true:

Theorem 3.3 *Consider the Boussinesq system (3.2). For every weak solution* u_w, θ_w *in* $C(\mathbf{R}^+;(L^2(\Omega))^4)$, Ω *a bounded open set in* $\mathbf{R}^3$ *with a large aspect ratio,* $\frac{L}{h} >> (T_1 - T_2)^2 L^3$, *where L is the width (radius) and h is the height of Ω; there exist a time t_0, such that there exists a unique strong solution (u, θ) in $C([t_0,\infty);(H^1(\Omega))^4)$, $t \geq t_0$, with initial data $u(t_0) = u_w(t_0)$ and $\theta(t_0) = \theta_w(t_0)$. Moreover, the system (3.2) possesses a global attractor in $(L^2(\Omega))^4$.*

Proof : We multiply the second equation in (3.2) above by θ and treat it in the same way as u in the proof of Lemma 2.1 to get

$$\frac{1}{2}\frac{d}{dt}|\theta(t)|^2 + \kappa|\nabla\theta(t)|^2 \leq \frac{(T_1-T_2)}{h}|\theta(t)||u_n(t)|,$$

and by adding and subtracting $\beta|\theta|$ and applying Poincaré's inequality, we get

$$\frac{d}{dt}|\theta(t)| + \beta|\theta(t)| + (\lambda_1^{1/2}\kappa - \frac{\beta}{\lambda_1^{1/2}})|\nabla\theta(t)| \leq \frac{(T_1-T_2)}{h}|u_n(t)|$$
$$\leq \frac{(T_1-T_2)}{h}(|u_0|e^{-\lambda_1\nu t} + \frac{Kh^{1/2}}{\lambda_1\nu}(1-e^{-\lambda_1\nu t})) \leq c_1 + c_2 e^{-\lambda_1\nu t}.$$

Then integrating with respect to t, we get

$$(\lambda_1^{1/2}\kappa - \frac{\beta}{\lambda_1^{1/2}})\int_0^t e^{-\beta(t-s)}|\nabla\theta(s)|ds \leq |\theta(0)|e^{-\beta t} + \frac{c_1}{\beta}(1-e^{-\beta t}) + c_2\frac{(e^{-\beta t}-e^{-\lambda_1\kappa t})}{(\lambda_1\kappa-\beta)}.$$

Next we multiply the u equation in (3.2) by Δu and integrate over Ω. By integration by parts and Schwarz's inequality

$$\frac{1}{2}\frac{d}{dt}|\nabla u(t)|^2 + \nu|\Delta u(t)|^2 \leq g\alpha|\nabla\theta(t)||\nabla u_n(t)| + |u(t)\cdot\nabla u(t)||\Delta u(t)|$$
$$\leq g\alpha|\nabla\theta(t)||\nabla u_n(t)| + C(1+\lambda_1^{-1}+\lambda_1^{-2})^{3/8}|u(t)|^{1/4}|\nabla u(t)||\Delta u(t)|^{7/4},$$

by the Gagliardo-Nirenberg inequalities, where we have used that the H^2 Sobolev norm is bounded by

$$\|u\|_2 \leq C(|u(t)|^2 + |\nabla u(t)|^2 + |\Delta u(t)|^2)^{1/2} \leq C(1+\lambda_1^{-1}+\lambda_1^{-2})^{1/2}|\Delta u(t)|,$$

by Poincaré's inequality. We use Young's inequality to eliminate $|\Delta u|^{7/4}$, namely

$$((\frac{\nu}{2})^7 C_0)^{1/8}|u|^{1/4}|\nabla u||\Delta u|^{7/4} \leq C_0|u|^2|\nabla u|^8 + \frac{\nu}{2}|\Delta u|^2$$

so

$$\frac{1}{2}\frac{d}{dt}|\nabla u(t)|^2 + \frac{\nu}{2}|\Delta u(t)|^2 \le g\alpha|\nabla\theta(t)||\nabla u(t)| + C_0|u(t)|^2|\nabla u(t)|^8$$

and by Poincaré's inequality

$$\frac{d}{dt}|\nabla u(t)| + \frac{\lambda_1\nu}{2}|\nabla u(t)| \le g\alpha|\nabla\theta(t)| + C_0|u(t)|^2|\nabla u(t)|^7.$$

Then we integrate the equation

$$\frac{d}{dt}|\nabla u(t)| + \beta|\nabla u(t)| - C_0|u(t)|^2|\nabla u(t)|^7 \le g\alpha|\nabla\theta(t)|,$$

where $\beta = \lambda_1\nu/2$. We integrate from the initial time t_0, to get

$$|\nabla u(t)| - C_0\int_{t_0}^t |u(s)|^2|\nabla u(s)|^7 e^{-\beta(t-s)}ds \le$$
$$\le c_1 + c_2 e^{-\beta(t-t_0)} + c_3 e^{-\lambda_1\nu(t-t_0)},$$

by the above inequality for $\int|\nabla\theta|dt$. Now if $|\nabla u(s)|$ attains its maximum on the interval $[t_0, t]$ at $s = t$, then

$$\int_{t_0}^t |u(s)|^2|\nabla u(s)|^7 e^{-\beta(t-s)}ds \le |\nabla u(t)|^7\int_{t_0}^t |u(s)|^2 e^{-\beta(t-s)}ds \le$$
$$\frac{|\nabla u(t)|^7}{2}\int_{t_0}^t \left(|u_0|^2 e^{-3\beta s}e^{-\beta t} + \frac{K^2h}{\lambda_1^2\nu^2}(1-e^{-2\beta s})^2 e^{-\beta(t-s)}\right)ds$$
$$\le \frac{|\nabla u(t)|^7}{6\beta}\left(|u_0|^2(e^{-3\beta t_0} - e^{-3\beta t})e^{-\beta t} + 3\frac{K^2h}{\lambda_1^2\nu^2}(1-e^{-\beta(t-t_0)})\right),$$

where we have used the bound

$$|u(t)| \le |u_0|e^{-\lambda_1\nu t} + \frac{Kh^{1/2}}{\lambda_1\nu}(1-e^{-\lambda_1\nu t}),$$

from Lemma 3.2 and the inequality $ab \le a^2/2 + b^2/2$. Thus, if we put $|\nabla u(t)| = v(t)$, we obtain the inequality

$$v(t) - av^7(t) \le M.$$

The constants a and M are,

$$a = \frac{K^2h}{2\lambda_1^2\nu^2\beta} = g^2\alpha^2\frac{(T_1-T_2)^2}{6\lambda_1^2\nu^2\beta}L^2h$$

and

$$M = 2c_1 = 2\frac{(T_1 - T_2)K}{h^{1/2}\lambda_1\nu} = 2g\alpha(T_1 - T_2)^2 L/3^{1/2}\lambda_1\nu h^{1/2}.$$

This means that for a large aspect ratio $L/h >> (T_1 - T_2)^2 L^3$, a becomes small. Now we repeat the arguments from the proof of Theorem 2.2 and conclude that the function $F(v) = v - av^7$ is concave and attains its maximum at $v_{max} = 1/(7a)^{1/6}$. This maximum is

$$F(v_{max}) = \frac{6}{7}\frac{1}{(7a)^{1/6}},$$

so that $v(t)$ can not escape beyond its maximum, see Figure 1. Moreover arguing as in Theorem 2.2 we conclude that the derivative of F is positive at the point where F first reaches M and this gives us the bound

$$v \leq \frac{7M}{6}.$$

The last step is to get a bound for $|\nabla\theta|$. We multiply the θ equation in (3.2) by $\Delta\theta$, to get

$$\frac{1}{2}\frac{d}{dt}|\nabla\theta(t)|^2 + \kappa|\Delta\theta(t)|^2 \leq \frac{(T_1 - T_2)}{h}|\nabla\theta(t)||\nabla u(t)| + |u(t)\cdot\nabla\theta(t)||\Delta\theta(t)|$$

by Schwarz's inequality

$$\leq \frac{(T_1 - T_2)}{h}|\nabla\theta(t)||\nabla u(t)| + |u|_6|\nabla\theta(t)|_3|\Delta\theta(t)|,$$

by Holder's inequality

$$\leq \frac{(T_1 - T_2)}{h}|\nabla\theta(t)||\nabla u(t)| + C(|\nabla u(t)|^2|\theta|/\delta^3 + \delta|\Delta\theta|)|\Delta\theta(t)|,$$

by Poincaré's and Sobolev's inequalities, because

$$|u|_6 \leq K\|u\| \leq C|\nabla u|,$$

as above, and be cause,

$$|\nabla\theta|_3 \leq C\|\nabla\theta\|_{1/2} \leq K(\frac{|\nabla\theta|}{\delta} + \delta|\Delta\theta|) \leq C(\frac{|\theta|}{\delta^3} + \delta|\Delta\theta|)$$

by two applications of interpolation, where δ is small. Thus

$$\frac{1}{2}\frac{d}{dt}|\nabla\theta(t)|^2+\kappa|\Delta\theta(t)|^2 \leq \frac{(T_1-T_2)}{h}|\nabla\theta(t)||\nabla u(t)|+\frac{C^2|\nabla u(t)|^4|\theta|^2}{\delta^6}+2\delta|\Delta\theta(t)|^2,$$

by Young's inequality. Now moving the $|\Delta\theta(t)|^2$ term over to the left hand side of the inequality and applying Poincaré's inequality, we get

$$\frac{1}{2}\frac{d}{dt}|\nabla\theta(t)|^2+\lambda_1(\kappa-3\delta)|\nabla\theta(t)|^2 \leq \frac{1}{2}\frac{d}{dt}|\nabla\theta(t)|^2+(\kappa-3\delta)|\Delta\theta(t)|^2 \leq c_3,$$

since

$$\frac{(T_1-T_2)}{h}|\nabla\theta(t)||\nabla u(t)| \leq \lambda_1\delta|\nabla\theta(t)|^2+\frac{(T_1-T_2)^2}{4h^2\lambda_1\delta}|\nabla u(t)|^2,$$

by the inequality $ab \leq a^2+b^2/4$. Thus

$$|\nabla\theta(t)|^2 \leq |\nabla\theta(t_0)|^2 e^{-\gamma(t-t_0)}+(2c_3/\gamma)(e^{-\gamma t_0}-e^{-\gamma t}),$$

where $\gamma = 2(\kappa-3\delta)$.

We can now put the estimates for $|\nabla u|$ and $|\nabla\theta|$ together to get an absorbing set

$$|\nabla u(t)|+|\nabla\theta(t)| \leq constant+\epsilon,$$

where ϵ is arbitrarily small for t large enough. Namely, by Lemma 3.2, there exists a t_j, such that $(u,\theta)(t_j) \in H^1(\Omega)$ and the global bound on the H^1 norm holds for $t = t_0 = t_j$. Combined with the local existence Theorem 3.1, the a priori bound above now gives the existence of a global solution in $H^1(\Omega)$, for $t \geq t_0$. The existence of a global attractor in $(L^2(\Omega))^4$ then follows from the compact embedding of $(H^1(\Omega))^4$ in $(L^2(\Omega))^4$. □

Corollary 3.1 *The Boussinesq system (3.2) has a maximal $\mathcal{B}$-attractor whose basin is a prevalent set in* $(L^2(\mathbf{R}^n)^{n+1})$, $n = 2, 3$.

The proof is a straight-forward application of Theorem 2.3

Theorem 3.3 says that in experiments in pattern formation where one has a large aspect ratio, it is only necessary to wait a short time to have global spatially smooth solutions. The smoothness follows from the smoothness of the nonlinear semigroup, see

Kreiss and Lorentz [21]. Corollary 3.1 says that what is observed in the experiments after the initial settling-down time, is a $\mathcal{B}$-attractor that is a component of the maximal $\mathcal{B}$-attractor. In other word this attractors is observed for an open set of initial conditions. (However, not necessarily an open set of parameter values.) It is then useful to know how long one has to wait and this time is easily estimated. It is the time it takes the exponentially decaying initial data to become comparable in size with the L^2 absorbing set. Namely,

$$\begin{aligned} t &= \frac{1}{\lambda_1 \nu} \ln \left[\frac{\lambda_1 \nu |u_0|}{K h^{1/2}} \right] \\ &= \frac{1}{\lambda_1 \nu} \ln \left[\frac{3^{1/2} \lambda_1 \nu |u_0|}{g\alpha(T_1 - T_2) L h^{1/2}} \right]. \end{aligned}$$

In experiments for Prandtl numbers close to one, this time is measured in hours for experiments that take days [17].

4 Rayleigh-Bénard convection

4.1 The physical background

Consider again the Boussinesq equations in the case of Rayleigh-Bénard convection

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u - \nu \Delta u + \nabla p = g\alpha(T - T_2), \\[2ex] \dfrac{\partial T}{\partial t} + (u \cdot \nabla) T - \kappa \Delta T = 0, \qquad\qquad x \in \Omega,\ t \in \mathbf{R}^+ \\[2ex] \operatorname{div} u = 0 \end{cases}$$

It is well-known that for $T_1 - T_2$ sufficiently small the fluid is quiescent and the temperature increases linearly in the vertical direction. This is a pure conduction state. It is also the unique $\mathcal{B}$-attractor in this parameter regime. As $T_1 - T_2$ increases beyond a certain threshold the pure conduction state becomes unstable and convection starts. An excellent way to visualize the parameter regions for different $\mathcal{B}$-attractors is to plot the

Rayleigh number as a function of the wavenumber ξ. (The wavenumber is the spatial Fourier variable of the rolls.) The neutral stability curve (NC) of the pure conduction state is then a quadradic curve in the $\xi - R$ plane, that forms the outline of the stability region for straight convection rolls, called the Busse balloon, see Figure 2. Inside the

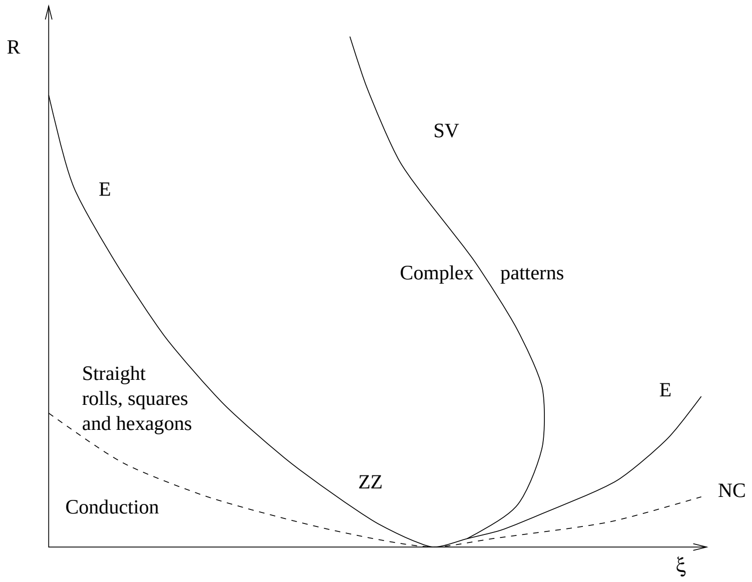

Figure 2: B-attractors for Prandtl number 0.71 (air).

balloon the fluid settles on a pattern of spatially periodic convection rolls, after initial transients. These solutions, form the Busse $\mathcal{B}$-attractors. There are seven possibilities, described by the amplitude equations of Stuart, Watson and Landau [39, 41], and the various cases arising from cubic terms of these equations have been desribed in [29, 34, 1]. Namely, up to rotation through an angle that is fixed by the initial conditions, one can have straight rolls with three orientations seperated by 120^o. Secondly, one can have a square pattern of solutions with three possible orientations corresponding to a combination of two orientations of the straight rolls. The last possible $\mathcal{B}$-attractor consists of a hexagonal pattern where the straight rolls with all three orientations have been superposed. Square and hexagonal patterns require the presence of quadradic terms in the amplitude equations, without them the $\mathcal{B}$-attractor consists of only straight rolls.

The straight rolls become unstable when the Rayleigh number is sufficiently large and this can happen either through a long-wave instability, Eckhaus (E), zig-zag (ZZ) or skew-varicose (SV) instability, see Figure 2. The first one is a modulation of the roll separation, the second a bending of the rolls and the third a combination of the two. Or the instability may happen through a short-wavelength instability such as a cross-roll or a knot instability. These instability curves can be found analytically, see [15], but they require an inclusion of a mean drift (flow) terms in the amplitude equations, see [14, 32]. Beyond these instability boundaries the $\mathcal{B}$-attractors consist of wavy rolls, knot convection and bimodal cells, one can also have time dependent patterns such as oscillatory blob convection, see [9]. Sufficiently far beyond these instability boundaries very complex spatial patterns exist and the global attractor may have more than one minimal component. In experiments performed in thin circular boxes, see Morris et al. [31, 2], with a large aspect ratio, 1. e. the height of the box is small compared to its radius, there exist minimal "chaotic" attractors consisting of jumbled broken pieces of convection rolls. Our techniques are designed to apply to the theoretical investigations of these experiments, but they may be of general use. For a discussion of the global attractor, it's $\mathcal{B}$-attractor and minimal $\mathcal{B}$ components see Milnor [30], Birnir [3, 4] and Birnir and Grauer [6].

The instabilities of the convection rolls at sufficiently large temperature gradients (Rayleigh numbers) are due to flow (fluctuations) on a small scale. This scale is driven by both the heat conduction and the large scale flow that couples to the small scale flow through the convective nonlinearity. In addition, the small scale flow is dissipated by the viscosity which acts on even smaller scales. In [7] we give these considerations a mathematical formulation by deriving an equation describing the small scale flow. Using this equation we recover the mean field flow, where we have averaged over the small scale. In particular this captures how the large scale convection and conduction contribute to the mean field.

4.2 Scaling and expansions

As a starting point we introduce a small "scaling" parameter $\epsilon > 0$ and consider the scaled system

$$
(4.1) \qquad \begin{cases} \dfrac{\partial u_\epsilon}{\partial t} + (u_\epsilon \cdot \nabla) u_\epsilon - \epsilon^\gamma \nu \Delta u_\epsilon + \nabla p_\epsilon = g\alpha(T_\epsilon - T_2), \\[2ex] \dfrac{\partial T_\epsilon}{\partial t} + (u_\epsilon \cdot \nabla) T_\epsilon - \epsilon^\gamma \kappa \Delta T_\epsilon = 0, & x \in \Omega,\ t \in \mathbf{R}^+, \\[2ex] \operatorname{div} u_\epsilon = 0, \end{cases}
$$

equipped with initial data

$$
u_\epsilon(x, 0) = a(x) \ \text{ and } \ T_\epsilon(x, 0) = b(x).
$$

The boundary conditions are periodic on the vertical surfaces $x_k = 0, l$ for $k < n$. Furthermore, on the horizontal surfaces $x_n = 0$ and $x_n = h$, the velocity u_ϵ vanishes (no-slip condition) and the temperature T_ϵ is kept constant, i.e.,

$$
T_\epsilon = T_1 \ \text{ on } \ x_n = 0, \ \ T_\epsilon = T_2 \ \text{ on } \ x_n = h.
$$

k is a positive constant. A small value of ϵ corresponds to a high Reynolds number and the solutions will likely become turbulent. We also consider the scaled system corresponding to (3.2), i.e.,

$$
(4.2) \qquad \begin{cases} \dfrac{\partial u_\epsilon}{\partial t} + (u_\epsilon \cdot \nabla) u_\epsilon - \epsilon^\gamma \nu \Delta u_\epsilon + \nabla p_\epsilon = g\alpha e_n \theta_\epsilon, \\[2ex] \dfrac{\partial \theta_\epsilon}{\partial t} + (u_\epsilon \cdot \nabla) \theta_\epsilon - \epsilon^\gamma \kappa \Delta \theta_\epsilon = \dfrac{(T_1 - T_2)}{h} (u_\epsilon)_n, & x \in \Omega,\ t \in \mathbf{R}^+, \\[2ex] \operatorname{div} u_\epsilon = 0. \end{cases}
$$

For the temperature θ_ϵ we get initial data $\theta_\epsilon(x, 0) = \theta_\epsilon^0(x)$ and the new boundary data $\theta_\epsilon = 0$ at $x_n = 0$ and at $x_n = h$. The initial and boundary data for u_ϵ remain unchanged. Moreover, u_ϵ and θ_ϵ and their gradients and p_ϵ are periodic as above.

It turns out that the value $\gamma = 3/2$ is critical. To see that we perform a multiple scales expansion technique of the unknown quantities u_ϵ, p_ϵ and θ_ϵ and assume that

$$u_\epsilon(x,t) = \epsilon^\rho \sum_{i=0}^{\infty} \epsilon^i u_i(x, \frac{x}{\epsilon^\mu}, t, \frac{t}{\epsilon^\beta}),$$

$$p_\epsilon(x,t) = \sum_{i=0}^{\infty} \epsilon^i p_i(x, \frac{x}{\epsilon^\mu}, t, \frac{t}{\epsilon^\beta}),$$

$$\theta_\epsilon(x,t) = \sum_{i=0}^{\infty} \epsilon^i \theta_i(x, \frac{x}{\epsilon^\mu}, t, \frac{t}{\epsilon^\beta}),$$

where u_i, p_i and θ_i are all assumed to be T^n-periodic with respect to $y \in \mathbf{R}^n$, $n = 2, 3$, T^n being the usual unit torus in $\mathbf{R}^n$. If we put $y = x/\epsilon^\mu$ and $\tau = t/\epsilon^\beta$, the chain rule transforms the differential operators as

$$\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} + \frac{1}{\epsilon^\beta}\frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial x} + \frac{1}{\epsilon^\mu}\frac{\partial}{\partial y}.$$

The question is: Can we preserve the structure from the original problem in the leading order approximation? One sees that if the viscosity and conductivity terms scales like $\epsilon^{3/2}$, then the choices $\mu = 1$ and $\rho = \beta = 1/2$ preserve all quantities. By substituting all this into the system (4.2) we can equate the powers of ϵ. This formal manipulation shows that the functions u_i and θ_i, $i = 0, 1, \ldots$ are all independent of t and that p_0 is independent of y. Formally this means that there are two scales in space and in order to preserve the evolutionary behaviour, in the leading order approximation, a chance of time scale $t \mapsto \frac{t}{\epsilon^{1/2}} = \tau$ becomes necessary, c.f. Lions [27]. It also follows, from the equations, that the functions p_i are independent of t. The leading order system reads

$$(4.3) \qquad \begin{cases} \dfrac{\partial u_0}{\partial \tau} + (u_0 \cdot \nabla_y) u_0 - \nu \Delta_y u_0 + \nabla_y p_1 = g\alpha e_n \theta_0 - \nabla_x p_0 \\[2ex] \dfrac{\partial \theta_0}{\partial \tau} + (u_0 \cdot \nabla_y)\theta_0 - \kappa \Delta_y \theta_0 = 0 \\[2ex] \mathrm{div_y} u_0 = 0, \quad \mathrm{div_x}(\displaystyle\int_{T^n} u_0 dy) = 0, \end{cases}$$

on $\Omega \times T^n \times \mathbf{R}^+$ with T^n-periodicity in y.

We see that the system (4.3) differs from the system (3.2) only by the additional forcing term coming from the "global" pressure gradient. Thus the local problem (on the ϵ-scale) has *two pressure gradients*, one local $\nabla_y p_1$ and one global $\nabla_x p_0$.

Remark 4 The scaling discussed above is isotropic but the convection rolls in Rayleigh-Bénard convection have a preferred direction, say along the x_2-axis. This commands the use of anisotropic scaling for the amplitude equations and it turns out that the scaling by $\epsilon^{3/2}$ is the one pertaining to the other directions direction, perpendicular to the orientation of the rolls. The details of this affine scaling will be spelled out in [7].

4.3 Existence results

In three dimensions the following global existence result holds true:

Theorem 4.1 *Consider the Boussinesq system (4.2). For every fixed value of $\epsilon > 0$ the following holds true: For every weak solution $u_{w,\epsilon}$, $\theta_{w,\epsilon}$ in $C(\mathbf{R}^+; (L^2)^4(\Omega))$, Ω a bounded open set in $\mathbf{R}^3$ with a large aspect ratio, $\frac{L}{h} >> (T_1 - T_2)^2 L^3$, where L is the length and h is the height of Ω; there exist a time t_0, such that there exists a unique strong solution u_ϵ, θ_ϵ in $C([t_0, \infty); (H^1)^4(\Omega))$, $t \geq t_0$, with initial data $u_\epsilon(t_0) = u_{w,\epsilon}(t_0)$ and $\theta_\epsilon(t_0) = \theta_{w,\epsilon}(t_0)$. Moreover, the system (4.2) possesses a global attractor in $(L^2)^4(\Omega)$.*

Proof: Theorem 4.1 follows from Theorem 3.3 if we set the viscosity and heat conductivity in Theorem 3.3 equal to $\epsilon^{3/2}\nu$ and $\epsilon^{3/2}\kappa$, respectively.

Remark 5 The existence of a pressure $p_\epsilon \in C([t_0,\infty);(H^1)(\Omega)/\mathbf{R})$ follows by the same arguments as those in Remark 2 after the proof of Theorem 2.2.

We conclude with the existence result for the two-scale system (4.3)

Theorem 4.2 *Consider the system (4.3) with initial data $u_0(x,y,0)=a(x,y)$ and $\theta_0(x,y,0)=b(x,y)$, with zero mean over T^n. For almost every $x\in\Omega$ the system has a unique global strong solution $u_0\in C([t_0,\infty);(H^1(T^n))^n)$, $n=2,3$, and $\theta_0\in C([t_0,\infty);H^1(T^n))$. Moreover, $p_1\in C([t_0,\infty);H^1(T^n)/\mathbf{R})$. Integrated over T^n, $(u_0)_i$, θ_0 and p_1 all belong to $L^2(\Omega)$. Finally, $p_0\in C([t_0,\infty);H^1(\Omega))$. In two dimensions $t_0=0$ and in three dimensions $t_0=t_0(|a|,|b|)>0$, in general. Moreover for almost every $x\in\Omega$, the system (4.3) possesses a global attractor in $(L^2(T^n))^n\times L^2(T^n)$, $n=2$, 3.*

The proof in the two-dimensional case is relatively straightforward and details can be found in Foias et al. [10].

Proof: In the three-dimensional case the proof is similar to the proof of Theorem 3.3 but simpler. First we multiply the first equation in (4.3) by θ and integrate by parts to get

$$|\theta_0(t)|\le |b|e^{-\gamma t}$$

after integration in t, where $\gamma=\lambda_1\kappa$. In [7] we show that

$$g\alpha e_3\theta_0-\nabla_x p_0=g\alpha\pi_2(e_3\theta_0),$$

where $\pi_2(e_3\theta_0)$ denotes the projection onto the divergence free part of $e_3\theta_0$. Hence,

$$|g\alpha e_3\theta_0-\nabla_x p_0|\le g\alpha|\theta_0|\le g\alpha|b|e^{-\gamma t}.$$

This means that for t sufficiently large the forcing in the first equation of (4.3) becomes small. Thus we obtain the existence of unique global solution $u_0\in C([t_0,\infty);(H^1(T^n))^3)$

to the first equation of (4.3), by Theorem 2.2, and an absorbing set in this space. Since $u_0(t) \in (H^1(T^n))^3$ we also obtain global existence for θ_0 in the second equation of (4.3), just as in the proof of Theorem 3.3 and the existence of an absorbing set in this space as well. The existence of the attractors follows from the fact that the equations possess an absorbing set in $(H^1(T^n))^3 \times H^1(T^n)$ which is compactly embedded in $(L^2(T^n))^3 \times L^2(T^n)$.

. $\square$

Remark 6 In [7], we prove, by using homogenization theory, that the solutions of the scaled system (4.2) converges in the two-scale sense, see [33], to the unique solution to the system (4.3).

Remark 7 By averaging the system (4.3) over the unit torus in local variable y we obtain the mean-field corresponding to the scaled system (4.2). The understanding of the effects of this field on the system is crucial in the theoretical understanding of the complex patterns in Rayleigh-Bénard convection, see [7].

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