A FINITE DIMENSIONAL ATTRACTOR OF THE MOORE-GREITZER PDE MODEL *

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Abstract. The Moore-Greitzer PDE model with viscosity is presented and the equations rewritten as an evolution equation on a Hilbert space. It is proven that the model has a unique global solution which is smooth in space and time. Furthermore it is proven that there exists a global attractor, i.e a compact set which attracts all bounded sets. Finally it is shown that the attractor has a finite fractal dimension.

Key words. Moore-Greitzer equation, jet engine model, existence and uniqueness of solutions, global attractor, Hausdorff dimension, fractal dimension

AMS subject classifications. 34D45, 34K15, 35B40, 35K55

1. Introduction. In recent years a lot of attention has been devoted to the study of air flow through turbomachines. The main reason for this interest is that when a turbomachine, such as a jet engine, operates close to its optimal operating parameter values, the flow can become unstable. These instabilities put a large stress on the engine and in some cases the engine needs to be turned off in order to recover original operation. For this reason jet engines are currently operated away from their optimal operating parameter values.

A jet engine can be thought of as a compressor where the incoming air is compressed by alternating rings of rotating blades and stationary blades. The mixture of fuel and compressed air is then ignited and the resulting combustion generates thrust that propels the aircraft. There are two types of instabilities that occur in the flow through the compressor. They are called *surge* and *stall*. Surge is characterized by large oscillations of the mean mass flow through the engine. During part of the cycle, the mean mass flow may become negative, thrusting air out the front of the engine. This puts a large stress on the components of the engine and seriously impairs its performance. When stall occurs, there are regions of relatively low air flow that form at isolated locations around the rim of the compressor. Here too, the phenomenon can be so pronounced that the flow in these isolated regions is reversed. Again this causes a large stress on the components of the engine and reduces its performance. Stall is, however, not as devastating as surge and it can be recovered without shutting the engine off.

The goal is to stabilize the flow closer to the optimal parameter values. This would result in better fuel efficiency, lighter engines and better safety. To facilitate this, a better understanding of the behaviour of the air flow during stall and surge is necessary. The current paper is the first in, what we hope will be, a series of papers, analyzing the attractor of a turbomachine model with the intent of controlling the instabilities.

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Moore and Greitzer published in 1986 a PDE model for turbomachines which has been very successful [13]. A substantial amount of work has been done on finite Galerkin approximations of that model since, see e.g. [12] [11]. These approximations have been shown to agree somewhat with experiments but have never been justified analytically.

In what follows we first describe the physical model briefly and rewrite it as an evolution equation. We prove the existence of a unique local solution and prove that it is smooth in both space and time. A priori bounds then give us global solutions. We then define the global attractor, prove its existence and prove that it is finite dimensional.

1.1. The Physical Model. The PDE model proposed by Moore and Greitzer [13] to describe flow disturbances in turbomachines is Laplace's equation

$$\phi'_{\eta\eta} + \phi'_{\theta\theta} = 0,$$

on the semi-infinite cylinder $(\eta, \theta) \in (-\infty, 0) \times [0, 2\pi]$ with zero boundary conditions at $\eta = -\infty$ and nonlinear nonlocal boundary conditions (1.1) at $\eta = 0$ which couple Laplace's equation to two ODE's, (1.2) and (1.3).

$$\text{(1.1)}\quad \Psi(t) = \psi_c(\Phi + \phi'_\eta|_{\eta=0}) - l_c\frac{d\Phi}{dt} - m\phi'_t|_{\eta=0} - \frac{1}{2a}(2\phi'_{t\eta} + \phi'_{\theta\eta})|_{\eta=0} + \frac{\nu}{2}\phi'_{\eta\theta\theta}|_{\eta=0}$$

$$\text{(1.2)}\quad l_c\frac{d\Phi}{dt} = -\Psi(t) + \frac{1}{2\pi}\int_0^{2\pi} \psi_c(\Phi + \phi'_\eta|_{\eta=0})d\theta$$

$$\text{(1.3)}\quad l_c\frac{d\Psi}{dt} = \frac{1}{4B^2}(\Phi(t) - F_T^{-1}(\Psi))$$

Here we have included a viscosity term in equation (1.1) as was first suggested by Adomaitis and Abed [9].

The variables θ, η, t represent circumferential coordinate, axial coordinate and time respectively. Φ and Ψ are the mean flow and pressure rise over the compressor respectively. ϕ' is the potential for the disturbance from the mean flow Φ. The two functions ψ_c and F_T^{-1} describe the pressure change over the compressor and the throttle respectively. We will assume that ψ_c is a cubic polynomial in $\Phi + \varphi$ with a negative leading coefficient and that F_T^{-1} is a smooth function of Ψ such that

$$F_T^{-1}(\Psi) = \gamma\Psi|\Psi|^{-1/2}$$

outside a small neighborhood of $\Psi = 0$. We also assume that the graphs of $\psi_c(x)$ and $F_T^{-1}(x)$ intersect for $x > 0$. The assumption that ψ_c is a cubic polynomial can be relaxed somewhat.

Following Banaszuk *et al.* [10] one can rewrite the model so that it becomes an evolution equation on a Banach space. To be more specific, we will rewrite it as a parabolic equation on the unit circle coupled with two ODE's.

First assume that Laplace's equation has a solution of the form

$$\phi' = \sum_{n\in\mathbf{Z}\setminus\{0\}} \alpha_n(t)\mathrm{e}^{|n|\eta - in\theta}.$$

If the flow disturbance at the compressor face is defined as $\varphi = \phi'_\eta|_{\eta=0}$, then it can be written as

$$\varphi = \sum_{n\in\mathbf{Z}\setminus\{0\}} a_n(t)\mathrm{e}^{|n|\eta - in\theta}\Big|_{\eta=0}$$

where $a_n = \alpha_n|n|$. Substituting equation (1.2) into equation (1.1), eliminating ϕ' by introducing φ and the operator K, which acts in Fourier spaces according to the formula

$$K(\Phi + \sum_{n\in\mathbf{Z}\setminus\{0\}} \varphi_n e^{in\theta}) = \Phi + \sum_{n\in\mathbf{Z}\setminus\{0\}} (1+\frac{am}{|n|})\varphi_n e^{in\theta},$$

we can rewrite the Moore-Greitzer model as an evolution equation of the form

$$\frac{\partial}{\partial t}\varphi = K^{-1}\left(\frac{\nu}{2}\frac{\partial^2\varphi}{\partial\theta^2} - \frac{1}{2}\frac{\partial\varphi}{\partial\theta} + a(\psi_c(\Phi+\varphi) - \overline{\psi_c})\right) \tag{1.4}$$

$$\frac{d\Phi}{dt} = \frac{1}{l_c}(\overline{\psi_c} - \Psi) \tag{1.5}$$

$$\frac{d\Psi}{dt} = \frac{1}{4l_cB^2}(\Phi - F_T^{-1}(\Psi)) \tag{1.6}$$

where

$$\overline{\psi_c} = \frac{1}{2\pi}\int_0^{2\pi} \psi_c(\Phi(t) + \varphi(t,\theta))d\theta.$$

LEMMA 1.1. *The operators K, $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial\theta}$ commute on $C^1(0,T;H^1)$. K and K^{-1} are bounded and positive operators.*

Proof. First note that K and $\frac{\partial}{\partial\theta}$ are mutually diagonalizable in the Sobolev space H^1 with the basis $\{e^{in\theta}\}_{n\in\mathbf{Z}}$ and hence they commute. Representing an element $x \in C^1(0,T;H^1)$ by

$$x = \sum_{n\in\mathbf{Z}} c_n(t)e^{in\theta}$$

we can differentiate under the sum with respect to t and hence K commutes with $\frac{\partial}{\partial t}$. The spectrum $\sigma(K)$ lies in $[1, 1+am]$ so clearly K is a positive bounded operator. It is straight forward to verify that the same holds for K^{-1} and $\sigma(K^{-1}) \subset [1-\frac{am}{1+am}, 1]$. $\square$

If the problem (1.4,1.5,1.6) has a unique solution with $\varphi \in H^1$ and $(\Phi,\Psi) \in \mathbf{R}^2$, then the coefficients a_n will be uniquely determined and hence ϕ' will be completely determined on the boundary $\eta = 0$. The problem of solving Laplace's equation with Dirichlet boundary conditions is well posed and hence there exists a unique solution to the Moore-Greitzer model, $x = (\phi', \Phi, \Psi)$.

1.2. Notation and Prerequisites. First observe that if the initial value for equation (1.4) has zero average, then the solution φ will also have zero average for all t. We denote by $|\cdot|_p$, $1 \le p \le \infty$, and $\|\cdot\|_k$, $k = 1, 2, \ldots$ the norms on the spaces $\bar{L}^p(S^1)$ and the Sobolev spaces $\bar{H}^k(S^1)$ of functions with zero mean. The norm $\|\cdot\|_k$ is given by

$$\|x\|_k^2 = \int_0^{2\pi} |\frac{\partial^k}{\partial\theta^k}x(\theta)|^2 d\theta.$$

We let $X^k = \bar{H}^k \times \mathbf{R}^2$, this is a Hilbert space when equipped with the norm

$$\|(\varphi,z)\|_{X^k}^2 = \gamma_0\|\varphi\|_k^2 + \sum_{i=1}^{2}\gamma_i|z_i|^2$$

and with the corresponding inner product. Here $\gamma_0, \gamma_1, \gamma_2 > 0$ are constants to be specified later, k is an integer and $\varphi \in \bar{H}^k, z \in \mathbf{R}^2$.

The system (1.4-1.6) can be written more compactly as an abstract evolution equation in X^k

$$\frac{dx}{dt} = Ax + f(x), \qquad x(0) = x_0, \tag{1.7}$$

where

$$A(\varphi, \Phi, \Psi) := (K^{-1}(\frac{\nu}{2}\frac{\partial^2 \varphi}{\partial \theta^2} - \frac{1}{2}\frac{\partial \varphi}{\partial \theta}), 0, 0) \tag{1.8}$$

$$f(g, \Phi, \Psi) := (aK^{-1}(\psi_c(\Phi + \varphi) - \overline{\psi_c}), \frac{1}{l_c}(\overline{\psi_c} - \Psi), \frac{1}{4l_c B^2}(\Phi - F_T^{-1}(\Psi)). \tag{1.9}$$

The domain of A is $D(A) = X^2$. One easily verifies that A has a complete set of eigenvectors χ_n and we denote the corresponding eigenvalues by λ_n. These are given by

$$\lambda_n = (-\frac{\nu}{2}[\frac{n}{2}]^2 - i[\frac{n}{2}]\frac{1}{2})(1 - \frac{am}{[\frac{n}{2}] + am})$$

$$\chi_n(\theta) = \begin{cases} (0, e_n) & : \text{ if } \quad n = 0, 1 \\ (\mathrm{e}^{i[\frac{n}{2}]\theta}, 0) & : \text{ if } \quad n \quad \text{odd} \\ (\mathrm{e}^{-i[\frac{n}{2}]\theta}, 0) & : \text{ if } \quad n \quad \text{even} \end{cases}$$

where $[s]$ denotes the integer part of s and $\{e_0, e_1\}$ is the standard basis in $\mathbf{R}^2$.

We will need the following two lemmas.

LEMMA 1.2. *The linear operator A generates a continuous semigroup $S(t)$ on X^0.*

Proof. We first need to verify that A is a closable operator. To this end, it suffices to show that the domain of the adjoint, $D(A^*)$, is dense in X^0. We can formally calculate A^*

$$A^*(\varphi, \Phi, \Psi) := (K^{-1}(\frac{\nu}{2}\frac{\partial^2 \varphi}{\partial \theta^2} + \frac{1}{2}\frac{\partial \varphi}{\partial \theta}), 0, 0)$$

by integration by parts. This operator is well defined on X^2 which is dense in X^0 and thus A is closable.

If $d(\lambda)$ denotes the distance of λ from the spectrum of $A, \sigma(A)$, then $\|\lambda I - A\| \geq d(\lambda)$ from which one can conclude that $\|(\lambda I - A)^{-1}\| \leq d(\lambda)^{-1}$. Since $\sigma(A)$ lies in the left half of the complex plane we have that the resolvent is bounded for all positive λ

$$\|(\lambda I - A)^{-1}\| \leq d(\lambda)^{-1} \leq \lambda^{-1}$$

and thus

$$\|(\lambda I - A)^{-m}\| \leq \|(\lambda I - A)^{-1}\|^m \leq \lambda^{-m}.$$

It now follows from the Hille-Yosida Theorem that A generates a C^0-semigroup $S(t)$. □

An arbitrary $x \in X^k$ can be expanded in the basis of eigenvectors of A

$$x = \sum_{n=0}^{\infty} \beta_n \chi_n,$$

then the norm of $x = (\varphi, z)$ is given by

$$\|x\|^2_{X^k} = \gamma_0 \sum_{n=2}^{\infty} ([\frac{n}{2}]^k |\beta_n|)^2 + \gamma_1 |z_1|^2 + \gamma_2 |z_2|^2.$$

We will in what follows use the notation K for both the operator as defined above as well as for the operator (K, I) where I is the identity in $\mathbf{R}^2$.

LEMMA 1.3. *For $x = (\varphi, z) \in X^k$ the following inequalities hold*

$$\begin{array}{lll} \mathrm{Re}\langle x, KAx\rangle_{X^k} & \leq & -\delta \|\varphi\|^2_{k+1} \\ \|S(t)x\|_{X^{k+1}} & \leq & t^{-3/4} C \|x\|_{X^k} \\ \|S(t)x\|_{X^k} & \leq & \|x\|_{X^k} \\ \|(S(t) - I)x\|_{X^k} & \leq & t^{1/4} C \|x\|_{X^{k+1}} \end{array}$$

for $t > 0$ small and some positive constants δ, C.

Proof. Without loss of generality we may assume that the constants in the norm as defined above are $\gamma_i = 1$ for $i = 0, 1, 2$. For the first inequality observe that

$$\begin{array}{lll} \mathrm{Re}\langle x, KAx\rangle_{X^k} & = & \sum_{n=2}^{\infty} (1 + \frac{am}{[\frac{n}{2}]}) \mathrm{Re}\lambda_n ([\frac{n}{2}]^k |\beta_n|)^2 \\ & = & \sum_{n=2}^{\infty} (1 + \frac{am}{[\frac{n}{2}]}) (\frac{\mathrm{Re}\lambda_n}{[n/2]^2}) ([\frac{n}{2}]^{k+1} |\beta_n|)^2 \\ & \leq & -\delta \|\varphi\|^2_{k+1} \end{array}$$

and

$$\delta = \min_n \frac{-\mathrm{Re}\lambda_n}{[n/2]^2} (1 + \frac{am}{[\frac{n}{2}]}) \geq \frac{\nu}{4} (1 - \frac{am}{1 + am}).$$

The action of the semigroup can be written in the following way

$$S(t)x = \sum_{n=0}^{\infty} \mathrm{e}^{\lambda_n t} \langle \chi_n, x\rangle \chi_n$$

and we can proceed to get

$$\begin{array}{lll} \|S(t)x\|^2_{X^{k+1}} & = & \sum_{n=2}^{\infty} ([\frac{n}{2}]^{k+1} \mathrm{e}^{\mathrm{Re}\lambda_n t} |\beta_n|)^2 + |z_1|^2 + |z_2|^2 \\ & \leq & (\sum_{n=2}^{\infty} [\frac{n}{2}]^2 \mathrm{e}^{2\mathrm{Re}\lambda_n t}) (\sum_{n=2}^{\infty} ([\frac{n}{2}]^k |\beta_n|)^2) + |z_1|^2 + |z_2|^2 \\ & \leq & \sum_{n=2}^{\infty} ([\frac{n}{2}]^2 \mathrm{e}^{-\nu [\frac{n}{2}]^2 t}) \|\varphi\|^2_k + |z_1|^2 + |z_2|^2. \end{array}$$

By estimating

$$\begin{array}{lll} \sum_{n=2}^{\infty} [\frac{n}{2}]^2 \mathrm{e}^{-\nu [\frac{n}{2}]^2 t} & = & \frac{2}{t\nu} \sum_{n=1}^{\infty} \nu t n^2 \mathrm{e}^{-\nu [\frac{n}{2}]^2 t} \\ & \leq & \frac{2}{t\nu} \int_0^{\infty} t\nu s^2 \mathrm{e}^{-t\nu s^2} ds \\ & = & \frac{2}{(t\nu)^{3/2}} \int_0^{\infty} r^2 \mathrm{e}^{-r^2} dr \\ & = & t^{-3/2} C \end{array}$$

where C is a constant, we get for $t \leq C^{2/3}$

$$\|S(t)x\|^2_{X^{k+1}} \leq t^{-3/2}C\|x\|^2_{X^k}.$$

Taking the square root of both sides will give the second inequality.

The third inequality follows from the fact that the real part of the spectrum of A lies on the negative real axis and from the simple representation of the semigroup that we have.

Let x be represented as a linear combination of eigenfunctions as before. We now get

$$\begin{aligned}
\|(S(t)-I)x\|^2_{X^k} &= \textstyle\sum_{n=2}^{\infty}([\frac{n}{2}]^k(\mathrm{e}^{\mathrm{Re}\lambda_n t}-1)|\beta_n|)^2 \\
&\leq \textstyle\sum_{n=2}^{\infty}(\frac{\mathrm{e}^{\mathrm{Re}\lambda_n t}-1}{[\frac{n}{2}]})^2\sum_{n=2}^{\infty}([\frac{n}{2}]^{k+1}|\beta_n|)^2 \\
&\leq \textstyle\sum_{n=2}^{\infty}(\frac{\mathrm{e}^{-\frac{\nu}{2}[\frac{n}{2}]^2 t}-1}{[\frac{n}{2}]})^2\|x\|^2_{X^{k+1}}.
\end{aligned}$$

By estimating

$$\begin{aligned}
\textstyle\sum_{n=2}^{\infty}(\frac{\mathrm{e}^{-\frac{\nu}{2}[\frac{n}{2}]^2 t}-1}{[\frac{n}{2}]})^2 &= \textstyle\frac{t\nu}{2}\sum_{n=1}^{\infty}(\frac{\mathrm{e}^{-t\frac{\nu}{2}n^2}-1}{n\sqrt{\frac{t\nu}{2}}})^2 \\
&\leq \textstyle\frac{t\nu}{2}\int_0^{\infty}(\frac{\mathrm{e}^{-t\frac{\nu}{2}s^2}-1}{s\sqrt{\frac{t\nu}{2}}})^2 ds \\
&= \textstyle\sqrt{\frac{t\nu}{2}}\int_0^{\infty}(\frac{\mathrm{e}^{-r^2}-1}{r})^2 dr \\
&= \sqrt{t}C
\end{aligned}$$

where C is a constant, we get

$$\|(S(t)-I)x\|^2_{X^k} \leq \sqrt{t}C\|x\|^2_{X^{k+1}}$$

which gives us the last inequality. □

2. Existence and Regularity of Local Solutions. The local problem 1.7 is well posed and the solutions are smooth functions. This follows from Henry [3] since the operator A is sectorial and $f \in C^{\infty}$. For completeness we include here the proof of local existence of smooth solutions.

THEOREM 2.1. *For any $x_0 \in X^1$, there exists a unique solution of (1.7) $x(t) \in C([0,T];X^1)$, for some $T > 0$, such that $x(0) = x_0$. Furthermore, x is smooth in its arguments.*

Proof. We will prove the existence by using the contraction principle. We use that f is a C^1 function from X^k to X^k for $k > \frac{1}{2}$.

Define the set B by

$$B_M = \{u \in C([0,T];X^1) | \sup_{0\leq t\leq T} \|u(t)-u_0\|_{X^1} \leq M\}$$

where $T > 0$ is to be specified later and $M > 3\|x_0\|_{X^1}$ is a finite constant. Let us also define a mapping

$$\mathcal{F}u = S(t)u_0 + \int_0^t S(t-\tau)f(u(\tau))d\tau.$$

For $u \in B_M$ we have

$$\begin{aligned}
\|\mathcal{F}u-u_0\|_{X^1} &\leq \textstyle\|(S(t)-I)u_0\|_{X^1} + \int_0^t\|S(t-\tau)f(u(\tau))\|_{X^1}d\tau \\
&\leq \textstyle\|S(t)u_0\|_{X^1} + \|u_0\|_{X^1} + \int_0^t\|f(u(\tau))\|_{X^1}d\tau
\end{aligned}$$

by Lemma 1.3. Since f is C^1, it is locally bounded. We conclude that

$$\begin{aligned} &\leq 2\|u_0\|_{X^1} + Ct\|u\|_{C([0,T];X^1)} \\ &\leq M, \end{aligned}$$

for small enough t. Let $T > 0$ be small enough so that this holds for all t such that $0 \leq t \leq T$. Then $\mathcal{F}$ maps B_M into itself. B_M is a closed subset of a complete metric space, therefore it is a complete metric space.

Now consider for $u, v \in B_M$

$$\begin{aligned} \|\mathcal{F}u - \mathcal{F}v\|_{X^1} &= \|\textstyle\int_0^t S(t-\tau)(f(u(\tau)) - f(v(\tau)))d\tau\|_{X^1} \\ &\leq \textstyle\int_0^t \|S(t-\tau)(f(u(\tau)) - f(v(\tau)))\|_{X^1} d\tau \\ &\leq C\textstyle\int_0^t (t-\tau)^{-3/4}\|f(u(\tau)) - f(v(\tau))\|_{X^1} d\tau \\ &\leq C'\textstyle\int_0^t (t-\tau)^{-3/4}\|u(\tau) - v(\tau)\|_{X^1} d\tau \\ &\leq 4C'\|u - v\|_{C([0,T];X^1)} t^{1/4} \\ &\leq \tfrac{1}{2}\|u - v\|_{C([0,T];X^1)} \end{aligned}$$

if we make T small enough. Here we used that f is locally bounded and we used Lemma 1.3. Hence

$$\|\mathcal{F}u - \mathcal{F}v\|_{C([0,T];X^1)} \leq \frac{1}{2}\|u - v\|_{C([0,T];X^1)}$$

and $\mathcal{F}$ is a strict contraction on B_M and hence it has a unique fixed point. The constant M is arbitrary so there exists a unique mild solution of equation (1.7) with initial condition in X^1.

To prove the regularity of u let us now assume that $u \in C([0,T];X^k)$, $k \geq 1$ is a mild solution of (1.7), i.e. a fixed point of $\mathcal{F}$. For $t > 0$ we have

$$\begin{aligned} \|\mathcal{F}u\|_{k+1}(t) &\leq \|S(t)u_0\|_{k+1} + \|\textstyle\int_0^t S(t-\tau)f(u(\tau))d\tau\|_{k+1} \\ &\leq Ct^{-3/4}\|u_0\|_k + \textstyle\int_0^t \|S(t-\tau)f(u(\tau))\|_{k+1} d\tau \\ &\leq Ct^{-3/4}\|u_0\|_k + C\textstyle\int_0^t (t-\tau)^{-3/4}\|f(u(\tau))\|_k d\tau \\ &\leq Ct^{-3/4}\|u_0\|_k + C'\textstyle\int_0^t (t-\tau)^{-3/4}\|u(\tau)\|_k d\tau \\ &\leq Ct^{-3/4}\|u_0\|_k + 4C't^{1/4}\|u\|_{C([0,T];X^k)}. \end{aligned}$$

Here we used Lemma 1.3 and the fact that f is locally bounded. This shows that the X^{k+1} norm of u is bounded for all t such that $0 < t < T$. If we consider the initial value problem with initial value $u(t) \in X^k$, then the solution will be in $C((t,T];X^{k+1})$. This too holds for any $0 < t \leq T$ and thus $u \in C((0,T];X^{k+1})$. By induction and Sobolev Embedding Theorem we get that u is smooth in the space variable. Furthermore, we see from equation (1.7) that the time derivative of u is continuous and smooth in space. Iterated differentiation of the equation with respect to time and substituting the result back into equation (1.7) to get rid of the mixed derivatives one concludes that u is smooth in the time variable as well. □

Remark. In the case when the nonlinear function f depends smoothly on some parameters and is smooth for all parameter values in question, then one can verify that the solution also depends smoothly on these parameters. This follows from the continuous dependence of the solution on initial data. Thus the local initial value problem is well posed.

3. A Priori Bounds and Global Existence. To get global existence it suffices to show that the solutions stay bounded for all time. Our goal is however to prove the existence of a global attractor, which we will define shortly. We therefore need to get more than just existence for all time, we need to show that there exist absorbing sets in the appropriate spaces.

A bounded set D is said to be an *absorbing set* for an evolution equation, if for all bounded sets R there exists a time t_R such that for all $t > t_R$ we have $S(t)R \subset D$. It is clear that if an evolution equation has an absorbing set in the same space that its initial value problem is well posed in, then the solutions will stay bounded for all time and global existence follows.

We now state two lemmas which will give us *a priori* bounds in X^0 and X^1, respectively. The first one gives us the existence of an absorbing set in X^0. The second one gives us that the flow disturbance φ decays exponentially into a bounded set in $\bar{H}^1$. From this we conclude that we have absorbing set in X^1.

LEMMA 3.1. *There exists an absorbing set D for equation (1.7) in X^0.*

Proof. We want to start by estimating $\langle x, Kf(x)\rangle_{X^0}$. First we specify the constants in the norm, $\gamma_0 = \frac{1}{a}, \gamma_1 = l_c$ and $\gamma_2 = 4l_cB^2$. With this choice we get cancellation of some cross terms.

In the inner product on X^0 which corresponds to this norm we get

$$\begin{aligned}
\langle x, Kf(x)\rangle_{X^0} &= \langle \varphi, \psi_c(\varphi+\Phi)\rangle_{\bar{L}^2} + \langle \Phi, \psi_c(\varphi+\Phi)\rangle_{\bar{L}^2} - F_T^{-1}(\Psi)\Psi \\
&= \langle (\varphi+\Phi), \psi_c(\varphi+\Phi)\rangle_{\bar{L}^2} - F_T^{-1}(\Psi)\Psi.
\end{aligned}$$

Here we have included the operator K to cancel the action of its inverse.

Since ψ_c is a cubic polynomial with a negative leading coefficients there exist positive constants, C_1, C_2, such that $y\psi_c(y) \leq -C_1 y^2 + C_2$. Using this we get

$$\begin{aligned}
\langle x, Kf(x)\rangle &\leq -C_1|\varphi+\Phi|_2^2 + C_2 - F_T^{-1}(\Psi)\Psi \\
&= -C_1|\varphi|_2^2 - C_1|\Phi|^2 - \gamma|\Psi|^{3/2} + C_2 \\
&\leq -C'\|x\|_{X^0}^{3/2} + C_2.
\end{aligned}$$

Here we used the fact that φ has zero average and we estimated the function F_T^{-1}. Furthermore, we assumed that $|\varphi|_2^2, |\Phi|^2 \geq 1$. If this is not the case, one can increase the constant C_2 and the estimate still holds.

We differentiate the following norm to get

$$\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}\|K^{\frac{1}{2}}x\|_{X^0}^2 &= \mathrm{Re}\langle x, KAx\rangle + \langle x, Kf(x)\rangle \\
&\leq -\delta\|x\|_{X^1}^2 - C'\|x\|_{X^0}^{3/2} + C_2
\end{aligned}$$

by Lemma 1.3. The inequality

$$\|K^{\frac{1}{2}}\|^{-1}\|K^{\frac{1}{2}}x\|_{X^0} \leq \|x\|_{X^0} \leq \|K^{\frac{1}{2}}x\|_{X^0}$$

now enables us to get the follwing estimate

$$\frac{1}{2}\frac{d}{dt}\|K^{\frac{1}{2}}x\|_{X^0}^2 \leq -C'\|K^{\frac{1}{2}}\|^{-3/2}\|K^{\frac{1}{2}}x\|_{X^0}^{3/2} + C_2$$

where we have ignored the first term in the above estimate. For $\epsilon > 0$ let $D_\epsilon = \{x \in X^0 | \|K^{\frac{1}{2}} x\|_{X^0} \leq (\frac{C_2+\epsilon}{C'})^{2/3} \|K^{\frac{1}{2}}\|\}$. Then D_ϵ is a bounded closed set and for $x \in X^0 \backslash D_\epsilon$ we have

$$\frac{1}{2}\frac{d}{dt}\|K^{\frac{1}{2}} x\|_{X^0}^2 \leq -\epsilon$$

and $\|K^{\frac{1}{2}} x\|_{X^0}^2$ decays at least linearly outside D_ϵ. Since $\|x\|_{X^0} \leq \|K^{\frac{1}{2}} x\|_{X^0}$ we conclude that D_ϵ is an absorbing set. □

The existence of an absorbing set in X^0 implies that the ODE variable z will be absorbed in a bounded set in $\mathbf{R}^2$ in finite time. To get an absorbing set in $X^k, k \geq 1$ it therefore suffices to show that the flow disturbance φ is absorbed into some bounded ball in finite time for all bounded z.

LEMMA 3.2. *There exists an absorbing set for equation (1.7) in $X^k, k = 1, 2$. Furthermore, the solutions $x = (\varphi, z)$ satisfy the* a priori *bound*

$$\|\varphi\|_k^2(t) \leq \|K^{\frac{1}{2}}\|^2 \|\varphi\|_k^2(0) \mathrm{e}^{-t} + (1 - \mathrm{e}^{-t}) C \tag{3.1}$$

where the constant C is independent of the initial conditions.

Proof. Let us denote the first coordinate of f in equation (1.7) by f_1.

Consider the case $k = 1$. Without loss of generality we may assume that $x(t)$ lies in an absorbing set D_ϵ in X^0 for $t \geq 0$. If not, we find a time t_0 so that it does for $t \geq t_0$ and then take $x(t_0)$ as our new initial condition.

Since ψ_c is a cubic polynomial with a negative leading coefficient, $\sup_{\phi \in \mathbf{R}} \psi_c'(\phi)$ is a finite constant. We define $M = \sup_{\phi \in \mathbf{R}} \psi_c'(\phi)$, then

$$\begin{aligned} \langle \varphi, K f_1(x) \rangle_{\bar{H}^1} &= \int_0^{2\pi} \psi_c'(\varphi + \Phi) \varphi_\theta^2 d\theta \\ &\leq M \|\varphi\|_1^2 \end{aligned}$$

where φ_θ denotes $\frac{\partial}{\partial \theta}$.

As before we differentiate the following norm

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|K^{\frac{1}{2}}\varphi\|_1^2 &= \mathrm{Re}\langle \varphi, KA\varphi \rangle_{\bar{H}^1} + \langle \varphi, K f_1(x) \rangle_{\bar{H}^1} \\ &\leq -\delta \|\varphi\|_2^2 + M \|\varphi\|_1^2 \\ &\leq -\delta \|K^{\frac{1}{2}}\|^{-2} \|K^{\frac{1}{2}}\varphi\|_2^2 + M \|K^{\frac{1}{2}}\|^{-2} \|K^{\frac{1}{2}}\varphi\|_1^2 \end{aligned}$$

using Lemma 1.3.

We can bound the $\bar{H}^1$ norm by an interpolation of the form

$$\|\cdot\|_1^2 \leq \frac{1}{2\sigma}\|\cdot\|_0^2 + \frac{\sigma}{2}\|\cdot\|_2^2.$$

So if we choose $\sigma = \frac{\delta \|K^{\frac{1}{2}}\|^{-2}}{(M\|K^{\frac{1}{2}}\|^{-2}+1)}$, then

$$\frac{d}{dt}\|K^{\frac{1}{2}}\varphi\|_1^2 \leq -\|K^{\frac{1}{2}}\varphi\|_1^2 + C - \frac{\delta}{2}\|K^{\frac{1}{2}}\|^{-2}\|K^{\frac{1}{2}}\varphi\|_2^2$$

where

$$C = \sup_{x \in D_\epsilon} \frac{1}{2\sigma} M \|K^{\frac{1}{2}}\|^{-2} \|K^{\frac{1}{2}}\varphi\|_0^2(s).$$

Neglecting the last term we get the following estimate

$$\frac{d}{dt}\|K^{\frac{1}{2}}\varphi\|_1^2 \leq -\|K^{\frac{1}{2}}\varphi\|_1^2 + C.$$

Solving this ODE inequality yields

$$\|K^{\frac{1}{2}}\varphi\|_1^2(t) \leq \|K^{\frac{1}{2}}\varphi\|_1^2(0)\mathrm{e}^{-t} + C(1-\mathrm{e}^{-t})$$

and thus

$$\|\varphi\|_1^2(t) \leq \|K^{\frac{1}{2}}\|^2\|\varphi\|_1^2(0)\mathrm{e}^{-t} + C(1-\mathrm{e}^{-t}).$$

It follows that for $\epsilon > 0$, the ball in X^1 of radius $C+\epsilon$ is an absorbing set in X^1.

The key estimate in this proof was the bound

$$\langle\varphi, Kf_1(x)\rangle_{\bar{H}^1} \leq M\|\varphi\|_1^2.$$

Once we have obtained the same type of bound for the case $k=2$, we can proceed in exactly the same way as before to get an absorbing set in X^2. To this end consider the following estimate

$$\begin{aligned}
\langle\varphi, Kf_1(x)\rangle_{\bar{H}^2} &= \langle\varphi_{\theta\theta}, \psi_c'(\Phi+\varphi)\varphi_{\theta\theta}\rangle_{\bar{L}^2} + \langle\varphi_{\theta\theta}, \psi_c''(\Phi+\varphi)\varphi_\theta^2\rangle_{\bar{L}^2} \\
&\leq |\psi_c'(\Phi+\varphi)|_\infty^2\|\varphi\|_2^2 + \tfrac{1}{2}(\|\varphi\|_2^2 + |\psi_c''(\Phi+\varphi)\varphi_\theta^2|_2^2) \\
&\leq |\psi_c'(\Phi+\varphi)|_\infty^2\|\varphi\|_2^2 + \tfrac{1}{2}(\|\varphi\|_2^2 + |\psi_c''(\Phi+\varphi)|_\infty^2|\varphi_\theta|_2^2|\varphi_\theta|_\infty^2) \\
&\leq |\psi_c'(\Phi+\varphi)|_\infty^2\|\varphi\|_2^2 + \tfrac{1}{2}(\|\varphi\|_2^2 + |\psi_c''(\Phi+\varphi)|_\infty^2\|\varphi\|_1^2\|\varphi\|_2^2) \\
&\leq M(\|\varphi\|_1)\|\varphi\|_2^2.
\end{aligned}$$

Here we have made extensive use of the Sobolev Imbedding theorem. The function M is bounded on bounded sets, so if D_ϵ is a bounded absorbing set in X^1 then

$$\tilde{M} = \sup_{x\in D_\epsilon} M(\|\varphi\|_1)$$

is a finite constant and the estimate

$$\langle\varphi, Kf_1(x)\rangle_{\bar{H}^2} \leq \tilde{M}\|\varphi\|_2^2$$

will allow us to proceed as before, with X^2 in the place of X^1 and X^1 in the place of X^0, to obtain a bounded absorbing set in X^2. □

The local existence of solutions together with the a priori bounds give us global existence and we can now state the following theorem

THEOREM 3.3. *The initial value problem (1.7) has a unique solution in the space $C([0,\infty); X^1)$. Furthermore, the solution is smooth in both space and time variables for $t > 0$.*

Remark. The existence of a global solution in X^1 and the smoothness of the local solution guarantees that the solution is smooth for all time. Absorbing sets in X^1 and X^2 do not imply the existence of an absorbing set in X^k in general. This can however be proven by induction. The essential estimate one needs is of the type

$$\langle\varphi, Kf_1(x)\rangle_{\bar{H}^k} \leq M(\|\varphi\|_{k-1})\|\varphi\|_k^2$$

where the function M is bounded on bounded sets. This argument is the same as in Lemma 3.2.

Remark. Although we have assumed that ψ_c is a cubic polynomial with a negative leading coefficient, the above analysis goes through for a more general ψ_c. In fact, all we needed was that ψ_c was smooth, $\sup \psi_c' < \infty$ and $\psi_c(y)y \leq -C_1y^2 + C_2$ for some positive constants C_1, C_2.

4. Global Attractor.

4.1. Existence of the Attractor.

We now prove the existence of the global attractor. Let us first give some definitions.

The *ω-limit set* of a set B is defined by

$$\omega(B) = \cap_{s\geq 0}\overline{\cup_{t\geq s}T(s)B}$$

An invariant set $\mathcal{A}$ is said to be a *global attractor* if $\mathcal{A}$ is a maximal compact invariant set which attracts each bounded set $B \subset X$. Observe that $\omega(B) \subset \mathcal{A}$.

The following theorem can be found in Hale [2].

THEOREM 4.1. *Let X be a Banach space and $\{T(t)\}_{t\geq 0}$ be a continuous semi-group on X. If $T(t)$ has a compact absorbing set D, then $\mathcal{A} = \omega(D)$ is a global attractor. The global attractor $\mathcal{A}$ is connected, if D is convex.*

A more general theorem on the existence of global attractors can be found in Birnir and Grauer [1].

COROLLARY 4.2. *Equation (1.7) has a global attractor $\mathcal{A}$ which is connected and consists of spatially smooth functions.*

Proof. There exists an absorbing set in X^1. For any initial condition in X^1, there exists a time T_1 after which the solution $x(t)$ lies in the absorbing set in X^1 by Lemma 3.2. Now take $x(T_1)$ as our initial condition. $x(t)$ lies in X^2 for $t > T_1$ by Theorem 2.1 and we now get an absorbing set D in X^2 by Lemma 3.2 and there exists a time T_2 after which the solution lies in D. D is compact in X^1 and it is an absorbing set for all initial conditions in X^1. D can be chosen to be convex. Hence, by Theorem 3.3 we have a connected global attractor $\mathcal{A}$. Since the attractor is invariant we know that $T(1)\mathcal{A} = \mathcal{A}$, but then by Theorem 3.3, $\mathcal{A} \subset X^k$ for all $k \geq 1$ and thus by the Sobolev embedding theorem, $\mathcal{A}$ consists of spatially smooth functions. □

Remark. $\mathcal{A}$ consists of smooth functions, but as a set it may not be smooth.

4.2. Dimension of the Attractor.

Methods for estimating the dimension of the attractor $\mathcal{A}$ were developed by Ruelle [5, 6] and later by Lieb [4] and Temam [7]. One typically studies the first variation of the evolution equation, i.e. analyzing the linearized equation, linearized about a solution to the nonlinear equation (1.7). If one can show that for all n large enough the n-dimensional volume element will decay in the linearized flow one can conclude that the dimension of the attractor is finite.

Let x denote a solution of the nonlinear evolution equation (1.7) with initial conditions $x(0) = x_0$. Assume that this solution lies in the attractor, $x \in \mathcal{A}$. We linearize the equation about this solution and denote this linearization by

$$\dot{y} = Ay + B(x)y, \qquad y(0) = y_0 \tag{4.1}$$

where $B(x)$ is the Fréchet derivative of f.

Let us now consider the n-dimensional volume element $\|y_1 \wedge ... \wedge y_n\|$, see Ruelle [5]. The time evolution of an n-dimensional volume element is governed by the equation

$$\frac{d}{dt}\ln \|y_1 \wedge ... \wedge y_n\|_{X^1} = \sum_{j=1}^{n} \mathrm{Re}\langle u_j, (A + B(x))u_j\rangle$$

where $u_1(t), ..., u_n(t)$ is an orthonormal basis that spans span$[y_1(t), ..., y_n(t)]$. If we can show that all n-dimensional volume elements decay in time we can conclude that the Hausdorff dimension and the fractal dimension of the global attractor $\mathcal{A}$ is bounded, see Temam [7] and Constantin *et al.* [8].

THEOREM 4.3. *The global attractor of equation (1.7) has a finite fractal dimension d_F and a finite Hausdorff dimension d_H. The dimensions are bounded by*

$$d_H \leq n_0 \qquad d_F \leq 2n_0$$

where n_0 is the smallest odd integer such that

$$2^{1/3}\sqrt{\frac{24L(1+am)}{\nu}+1} \leq n_0.$$

$L < \infty$ is a constant.

Proof. It is straight forward to verify that the linear equation (4.1) is well posed in X^1. It is known that $f: X^1 \to X^1$ is a C^1 function and thus $B(x)$ is a continuous linear operator from X^1 to X^1. The global attractor $\mathcal{A}$ is compact and thus we can bound the norm of $B(x)$ uniformly with a finite constant L

$$\sup_{x\in\mathcal{A}} \|B(x)\| = L < \infty.$$

Now consider the evolution of the n-dimensional volume element

$$\begin{array}{rcl} \frac{d}{dt}\ln\|y_1\wedge ... \wedge y_n\|_{X^1} & = & \sum_{j=1}^n \mathrm{Re}\langle u_j, (A+B(x))u_j\rangle \\ & \leq & \sum_{j=1}^n (\mathrm{Re}\lambda_{j-1} + L) \end{array}$$

It then becomes clear that all n-dimensional volume element decay whenever n is large enough to make the sum

$$\sum_{j=1}^n (\mathrm{Re}\lambda_{j-1} + L) = -\sum_{j=0}^{n-1} \frac{\nu}{2}[\frac{j}{2}]^2(1 - \frac{am}{[\frac{j}{2}]+am}) + nL$$

negative. Let us assume that n is odd, then we can estimate the sum

$$\begin{array}{rcl} -\sum_{j=0}^{n-1}\frac{\nu}{2}[\frac{j}{2}]^2(1-\frac{am}{[\frac{j}{2}]+am}) + nL & \leq & -\frac{\nu}{2}\frac{1}{1+am}\sum_{j=2}^{n-1}[\frac{j}{2}]^2 + nL \\ & = & -\nu\frac{1}{1+am}\sum_{j=1}^{\frac{n-1}{2}} j^2 + nL \\ & = & -\nu\frac{1}{1+am}\frac{1}{6}\frac{n-1}{2}(\frac{n-1}{2}+1)(2\frac{n-1}{2}+1) + nL \\ & = & -\frac{\nu}{24(1+am)}n(n-\sqrt{\frac{24L(1+am)}{\nu}+1}) \\ & & \cdot(n+\sqrt{\frac{24L(1+am)}{\nu}+1}). \end{array}$$

Using the inequality

$$-cz(z-d)(z+d) \leq -\frac{2c}{3}z^3 + \frac{2c}{3}d^3$$

which follows from Young's inequality, we can finally obtain the estimate

$$\sum_{j=1}^n (\mathrm{Re}\lambda_{j-1} + L) \leq -\frac{\nu}{36(1+am)}n^3 + \frac{\nu}{36(1+am)}\left(\frac{24L(1+am)}{\nu}+1\right)^{3/2}.$$

By Temam, Theorem VI.2.1 [7], if we let n_0 be the smallest positive odd integer such that

$$2^{1/3}\sqrt{\frac{24L(1+am)}{\nu}+1} \leq n_0$$

then the Hausdorff dimension of $\mathcal{A}$ is bounded by n_0 and the fractal dimension is bounded by $2n_0$ and the proof is complete. □

Appendix: Fractal and Hausdorff Dimensions of Sets. For completeness we include here the definitions of fractal and Hausdorff dimensions of a compact subset $U \subset X$, where X is a Hilbert space. Let $l \in \mathbf{R}, \epsilon > 0$ and define

$$\mu_H(U,l,\epsilon) = \inf \sum_{i\in I} r_i^l$$

where the infimum is taken over all coverings $\{B_i\}_{i\in I}$ of the set U, B_i are the balls in X of radius $r_i \leq \epsilon$. We define

$$\mu_H(U,l) = \lim_{\epsilon\to 0} \mu_H(U,l,\epsilon).$$

$\mu_H(U,l)$ is called the *l-dimensional Hausdorff measure* of the set U. The *Hausdorff dimension* of the set U is the unique number l' such that

$$\mu_H(U,l) = \begin{cases} \infty & : \quad l < l' \\ 0 & : \quad l > l' \end{cases}$$

The Hausdorff dimension of a set U is denoted by $d_H(U)$.

Let $N_\epsilon(U)$ be the smallest number of balls of radius ϵ which can cover U. The fractal dimension of U, denoted by $d_F(U)$, is defined as

$$d_F(U) = \inf\{l > 0, \mu_F(U,l) = 0\}$$

where

$$\mu_F(U,l) = \limsup_{\epsilon\to 0} \epsilon^l N_\epsilon(U).$$

Thus the difference between the Hausdorff dimension and the fractal dimension is that the radii of the balls are fixed for the fractal dimension whereas they are allowed to vary for the Hausdorff dimension, before we let ϵ tend to zero. From the fact that $\mu_F(U,l) \geq \mu_H(U,l)$ we can deduce that the fractal dimension of a set is always greater or equal to its Hausdorff dimension. In general they are not equal.

The Hausdorff and fractal dimensions have a number of nice properties. Let U, V be compact subsets of X.

- If U is a smooth m-dimensional submanifold of X then $d_H(U) = m = d_F(U)$.
- The dimensions are monotonic, i.e. if $U \subset V$ then $d_H(U) \leq d_H(V)$ and $d_F(U) \leq d_F(V)$.
- For two sets U, V,

$$\begin{aligned} d_H(U \cup V) &= max\{d_H(U), d_H(V)\} \\ d_F(U \cup V) &= max\{d_F(U), d_F(V)\}. \end{aligned}$$

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