THE HOMOLOGY OF STRING ALGEBRAS I

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Abstract. We show that string algebras are 'homologically tame' in the following sense: First, the syzygies of arbitrary representations of a finite dimensional string algebra Λ are direct sums of cyclic representations, and the left finitistic dimensions, both little and big, of Λ can be computed from a finite set of cyclic left ideals contained in the Jacobson radical. Second, our main result shows that the functorial finiteness status of the full subcategory $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ consisting of the finitely generated left Λ-modules of finite projective dimension is completely determined by a finite number of, possibly infinite dimensional, string modules – one for each simple Λ-module – which are algorithmically constructible from quiver and relations of Λ. Namely, $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in Λ-mod precisely when all of these string modules are finite dimensional, in which case they coincide with the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the corresponding simple modules. Even when $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ fails to be contravariantly finite, these 'characteristic' string modules encode, in an accessible format, all desirable homological information about Λ-mod.

1. Introduction

The representation theory of the Lorentz group is intimately linked to that of a certain string algebra Λ (for a definition of string algebras see Section 2), as was observed and exploited by Gelfand and Ponomarev in [**17**]. In particular, it was proved there that this algebra – along with a class of close relatives – has tame representation type; in fact, its finite dimensional indecomposable representations were explicitly pinned down. In a sequence of articles by Ringel [**29**], Bondarenko [**6**], Donovan-Freislich [**12**], Butler-Ringel [**8**], and others, the class of algebras amenable to techniques derived from the Gelfand-Ponomarev archetype was subsequently found to be much larger and, moreover, to be related to further classical scenarios, such as the representation theory of dihedral groups. This development ultimately led to a well-rounded representation-theoretic picture of the extended class of algebras on which we concentrate here, the class of string algebras. Among other tools, Auslander-Reiten methods were employed to place the finite dimensional indecomposable objects into a tightly knit categorical context. In tandem, certain portions of the infinite dimensional representation theory were rendered accessible. However, in spite of the availability of a full classification of the finitely generated indecomposable representations of string algebras, their homological properties, known to vary widely (see, e.g., [**25**]), remained far from understood.

Our goal is to supplement the structural information with equally precise homological data. For a more detailed preview of our results, let Λ be a finite dimensional string algebra over an algebraically closed field K. We start by showing how the homological dimensions – global and finitistic – can be obtained from a finite collection of cyclic modules contained in the Jacobson radical J of Λ (Theorem 3). Then we turn to two far more deep-seated problems concerning the category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ that has as objects the finitely generated modules of finite projective dimension. These problems are as follows: (I) Can the internal structure of the objects in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ be characterized, so as to distinguish them from those in $\Lambda\text{-mod} \setminus \mathcal{P}^{<\infty}(\Lambda\text{-mod})$? – here Λ-mod stands for the category of all finitely generated left Λ-modules – and (II) How is the category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ embedded in Λ-mod, in terms of maps entering or leaving $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$? It is our answer to the second question which displays the homological mechanisms of Λ; in particular, it entails a solution to the first problem.

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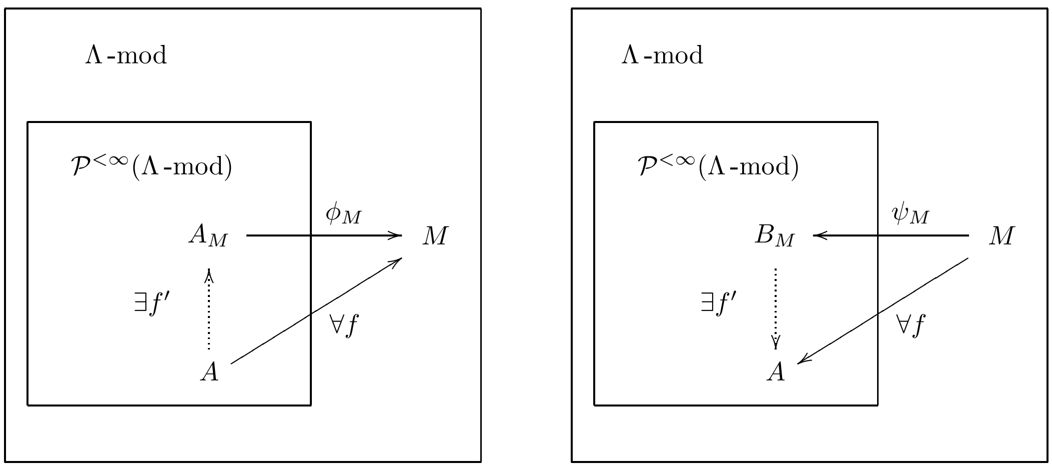
To tackle Problem II, we establish a readily checkable characterization of contravariant finiteness of the category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ in Λ-mod and describe the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the simple modules $S_1, \dots, S_n$ in the positive case (cf. Section 4 for the relevant definitions); in fact, our description yields a procedure for constructing them (Theorem 5 and Proposition 14). Existence of such minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the S_i and – existence provided – their structure are known to have far-reaching consequences for the homology of Λ (see, e.g. [**3**], [**1**], [**24**]); those which have direct impact on our present investigation are reviewed in Section 4.

To appreciate how contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ relates to our second problem, recall that this condition implies functorial finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$, i.e., dually defined left $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the objects in Λ-mod come as free byproducts (see [**24**]). Suppose, for the moment, that $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite, and fix an object $M \in \Lambda\text{-mod}$. The key roles played by the minimal right $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation $\phi_M : A_M \to M$ and the minimal left approximation $\psi_M : M \to B_M$ of M can be visualized as follows:



In other words, A_M is minimal with the property that all homomorphisms from any object $A \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ to M pass through the way station A_M via ϕ_M, and consequently, the problem of controlling all maps in $\mathrm{Hom}_\Lambda(\mathcal{P}^{<\infty}(\Lambda\text{-mod}), M)$ boils down to describing the approximation ϕ_M and understanding the internal maps of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$. The map ψ_M plays a dual role. On the side, we mention that functorial finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ guarantees the existence of almost split sequences in that category (see [**2**] and [**3**]).

Whether or not $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in Λ-mod, one can associate with each simple Λ-module S_i a representation A_i in the category $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ of all, not necessarily finite dimensional, left Λ-modules of finite projective dimension, together with a canonical map $\varphi_i : A_i \to S_i$, which is indicative of the map-theoretic 'location' of S_i relative to $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ (Theorem 5). In fact, $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite if and only if all of the A_i are finite dimensional, in which case the maps φ_i are the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the simple modules. Otherwise, the A_i are still 'phantoms' of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the S_i in the sense of [**18**] (see Section 4 for a reminder). Roughly, this means that each A_i exhibits, in the tightest possible format, the relations of those modules $M \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ which map onto S_i; for instance, it is precisely when S_i has finite projective dimension that $A_i \cong S_i$. All of these constructions are algorithmic. Indeed, even when the modules A_i are infinite dimensional, they can be constructed in a predictable number of steps, growing polynomially with $\dim_K \Lambda$, due to their periodicity properties.

In the present work, we primarily focus on the homology of the 'little' category Λ-mod – even though we need to resort to infinite dimensional modules to fully understand the latter – whereas a sequel to this paper will address additional phenomena arising in the homology of the 'big' category Λ-Mod.

As mentioned at the outset, the class of string algebras has developed into a showcase for representation-theoretic methods, thus attesting to the 'state of the art' on various fronts. Moreover, algebras degenerating to string algebras play a pivotal role in understanding more general classes of algebras. To tie the present investigation into the context of existing work, we add a relatively short, chronologically ordered list of further references providing historical background and samples of different lines of approach: [**37**], [**15**], [**35**], [**39**], [**13**], [**10**], [**11**], [**30**], [**5**], [**16**], [**26**], [**33**], [**38**], [**34**], [**31**], [**32**], [**7**].

2. Prerequisites and conventions

Consistently, $\Lambda = K\Gamma/I$ will denote a finite dimensional *string algebra* over an algebraically closed field K. This means that Λ is a *monomial* relation algebra (that is, the admissible ideal I of the path algebra $K\Gamma$ can be generated by paths), and Λ is *special biserial*. The latter amounts to the combination of the following two conditions on Γ and I: at most two arrows enter and at most two arrows leave any given vertex of Γ, and, for any arrow α in Γ, there is at most one arrow β with $\alpha\beta \notin I$ and at most one arrow γ with $\gamma\alpha \notin I$.

Our convention for multiplying paths is as follows: if p and q are paths in $K\Gamma$, then pq stands for 'p after q'. Correspondingly, a *right subpath* of a path p is a path u such that $p = u'u$ for some path u'; *left subpaths* of p are defined symmetrically. The set of vertices of Γ will be identified with a full set of primitive idempotents $e_1, \dots, e_n$ of Λ, and we will be referring to idempotents from this set whenever we mention primitive idempotents. Given any left Λ-module M, a *top element* of M is an element x with the property that $ex = x$ for some primitive idempotent e, in which case we will also call x a top element *of type* e of M.

Throughout, Λ-mod and Λ-Mod will stand for the categories of finite dimensional and arbitrary left Λ-modules, respectively, $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ will denote the full subcategory of Λ-mod having as objects the modules of finite projective dimension, while $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ will be the full subcategory of Λ-Mod consisting of all modules of finite projective dimension.

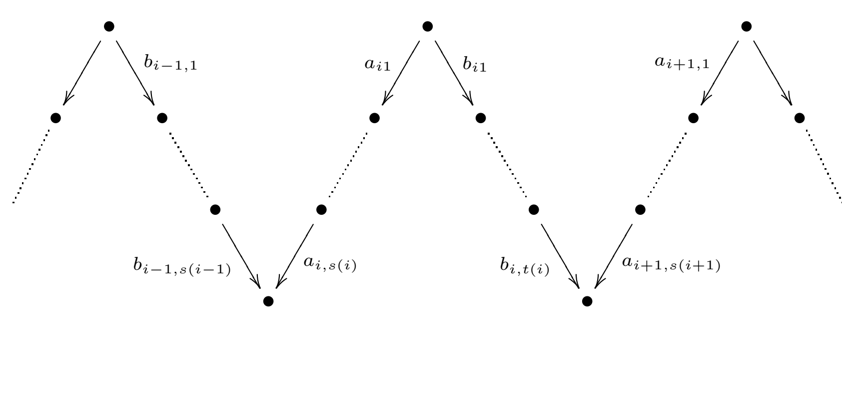
In essence, our conceptual and notational backdrop is that developed in successive steps in [**17**, **29**, **12**, **8**], but some modifications to the presentation of the relevant data will be more convenient for our purposes. As is common, our notion of a 'word' is based on the fixed presentation $\Lambda = K\Gamma/I$ as follows: *Syllables* are elements of the set $\mathcal{P} \sqcup \mathcal{P}^{-1}$, where $\mathcal{P}$ is the set of all paths in $K\Gamma \setminus I$ and $\mathcal{P}^{-1}$ consists of the formal inverses of the elements of $\mathcal{P}$. The paths of length 0, i.e., the vertices of Γ, are included in $\mathcal{P}$ and will be called the *trivial paths*; both these trivial paths and their inverses are called *trivial syllables.* For $p \in \mathcal{P}$, let $(p^{-1})^{-1} = p$, so that $(\mathcal{P}^{-1})^{-1} = \mathcal{P}$. *(Generalized) words* are $\mathbb{Z}$-indexed sequences of pairs of syllables $w = (p_i^{-1} q_i)_{i\in\mathbb{Z}}$ with $p_i, q_i \in \mathcal{P}$, which we also communicate as juxtapositions

$$\dots p_r^{-1} q_r \dots p_{-1}^{-1} q_{-1} p_0^{-1} q_0 p_1^{-1} q_1 \dots p_s^{-1} q_s \dots$$

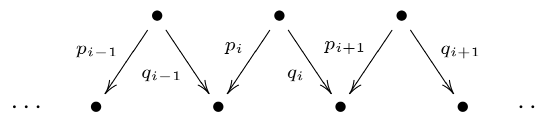
(note that syllables from $\mathcal{P}$ alternate with syllables from $\mathcal{P}^{-1}$) subject to the following constraints:

- For each $i \in \mathbb{Z}$, the starting points of p_i and q_i coincide, but the first arrows of p_i and q_i are distinct whenever both p_i and q_i are nontrivial.
- For each $i \in \mathbb{Z}$, the end points of q_i and p_{i+1} coincide, but the last arrows of q_i and p_{i+1} are distinct whenever both q_i and p_{i+1} are nontrivial.
- No trivial syllable occurs between two nontrivial syllables (i.e., the nontrivial syllables form a 'connected component').

A word $w = (p_i^{-1} q_i)_{i\in\mathbb{Z}}$ will be called *finite* in case, for all $i \gg 0$ and all $i \ll 0$, the syllables p_i^{-1} (and hence also the syllables q_i) are trivial; finite words are also communicated as finite juxtapositions $(p_i^{-1} q_i)$ in which all nontrivial syllables are preserved. More generally, we do not insist on recording trivial syllables; keep in mind that they can only occur at the left or right tail ends of a word. It is self-explanatory what we mean by a *left* or *right finite* word. Given a word $w = (p_i^{-1} q_i)_{i\in\mathbb{Z}}$, the inverse of w is defined as $w^{-1} = (q_{-i}^{-1} p_{-i})_{i\in\mathbb{Z}}$ carrying the pair of syllables $q_{-i}^{-1} p_{-i}$ in position i. With each word, we associate a (not necessarily finite) directed graph which records the nontrivial syllables. Namely, if $w = (p_i^{-1} q_i)_{i\in\mathbb{Z}}$ with $p_i = a_{is(i)} \dots a_{i1}$ and $q_i = b_{it(i)} \dots b_{i1}$, where the a_{ij} and b_{ij} are arrows, the graph of w is



where the trivial syllables make no appearance, and the nodes are identified with the primitive idempotents occurring as the starting and end points of the arrows a_{ij} and b_{ij}. When less detail is required, a simplified rendering of this graph will be preferred, namely:



Graphs of words will only play a role in the proof of our main theorem, while graphs of string and pseudo-band modules, as given below, will be essential throughout our discussion.

To prepare for the upcoming definition, note that, for any nontrivial two-syllable word $w = p^{-1}q$, there exists at most one arrow α such that $(\alpha p)^{-1}q$ is again a word; analogously, there exists at most one arrow β making $p^{-1}(\beta q)$ a word.

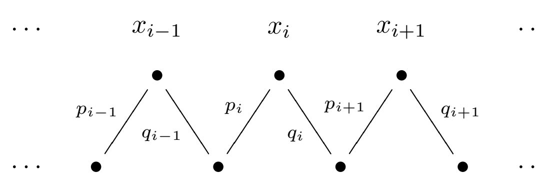
Each (generalized) word $w = (p_i^{-1}q_i)_{i\in\mathbb{Z}}$ gives rise to a *(generalized) string module* $\mathrm{St}(w)$ defined as follows: If w is trivial, say $w = e$, then $\mathrm{St}(w)$ is the simple module $\Lambda e/Je$. Now suppose that w is nontrivial, and let $\mathrm{supp}(w)$ be the set of all those integers j for which either p_j or q_j is nontrivial. If, moreover, the joint starting vertex of p_i and q_i is denoted by $e(i)$, then

$$\mathrm{St}(w) = \Bigl(\bigoplus_{i\in \mathrm{supp}(w)} \Lambda e(i)\Bigr) \Big/ C\,, \qquad \text{where}$$

$$C \;=\; \Bigl(\sum_{i,i+1\in \mathrm{supp}(w)} \Lambda\bigl(q_i e(i) - p_{i+1}e(i+1)\bigr)\Bigr) \;+\; C_{\mathrm{left}} \;+\; C_{\mathrm{right}},$$

with cyclic correction terms C_{left} and C_{right} defined as follows: $C_{\mathrm{left}} = 0$ if either $\mathrm{supp}(w)$ is unbounded on the negative $\mathbb{Z}$-axis or $l = \inf \mathrm{supp}(w)$ is an integer and $(\alpha p_l)^{-1}q_l$ fails to be a word for all arrows α; in the remaining case, where $l \in \mathbb{Z}$ and there exists a (necessarily unique) arrow α with the property that $(\alpha p_l)^{-1}q_l$ is again a word, we set $C_{\mathrm{left}} = \Lambda\alpha p_l e(l)$. The right-hand correction term C_{right} is defined symmetrically. Note that $\mathrm{St}(w)$ is finite dimensional over K precisely when w is a finite word. Moreover, $\mathrm{St}(w) \cong \mathrm{St}(w^{-1})$, a fact which allows us to pass back and forth between w and w^{-1} as convenience dictates. It is well-known that string modules are indecomposable; in the finite dimensional case, this is proved in [**17**, **29**, **12**, **8**], for infinite dimensional string modules in [**27**].

For $i \in \mathrm{supp}(w)$, let x_i be the residue class of $e(i)$ in $\mathrm{St}(w)$ in the above presentation. Clearly, the family $(x_i)_{i\in\mathrm{supp}(w)}$ consists of top elements which generate $\mathrm{St}(w)$ and are K-linearly independent modulo $J\,\mathrm{St}(w)$; by construction, they have the property that $q_ix_i = p_{i+1}x_{i+1}$, whenever q_i and p_{i+1} are both nontrivial. Any sequence of top elements of $\mathrm{St}(w)$ with the listed properties is called a *standardized sequence of top elements*. In the sense of [**19**] and [**20**], the module $\mathrm{St}(w)$ has a layered graph relative to any standardized sequence of top elements: It is the undirected variant of the (directed) graph of w, layered in such a fashion that the vertices in the i-th row from the top correspond to the simple composition factors in $J^{i-1}\,\mathrm{St}(w)/J^i\,\mathrm{St}(w)$. We will usually indicate the chosen sequence of top elements above the corresponding vertices in the first row of the graph as illustrated below.



The second class of representations of Λ which will be pivotal in our discussion slightly generalizes the classical 'band modules' (a generalization which will prove convenient in the proof of the main theorem). This class consists entirely of finite dimensional representations, but this time, they need not be indecomposable. For a description following the classical road, suppose that $v = p_0^{-1}q_0 \dots p_t^{-1}q_t$ is a finite word with $t \ge 0$ and p_0, q_t both nontrivial; by our conventions, this amounts to the same as to require that all p_i and q_i be nontrivial. We call v *primitive* if

- the juxtaposition $v^2 = vv$ is again a word (in which case all powers v^r are words), and
- v is not itself a power of a strictly shorter word.

In addition to the primitive word v, let r be a positive integer and $\phi : K^r \to K^r$ a cyclic automorphism (meaning that ϕ turns K^r into a cyclic $K[X]$-module) with Frobenius companion matrix

$$\begin{pmatrix} 0 & \cdots & 0 & c_1 \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & \vdots \\ 0 & \cdots & 1 & c_r \end{pmatrix}.$$

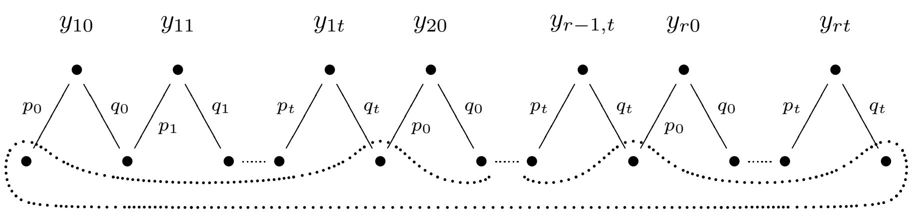
Then the *pseudo-band module* $\mathrm{Bd}(v^r, \phi)$ is defined as follows: Let $x_{10}, \dots, x_{1t}, x_{20}, \dots, x_{2t}, \dots, x_{r0}, \dots, x_{rt}$ be the standardized sequence of top elements of $\mathrm{St}(v^r)$ following the definition of a string module; in particular, $q_j x_{ij} = p_{j+1} x_{i,j+1}$ for $0 \le i \le r$ and $1 \le j < t$, and $q_t x_{it} = p_0 x_{i+1,0}$ for $i < r$. Then

$$\mathrm{Bd}(v^r, \phi) = \mathrm{St}(v^r)/\Lambda\big(q_r x_{rt} - \sum_{i=1}^{r} c_i p_0 x_{i0}\big)$$

by definition. If the residue classes of the x_{ij} in $\mathrm{Bd}(v^r, \phi)$ are denoted by y_{ij}, then the latter are top elements of $\mathrm{Bd}(v^r, \phi)$ which are K-linearly independent modulo the radical, generate $\mathrm{Bd}(v^r, \phi)$, and satisfy the above equations, as well as the additional one

$$q_t y_{rt} = \sum_{i=1}^{r} c_i p_0 y_{i0}.$$

Any sequence of top elements with these properties is, in turn, called a standardized sequence of top elements of the pseudo-band module $\mathrm{Bd}(v^r, \phi)$. Relative to such a sequence, $\mathrm{Bd}(v^r, \phi)$ can be depicted in the form



where the dotted line that encircles the vertices standing for the elements $q_t y_{rt}$, $p_0 y_{i0}, \dots, p_0 y_{r0}$ in the socle of $\mathrm{Bd}(v^r, \phi)$ indicates that the K-space spanned by these $r+1$ elements has dimension r only.

In case the automorphism ϕ of K^r is irreducible, the pseudo-band module $\mathrm{Bd}(v^r, \phi)$ is called a *band module*. It is readily seen that a pseudo-band module is a band module if and only if it is indecomposable (see, e.g., [**8**]). Note that, in contrast to the string case, the graph of a band module $\mathrm{Bd}(v^r, \phi)$ does not pin down the latter up to isomorphism, unless the scalars $c_1, \dots, c_r$ are recorded. Moreover, observe that, if we subject the pairs of syllables $p_i^{-1} q_i$ of the underlying primitive word v to a cyclical permutation resulting in $\hat{v}$ say, then $\mathrm{Bd}(v^r, \phi) \cong \mathrm{Bd}(\hat{v}^r, \phi)$.

The pivotal role played by the finite dimensional string and band modules is apparent from the following classification result, which will be used extensively in the sequel. In its present form, it was established by Butler and Ringel, but the ideas go back to Gelfand and Ponomarev who determined the finite dimensional representation theory of a somewhat more restricted class of algebras.

Theorem 0. (See [**17**, **29**, **6**, **12**, **8**]) *The finitely generated string and band modules are precisely the indecomposable objects of* Λ-mod. $\square$

3. Syzygies and the homological dimensions of string algebras

This short section provides a first installment of evidence that, not only from a representation-theoretic, but also from a homological viewpoint, string algebras show 'tame behavior'. Not only can the global dimension of a string algebra be computed algorithmically from quiver and relations as we will shortly see, but this is true more generally for its finitistic dimensions.

Our first proposition determines the syzygies of string and band modules. All of these are direct sums of cyclic string modules which can be described in terms of the string and band data. Since we will repeatedly invest this information in subsequent sections, we will be explicit, starting with the slightly cumbersome notation required to pin down syzygies. Suppose that p and q are paths, not both trivial, such that $p^{-1}q$ is a word, i.e., p and q both start in the same vertex e, and if both of these paths are nontrivial, they have distinct first arrows. Then, clearly, the string module $\mathrm{St}(p^{-1}q)$ is a factor module of Λe. Let $\mathfrak{p}$ and $\mathfrak{q}$ be the unique paths starting in the end points of p and q, respectively, with the property that $\mathrm{St}\big((\mathfrak{p}p)^{-1}(\mathfrak{q}q)\big) \cong \Lambda e$. In other words, if p is nontrivial, then $\mathfrak{p}$ is the longest path such that $\mathfrak{p}p \in K\Gamma \setminus I$, the path $\mathfrak{q}$ having an analogous description if q is nontrivial; if on the other hand, p is trivial, then q is not, and $\mathfrak{p}$ is the longest path starting in e which does not contain the first arrow of q as a right subpath (in particular, $\mathfrak{p} = e$ in case q is nontrivial and Λe is uniserial). Finally, given any nontrivial path u, let $u^{(0)}$ be the path of length ≥ 0 obtained from u through deletion of the first arrow, and set $u^{(0)} = 0 \in K\Gamma$ if u is trivial.

We observe that the first syzygy of any cyclic string module $\mathrm{St}(p^{-1}q)$ equals $\mathrm{St}(\mathfrak{p}^{(0)}) \oplus \mathrm{St}(\mathfrak{q}^{(0)})$ if p, q are not both trivial, and equals Je if $p = q = e$. Less obvious situations are addressed in the first proposition, the proof of which is immediate from the definitions and the graphical methods for determining syzygies developed in [**19**].

Proposition 1.

(1)(a) *Suppose w is a finite nontrivial word of the form $p_0^{-1}q_0 \dots p_t^{-1}q_t$ with q_0 and p_t nontrivial. Then the first syzygy of the finite dimensional string module $\mathrm{St}(w)$ is*

$$\mathrm{St}(\mathfrak{p}_0^{(0)}) \oplus \bigoplus_{i=0}^{t-1} \mathrm{St}(\mathfrak{p}_{i+1}^{-1}\mathfrak{q}_i) \oplus \mathrm{St}(\mathfrak{q}_t^{(0)}),$$

where the paths $\mathfrak{p}_i$, $\mathfrak{q}_i$, $\mathfrak{q}_0^{(0)}$, and $\mathfrak{q}_t^{(0)}$ are as introduced above.

(b) *Now suppose that $w = \dots p_{-1}^{-1}q_{-1}p_0^{-1}q_0p_1^{-1}q_1 \dots$ is a word with p_i, q_i nontrivial for all $i \in \mathbb{Z}$. The first syzygy of the infinite dimensional string module $\mathrm{St}(w)$ equals*

$$\bigoplus_{i\in\mathbb{Z}} \mathrm{St}(\mathfrak{p}_{i+1}^{-1}\mathfrak{q}_i).$$

(c) *If $w = p_0^{-1}q_0p_1^{-1}q_1 \dots$ with q_i nontrivial for $i \geq 0$, then the first syzygy of $\mathrm{St}(w)$ is*

$$\mathrm{St}(\mathfrak{p}_0^{(0)}) \oplus \bigoplus_{i=0}^{\infty} \mathrm{St}(\mathfrak{p}_{i+1}^{-1}\mathfrak{q}_i).$$

The left infinite case is symmetric.

(2) *If $v = p_0^{-1}q_0 \dots p_t^{-1}q_t$ is a primitive word with p_0 and q_t nontrivial, r a positive integer, and ϕ a cyclic automorphism of K^r, then the first syzygy of the pseudo-band module $\mathrm{Bd}(v^r, \phi)$ is*

$$\bigoplus_{i=0}^{t-1} \big(\mathrm{St}(\mathfrak{p}_{i+1}^{-1}\mathfrak{q}_i)\big)^r \oplus \big(\mathrm{St}(\mathfrak{p}_0^{-1}\mathfrak{q}_t)\big)^r.$$

Furthermore, the following statements are equivalent:

- (i) *For* some *positive integer r and* some *cyclic automorphism ϕ of K^r, the pseudo-band module $\mathrm{Bd}(v^r, \phi)$ belongs to $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$.*
- (ii) *For* any *positive integer r and* any *cyclic automorphism ϕ of K^r, the pseudo-band module $\mathrm{Bd}(v^r, \phi)$ belongs to $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$.*
- (iii) *The generalized string module $\mathrm{St}(\dots vvv \dots)$ belongs to $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$* □

In view of Theorem 0 (Section 2), we glean from Proposition 1 that the syzygies of any finitely generated Λ-module M are direct sums of cyclic string modules. Next we will see that this remains true even when we drop the requirement that M be finitely generated.

Proposition 2. *Every submodule of a projective left* Λ*-module is a direct sum of string modules of the form* $\mathrm{St}(p^{-1}q)$*, where* p *and* q *are paths.*

In particular, syzygies of arbitrary Λ*-modules are direct sums of cyclic string modules which embed in* J*, and all second syzygies are direct sums of uniserial modules.*

Proof. To verify the first assertion, let P be a projective left Λ-module and $M \subseteq P$ a submodule. Since all of the indecomposable projective left Λ-modules are cyclic string modules, it is harmless to assume that $M \subseteq JP$.

Write M as a directed union of finitely generated submodules, say $M = \bigcup_{i \in I} M_i$. Then all M_i are syzygies of finitely generated modules and, by the preceding remark, the M_i are direct sums of string modules with simple tops. Since (up to isomorphism) there are only finitely many string modules of of the form $\mathrm{St}(p^{-1}q)$, this entails that the direct sum $\bigoplus_{i \in I} M_i$ is Σ-algebraically compact. Therefore, by [**24**, Observation 3.1] or [**28**], the category $\mathrm{Add}\big(\bigoplus_{i \in I} M_i\big)$ of arbitrary direct sums of direct summands of the M_i is closed under direct limits. In particular, M is in turn a direct sum of cyclic string modules, as claimed.

The final statements are immediate consequences. $\square$

Recall that the left little and big finitistic dimensions of any finite dimensional algebra Δ are defined as

$$\mathrm{l\,fin\,dim}\,\Delta = \sup\{\mathrm{p\,dim}\,M \mid M \in \mathcal{P}^{<\infty}(\Delta\text{-mod})\}$$
$$\mathrm{l\,Fin\,dim}\,\Delta = \sup\{\mathrm{p\,dim}\,M \mid M \in \mathcal{P}^{<\infty}(\Delta\text{-Mod})\},$$

respectively. According to [**19**] or [**20**], the finitistic dimensions of any monomial relation algebra can be computed up to an error of 1 by means of a simple graphical method. However, even for monomial relation algebras, the little finitistic dimension may be strictly smaller than the big finitistic dimension [**20**], and, for more general finite dimensional algebras, the difference $\mathrm{l\,Fin\,dim}\,\Delta - \mathrm{l\,fin\,dim}\,\Delta$ attains arbitrarily high values in $\mathbb{N}$ [**36**]. Our first theorem excludes such 'pathologies' for string algebras and pins down the finitistic dimensions in terms of the cyclic string modules of finite projective dimension; the latter are finite in number and easy to construct from Γ and I. To that end, consider the set $\mathcal{T}$ of those cyclic string modules in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ which are contained in the radical J of Λ, that is,

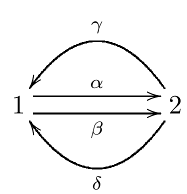
$$\mathcal{T} = \{\mathrm{St}(p^{-1}q) \in \mathcal{P}^{<\infty}(\Lambda\text{-mod}) \mid \mathrm{St}(p^{-1}q) \text{ embeds in } J\};$$

here the paths p and q may be trivial, and so, in particular, $\mathcal{T}$ includes all uniserial left modules of finite projective dimension contained in J. The proof of Theorem 3 is an immediate consequence of Proposition 2.

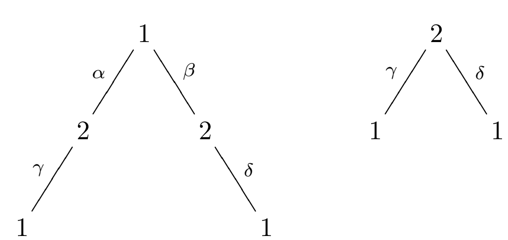
Theorem 3. $\mathrm{l\,fin\,dim}\,\Lambda = \mathrm{l\,Fin\,dim}\,\Lambda = t + 1$*, where* $t = \sup\{\mathrm{p\,dim}\,M \mid M \in \mathcal{T}\}$ *in case* $\mathcal{T}$ *is nonempty, and* $t = -1$ *otherwise.* $\square$

The following example shows that shrinking $\mathcal{T}$ to the set of all *uniserial* left modules from $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ contained in J does not leave the conclusion of Theorem 3 intact.

Example 4. Let $\Lambda = K\Gamma/I$, where Γ is the quiver



and $I \subseteq K\Gamma$ is the monomial ideal with the property that the indecomposable projective left Λ-modules have graphs



Then $\mathcal{T}$ is the singleton containing $\Lambda e_2 \cong \Lambda(\alpha-\beta)$, whence, by Theorem 3, the left little and big finitistic dimensions of Λ are equal to 1.

4. Background on contravariant finiteness and phantoms

We give a brief summary of that part of the theory of contravariant finiteness of a full subcategory $\mathcal{A} \subseteq \Lambda$-mod which will be relevant here. The two categories on which we will focus in the sequel are $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$, the full subcategory of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ consisting of the finite direct sums of string modules; note that the latter category depends, if only to a 'minor' degree, on the coordinatization (Γ, I) of the string algebra Λ.

Since the background material we need is quite general, we let Δ be any finite dimensional algebra for the moment, and $\mathcal{A} \subseteq \Delta$-mod a full subcategory which is closed under finite direct sums and direct summands. Recall from [**2**] that $\mathcal{A}$ is said to be *contravariantly finite* in Δ-mod in case each module $M \in \Delta$-mod has a (right) $\mathcal{A}$-*approximation*, that is a homomorphism $\varphi : A \to M$ with $A \in \mathcal{A}$ such that the induced sequence of functors

$$\mathrm{Hom}_\Delta(-, A)|_{\mathcal{A}} \longrightarrow \mathrm{Hom}_\Delta(-, M)|_{\mathcal{A}} \longrightarrow 0$$

is exact; in other words, the latter says that every map in $\mathrm{Hom}_\Lambda(B, M)$ with $B \in \mathcal{A}$ factors through φ. In the sequel, we will suppress the qualifier 'right'. Provided that M has an $\mathcal{A}$-approximation, there is a *minimal* such approximation which embeds, as a direct summand, into all other $\mathcal{A}$-approximations of M. In particular, such a minimal $\mathcal{A}$-approximation of M is unique up to isomorphism; it is therefore only a mild abuse of language to refer to it as *the minimal $\mathcal{A}$-approximation of M*. In case $\mathcal{A} = \mathcal{P}^{<\infty}(\Delta\text{-mod})$, the existence of approximations for the simple left Δ-modules S_i, $1 \le i \le n$, already guarantees contravariant finiteness of $\mathcal{P}^{<\infty}(\Delta\text{-mod})$. Moreover, in case of existence, the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations A_i of the S_i impinge on the structure of an arbitrary object in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ as follows (see [**1**]): A finitely generated Δ-module M has finite projective dimension precisely when it is a direct summand of a module X that has a finite filtration with successive factors in $\{A_1, \dots, A_n\}$. This structure theory was extended to $\mathcal{P}^{<\infty}(\Delta\text{-Mod})$, the category of *all* left Δ-modules of finite projective dimension by the authors in [**24**]: If $\mathcal{P}^{<\infty}(\Delta\text{-mod})$ is contravariantly finite, the objects of the 'big' category $\mathcal{P}^{<\infty}(\Delta\text{-Mod})$ are precisely those which are direct limits of objects X having filtrations $X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_m = 0$ with $X_i/X_{i+1} \cong A_{k(i)}$ for all $i < m$; in particular, the big and little left finitistic dimensions of Δ coincide in this situation and are attained on $\{A_1, \dots, A_n\}$. This illustrates the pivotal role played by the minimal approximations of the simple modules in case $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite.

$\mathcal{A}$-phantoms were introduced in [**18**], originally mainly with the aim of providing criteria for contravariant finiteness of $\mathcal{A}$. Here they serve a dual purpose: the one just mentioned and that of replacing minimal $\mathcal{A}$-approximations in case such approximations fail to exist. Our phantoms under (3) below are not to be confused with the phantom maps of algebraic topology, or the topology-inspired phantom maps in the modular representation theory of groups, as introduced by Benson and Gnacadja in [**4**].

Definition. Assume $\mathcal{A} \subseteq \Delta$-mod to be closed under direct summands and finite direct sums. Moreover, fix $M \in \Delta$-mod.

(1) Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$. A *relative $\mathcal{C}$-approximation of M in $\mathcal{A}$* is a homomorphism $f : A \to M$ with $A \in \mathcal{A}$ such that all maps in $\mathrm{Hom}_\Delta(\mathcal{C}, M)$ factor through f. In this situation, we will also refer to the module A as a relative $\mathcal{C}$-approximation of M in $\mathcal{A}$.

If $\mathcal{C} = \{C\}$, we write 'relative C-approximation' for short.

(2) A finitely generated module H is an *$\mathcal{A}$-phantom of M* in case there exists an object C in $\mathcal{A}$ with the property that H occurs as a subfactor of *every* relative C-approximation of M in $\mathcal{A}$.

More generally, a module $H \in \Delta$-Mod will be called an *$\mathcal{A}$-phantom* of M if each of its finitely generated submodules is a phantom in the sense just defined.

(3) Given an $\mathcal{A}$-phantom H of M and a subcategory $\mathcal{C}$ of $\mathcal{A}$, a homomorphism $\varphi : H \to M$ is called an *effective $\mathcal{C}$-phantom* of M, provided that H is a direct limit of objects in $\mathcal{C}$, and every map in $\mathrm{Hom}_\Delta(\mathcal{C}, M)$ factors through φ. In that case, we will also say that H is an effective $\mathcal{C}$-phantom of M.

We add a few comments to set up an intuitive backdrop for the concept of a phantom. The terminology is to evoke a 'phantom image' of an object, assembled from witness reports, to aid a search effort. In that spirit, a finitely generated module H is an $\mathcal{A}$-phantom of M precisely when there is a witness $C \in \mathcal{A}$ testifying to the effect that the source of any homomorphism in $\mathrm{Hom}_\Lambda(\mathcal{A}, M)$, which permits factorization of all maps in $\mathrm{Hom}_\Lambda(C, M)$, has an epimorphic image containing H.

In case $\mathcal{A}$-approximations of M exist, the minimal one, A say, is the only effective $\mathcal{A}$-phantom of M, and the class of all $\mathcal{A}$-phantoms of M coincides with the class of subfactors of A. So, in this situation, constructing $\mathcal{A}$-phantoms of M amounts to assembling 'phantom images' of the module A that one would like to track down; existence is recognized in the process if one can argue that the K-dimensions of such phantoms need to be bounded (cf. the existence theorem below). Otherwise, namely when M fails to have $\mathcal{A}$-approximations, $\mathcal{A}$-phantoms of M still provide minimal building blocks of objects through which all maps in $\mathrm{Hom}_\Lambda(\mathcal{A}, M)$ can be factored. Of course, effective phantoms of M hold the highest structural interest also in this case: Effective $\mathcal{C}$-phantoms, where $\mathcal{C} \subseteq \mathcal{A}$, take over the role of minimal approximations, in that they carry full complements of information on how $\mathcal{C}$ relates to M.

Note that, for any finite subcategory $\mathcal{C}$ of $\mathcal{A}$, relative $\mathcal{C}$-approximations of M in $\mathcal{A}$ exist; to obtain candidates, we only have to add up a sufficient number of copies of the objects in $\mathcal{C}$. Moreover, observe that, given a subclass $\mathcal{D} \subseteq \mathcal{C}$, every relative $\mathcal{C}$-approximation of M in $\mathcal{A}$ is also a relative $\mathcal{D}$-approximation. Hence calling for an object C in $\mathcal{A}$ such that H is a subfactor of *every* relative C-approximation of M in $\mathcal{A}$ – as we do in the definition of a finitely generated phantom – places strong 'minimality pressure' on H. On the other hand, the class of $\mathcal{A}$-phantoms of M is closed under subfactors and direct limits of directed systems, which often makes it enormous. This slack in the definition facilitates the search for phantoms and thus makes them an expedient tool in proving failure of contravariant finiteness: Indeed, the existence of $\mathcal{A}$-phantoms of unbounded lengths of a given module M signals non-existence of an $\mathcal{A}$-approximation of M. We conclude this sketch with the following existence result.

Theorem. (see [**18**]) *For M in Δ-mod and $\mathcal{A} \subseteq \Delta$-mod a subcategory as above, the following conditions are equivalent:*

(1) *M fails to have an $\mathcal{A}$-approximation.*

(2) *M has $\mathcal{A}$-phantoms of arbitrarily high finite K-dimensions.*

(3) *M has an $\mathcal{A}$-phantom of infinite K-dimension.*

(4) *There exists a countable subclass $\mathcal{C} \subseteq \mathcal{A}$ such that M has an effective $\mathcal{C}$-phantom of infinite K-dimension.* □

For a more extensive overview of contravariant finiteness results, we refer the reader to [**22**].

5. Statement of the main result, consequences, and examples

Given a generalized word $w = (p_i^{-1} q_i)_{i \in \mathbb{Z}}$, which is either trivial or has the property that $\mathrm{length}(p_0) + \mathrm{length}(q_0) > 0$, we will refer to the joint starting point of p_0 and q_0 as the *center* of w; denote this center by e. If we wish to emphasize the centered viewpoint, we will also refer to w as a *generalized word centered at* e. Each generalized word w centered at e comes paired with an obvious homomorphism $\varphi : \mathrm{St}(w) \to \Lambda e/Je$: Indeed, consider the standard presentation

$$\mathrm{St}(w) = \Bigg(\bigoplus_{i \in \mathrm{supp}(w)} \Lambda e(i) \Bigg) \Bigg/ \; C$$

as specified in Section 2; here $e(i)$ is again the starting vertex of p_i, q_i for $i \in \mathbb{Z}$. Let $x_i \in \mathrm{St}(w)$ be the residue class of $e(i)$. Then there exists a unique homomorphism $\varphi : \mathrm{St}(w) \to \Lambda e/Je$ with $\varphi(x_0) = e + Je$ and $\varphi(x_i) = 0$ for $|i| \geq 1$; we will refer to it as the *canonical map* of the centered word w.

Moreover, we call a generalized word w *left periodic*, resp. *right periodic*, in case $w = \ldots uuuw_1$, resp., $w = w_2 vvv \ldots$, with u, v either trivial or primitive, and w_1, w_2 left, resp. right, finite. (On the side, we point out that the set of those left and right periodic words which are twosided infinite coincides with the union of the 'periodic' and 'biperiodic $\mathbb{Z}$-words' introduced by Ringel in [**30**], while our left finite and right periodic words are 'periodic' or 'almost periodic $\mathbb{N}$-words' in Ringel's terminology.)

Finally, we recall that $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$ denotes the full subcategory of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ having as objects all finite direct sums of string modules of finite projective dimension.

We are now in a position to state our main theorem.

Theorem 5. *As before, let $\Lambda = K\Gamma/I$ be a string algebra with simple left modules $S_i = \Lambda e_i/Je_i$, $1 \le i \le n$. Then there exist centered generalized words $w_i = w(S_i)$, unique up to inversion, with the following properties:*

(I) *Each w_i is centered at e_i and is left and right periodic. (Moreover, the w_i can be effectively constructed from Γ and I in a number of steps which grows polynomially with $\dim_K \Lambda$.)*

(II) *Each of the generalized string modules $\mathrm{St}(w_i)$ has finite projective dimension, and the canonical map $\varphi_i : \mathrm{St}(w_i) \to S_i$ is an* effective *$\mathcal{S}^{<\infty}(\Lambda\text{-mod})$-phantom of S_i.*

(III) *The category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in Λ-mod if and only if the words $w_1, \dots, w_n$ are all finite. In the positive case, the canonical maps $\varphi_i : \mathrm{St}(w_i) \to S_i$ are the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the simple modules. More precisely, for each $i \in \{1, \dots, n\}$, the following conditions are equivalent:*

(i) *w_i is finite.*
(ii) *S_i has a $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation.*
(iii) *$\varphi_i : \mathrm{St}(w_i) \to S_i$ is the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of S_i.*
(iv) *S_i has an $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$-approximation.*
(v) *$\varphi_i : \mathrm{St}(w_i) \to S_i$ is the minimal $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$-approximation of S_i.*

Finally, if the equivalent conditions (i) – (v) *are satisfied, the top of* $\mathrm{St}(w_i)$ *has dimension at most* $4n$*.*

A proof will be given in Section 7. As will be further substantiated when we describe and explore the generalized words w_i of Theorem 5, they encode essentially all of the information required to understand the homology of Λ. Without providing an explicit algorithm, we mention that, in case of contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ in Λ-mod, the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of arbitrary string and band modules can readily be constructed from the $\mathrm{St}(w_i)$.

Theorem 5 remains unaffected if we replace the words w_i by their inverses; indeed, $\mathrm{St}(w) \cong \mathrm{St}(w^{-1})$ for any generalized word w, and if w is centered at e, the obvious isomorphism, namely the flip about the center, preserves the center and takes the canonical map $\mathrm{St}(w) \to \Lambda e/Je$ to the canonical map $\mathrm{St}(w^{-1}) \to \Lambda e/Je$. Consequently, we will invert whenever convenient. Note, moreover, that Theorem 5 guarantees uniqueness of any of the string modules $\mathrm{St}(w_i)$ up to isomorphism, whenever w_i is finite. On the other hand, the isomorphism type of $\mathrm{St}(w_i)$ may depend on the coordinatization in general, a fact which reflects the dependence of $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$ on the coordinatization; see Section 8, Example 23. Yet, this failure of uniqueness is 'minor', in that the graph of $\mathrm{St}(w_i)$, minus the labeling of the edges, is invariant up to a flip about the central axis; this is a consequence of Proposition 16 below. In categorical terms, if $w_1, \dots, w_n$ and $w'_1, \dots, w'_n$ are centered words having the properties described in Theorem 5 relative to two eligible coordinatizations of our string algebra Λ, there exists a Morita self-equivalence $F : \Lambda\text{-Mod} \to \Lambda\text{-Mod}$ such that $\mathrm{St}(w'_i) \cong F\big(\mathrm{St}(w_i)\big)$ for all i, and F carries the canonical epimorphisms $\varphi_i : \mathrm{St}(w_i) \to S_i$ to those of the centered words w'_i.

We conclude the section with two immediate consequences of the main theorem, followed by examples showing that neither can be extended to arbitrary special biserial algebras.

Corollary 6. *Suppose that Λ is a string algebra. If $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite, the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the simple left Λ-modules are string modules. In particular, they are indecomposable.* □

While, under the hypothesis of the corollary, the category of all finite direct sums of string modules is always closed with respect to minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations (we do not include a proof for this fact), indecomposability is not preserved in passing from a string module to its minimal approximation in general.

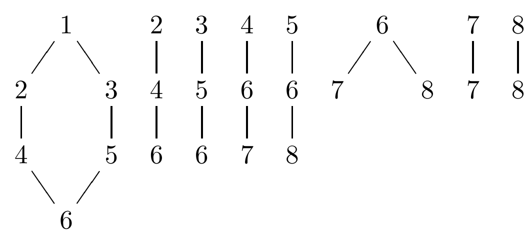
Actually, the property of being a string module may appear somewhat artificial, since it usually depends on the coordinatization of Λ. However, in light of Theorems 3 and 5, it becomes apparent that the concepts of 'string' and 'band' are more than devices permitting an explicit classification of the finitely generated indecomposable representations from quiver and relations of Λ. Indeed, we see that the class of all string modules determines (irrespective of the chosen coordinatization) the homological properties of the category Λ-mod.

Since the base field K does not enter into the structure of string modules and their syzygies, this, in turn, guarantees that the homology of string algebras is a purely combinatorial game which is governed by the graphs of the indecomposable projective modules alone. In particular, the various homological dimensions of a string algebra $\Lambda = K\Gamma/I$ are completely determined by the quiver Γ and any set of paths generating the ideal I. To see that this does not extend to arbitrary monomial relation algebras, compare, e.g., [**21**].

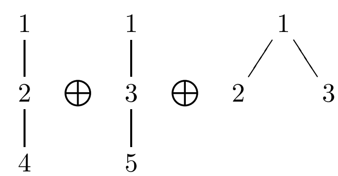
Corollary 7. *The class of string modules determines the homological dimensions of a string algebra $\Lambda = K\Gamma/I$, as well as the contravariant finiteness status of the subcategory $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$. In particular, these data depend only on the quiver Γ and the paths in I, not on the base field.* □

Both corollaries fail for special biserial algebras in general.

Example 8. We present a finite dimensional special biserial algebra Δ and a simple Δ-module S_1 whose $\mathcal{P}^{<\infty}(\Delta\text{-mod})$-approximation splits into three nontrivial summands. Suppose that Δ is a path algebra modulo relations over K with $\dim_K \Delta/J(\Delta) = 8$ such that the indecomposable projective left modules have the following graphs:

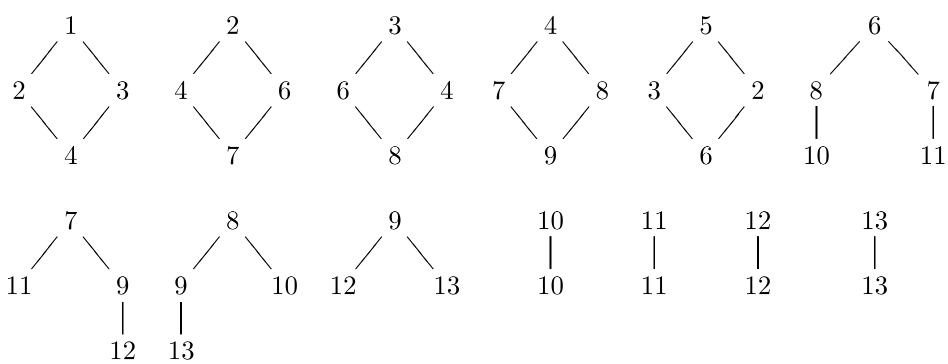


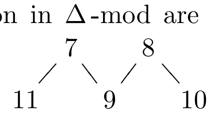
Clearly, Δ is a special biserial algebra. Moreover, it is not difficult to check that the minimal $\mathcal{P}^{<\infty}(\Delta\text{-mod})$-approximation of S_1 is the module with graph



□

Example 9. For the following finite dimensional special biserial algebra Δ, the category $\mathcal{S}^{<\infty}(\Delta\text{-mod})$ has finite representation type and is thus contravariantly finite – in particular, all simple modules have $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$-approximations – whereas $\mathcal{P}^{<\infty}(\Delta\text{-mod})$ fails to be contravariantly finite in Δ-mod. Let Δ be a path algebra with relations such that the graphs of the indecomposable projective left Δ-modules are



It is straightforward (if somewhat tedious) to check that the only finitely generated string modules of finite projective dimension in Δ-mod are the indecomposable projective modules Δe_6 through Δe_{13}, and the module with graph  . This shows that $\mathcal{S}^{<\infty}(\Delta\text{-mod})$ has finite type. Note, in particular, that there are no string modules of finite projective dimension which have a copy of S_1 in their tops, whence the zero map $0 \to S_1$ is the minimal $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$-approximation of S_1. On the other hand, S_1 fails to have a $\mathcal{P}^{<\infty}(\Delta\text{-mod})$-approximation; indeed, since each band module $\mathrm{Bd}(v^r, \phi)$, where v is the primitive word with graph $\begin{smallmatrix} & 1 & & 5 & \\ \swarrow & & \searrow \; \swarrow & & \searrow \\ 2 & & 3 & & 2 \end{smallmatrix}$, belongs to $\mathcal{P}^{<\infty}(\Delta\text{-mod})$, this amounts to a routine check with the aid of [**26**] for instance. □

6. The characteristic words of the simple modules

A *segment* of a generalized word $w = (p_i^{-1} q_i)_{i \in \mathbb{Z}}$ will be any subword of w, where we consider the syllables p_i^{-1} and q_i as indivisible building blocks of w. So, alternately expressed, the segments of w are just the connected components of syllables of w. The segments $q_0 p_1^{-1} q_1 \dots$ and $\dots p_{-1}^{-1} q_{-1} p_0^{-1}$ will be called the *principal segments* of w; often, we will refer to the former as the *principal right segment* of w, and to the latter as the *principal left segment* for ease of optical reference, even though the distinction of sides is not substantive (cf. the remarks following Theorem 5).

Moreover, it will be convenient to say that a generalized word w has finite projective dimension if the corresponding string module $\mathrm{St}(w)$ has this property.

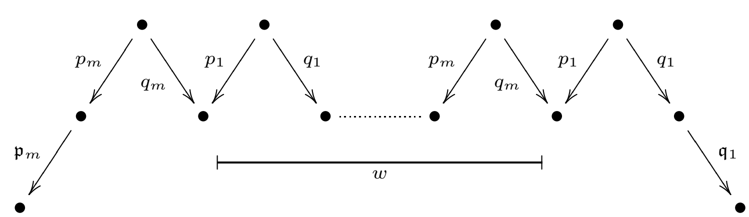
Given a simple module $S \in \Lambda$-mod, we will construct the (generalized) word $w = w(S)$ postulated in Theorem 5 as the limit of a sequence of successively growing segments of words of finite projective dimension. In doing so, we will observe that this process turns periodic after the construction of (at most) $4n + 1$ pairs of syllables, where n is the K-dimension of Λ/J. Clearly, segments of words of finite projective dimension may fail to inherit this property. However, we do have

Observations 10. *Let w be a nontrivial finite word.*

(1) *If w is a segment of a generalized word of finite projective dimension, then w is a segment of a* finite *word of finite projective dimension.*

(2) *If w is primitive and the band module* $\mathrm{Bd}(w^r, \phi)$ *has finite projective dimension for some r and ϕ, then, again, w is a segment of a finite word of finite projective dimension.*

Proof. We prove (2) and leave the similar argument backing part (1) to the reader. So let $w = p_1^{-1} q_1 \dots p_m^{-1} q_m$ be primitive with nontrivial flanking syllables p_1 and q_m. As in Proposition 1, we let $\mathfrak{p}_m$ (resp. $\mathfrak{q}_1$) be the longest paths in Γ such that $\mathfrak{p}_m p_m$ is a path in $K\Gamma \setminus I$ (resp., such that $\mathfrak{q}_1 q_1$ is a path in $K\Gamma \setminus I$). Invoking Proposition 1, we conclude that the word with graph



has finite projective dimension. □

In view of Observations 10, we will not run any risk of ambiguity if we henceforth simply refer to 'segments of words of finite projective dimension'.

In order to recognize the algorithmic nature of our construction, the reader should be familiar with the computation of projective dimensions of modules with tree graphs over monomial relation algebras (see [**19**]). We precede the description of the procedure with an easy auxiliary statement spelling out the mechanism of the individual steps. In view of Proposition 1, the proof is straight-forward and will be omitted.

The notationally somewhat involved second parts of statements (A) and (B) below, under the heading 'more detail', are only relevant for algorithmic purposes and do not impinge on the further development of the theory; the reader only interested in the latter is advised to skip them. Recall that, for any path p of positive length in $K\Gamma \setminus I$, we denote by $\mathfrak{p}$ the unique longest path with the property that the concatenation $\mathfrak{p}p$ is still a path in $K\Gamma \setminus I$.

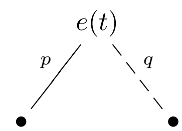
Observations 11.

(A) *Suppose that $u = q_{-t} p_{-(t-1)}^{-1} \cdots q_{t-1} p_t^{-1}$ with $t \geq 1$ is a segment of a centered word of finite projective dimension; so in particular, u is nontrivial if and only if the segment $p_0^{-1} q_0$ is nontrivial.*

Then there exist unique <u>shortest</u> paths q_t and p_{-t} such that $(p_{-t})^{-1} u q_t$ is again a segment of a word of finite projective dimension. In fact, the path q_t depends only on the first arrow of p_t in case the latter path *has positive length, and is trivial otherwise; symmetrically,* p_{-t} *depends only on the first arrow of* q_{-t} *in case that path has positive length, and is trivial otherwise.*

(Note that, in general, the choice $q_t = e(t)$ *will be ruled out, since all syllables to the right of a trivial syllable* q_i *with* $i \geq 0$ *are required to be trivial by the definition of a centered word. Analogous considerations apply to the left-hand side.)*

More detail: If $p_t = p_t e(t)$ *has positive length and* $\Lambda e(t)$ *has graph*



where p *contains* p_t *as a right subpath and* q *is a path of length* ≥ 0*, then* q_t *can be described as follows: It is the shortest right subpath of* q *such that, in writing* $q = \mathfrak{q}_t q_t$*, we either have*

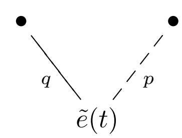
- $\mathrm{St}(p^{-1}q_t) \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$*, or else*
- $\mathrm{length}(q_t) \geq 1$*, and there exists a path* r *of positive length with the property that* $q_t r^{-1}$ *is a word and* $\mathrm{St}(\mathfrak{r}^{-1}\mathfrak{q}_t)$ *has finite projective dimension; here* $\mathfrak{r}$ *and* $\mathfrak{q}$ *relate to* r *and* q*, respectively, as indicated ahead of the lemma.*

The path p_{-t} *has an analogous description.*

(B) *Now suppose that* $u = p_{-(t-1)}^{-1} q_{-(t-1)} \cdots p_{t-1}^{-1} q_{t-1}$ *with* $t \geq 1$ *is a segment of a centered word of finite projective dimension.*

Then there exist unique <u>*longest*</u> *paths* p_t *and* q_{-t} *such that* $q_{-t} u (p_t)^{-1}$ *is a segment of a word of finite projective dimension. In fact, the path* p_t *depends only on the last arrow of* q_{t-1} *in case the latter path has positive length, and is trivial otherwise; symmetrically,* q_{-t} *depends only on the last arrow of* $p_{-(t-1)}$ *in case the latter path has positive length, and is trivial otherwise.*

More detail: If $q_{t-1} = \tilde{e}(t) q_{t-1}$ *has positive length and the injective hull of* $\Lambda\tilde{e}(t)/J\tilde{e}(t)$ *has graph*



where q *contains* q_{t-1} *as a left subpath, then* p_t *can be described as follows: it is the unique longest nontrivial subpath* r *of* p *such that*

- $\mathrm{St}(\mathfrak{r}^{-1}\mathfrak{q}_{t-1}) \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$*, if such a path* r *exists,*

and trivial otherwise.

Again, the path p_{-t} *can be described analogously.* ☐

In contrast to our usual practice, we will consistently record trivial syllables in the following construction.

12. Construction of the "characteristic word" of a simple module.

Start with a simple left module $S = \Lambda e/Je$ and construct a generalized word $w = w(S)$ centered at e as follows:

Step 0: Choose paths p_0 and q_0 starting in e and having minimal length with the property that $w_0 = p_0^{-1} q_0$ is a segment of a centered word of finite projective dimension. Note that, up to a swap of roles, p_0 and q_0 are uniquely determined by this requirement.

Step t, t $\geq$ 1: Suppose that the centered word $w_{t-1} = p_{-(t-1)}^{-1} q_{-(t-1)} \cdots p_{t-1}^{-1} q_{t-1}$ has already been constructed. According to part B of Observations 11, we first find the unique *longest* paths q_{-t} and p_t with the property that $q_{-t} w_{t-1} p_t^{-1}$ is a segment of a word of finite projective dimension and, according to part A of Observations 11, we then choose p_{-t} and q_t as the unique *shortest* paths such that $w_t = p_{-t}^{-1} q_{-t} w_{t-1} p_t^{-1} q_t$ is a segment of a word of finite projective dimension.

In Step $2n$, at the latest, we hit periodicity on both sides. We explain this for the principal right segment, the left-hand side behaving symmetrically. If q_{2n} has length zero, our claim is trivial, since in that case all further syllables on the right are paths of length zero. So suppose that $\mathrm{length}(q_{2n}) \geq 1$. Then the principal right segment of w_{2n}, that is, the word $q_0 p_1^{-1} q_1 \ldots p_{2n}^{-1} q_{2n}$ consists of $4n+1$ nontrivial syllables and thus gives rise to a string module with socle dimension $2n+1$, meaning that some simple, say $\Lambda\tilde{e}/J\tilde{e}$, occurs with multiplicity at least 3 in this socle. Hence at least two of the terminal arrows of the corresponding nontrivial paths q_i ending in $\tilde{e}$ coincide, say q_k and q_l with $k < l \le 2n$. Thus Observations 11 guarantee that $p_{k+1} = p_{l+1}$, $q_{k+1} = q_{l+1}$, $p_{k+2} = p_{l+2}$, and so forth.

This completes the description of the construction.

Even though the left and right periodic centered word $w = w(S)$ produced by this construction is only determined up to inversion, we will use the definite article in referring to it.

Definition 13. Given a simple left Λ-module $S = \Lambda e/Je$, the centered word $w = w(S)$ of Construction 12 will be called the *characteristic word* of S, and the module $\mathrm{St}(w)$ the *characteristic phantom* of S.

This terminology is justified by

Proposition 14. *The characteristic words of the simple left Λ-modules are (up to inversion) the centered words $w_1, \dots, w_n$ postulated in Theorem 5.*

Note that proving Proposition 14 will, at the same time, establish our main result, Theorem 5. This will be done in the next section. As is clear from their construction, the characteristic words of the simple modules are strongly interconnected. We record this fact in

Remark 15. *Let $w = (p_i^{-1} q_i)_{i \in \mathbb{Z}}$ be the characteristic word of a simple left Λ-module S and $e(i)$ the coinciding starting point of the paths p_i and q_i. If $i \ge 1$ and p_{i-1} is nontrivial, then the segment $q_i p_{i+1}^{-1} \dots$ of w is, up to re-indexing, one of the principal segments of the characteristic word of $\Lambda e(i)/Je(i)$. Analogously, if $i \le -1$ and q_{i-1} is nontrivial, then $\dots q_{i-2}^{-1} p_{i-1} q_{i-1}^{-1} p_i$ is, up to re-indexing, a principal segment of the characteristic word of $\Lambda e(i)/Je(i)$.* □

The paths arising as syllables of the characteristic words play a pivotal role in the structural makeup of arbitrary modules of finite projective dimension. This is reflected by the following facts.

Proposition 16. *Let $w = (p_i^{-1} q_i)_{i \in \mathbb{Z}}$ be the characteristic word of $S = \Lambda e/Je$, and again denote the joint starting point of p_i and q_i by $e(i)$. Moreover, let M be any object of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$.*

(A) *Suppose that x is a top element of M. If x is of type $e = e(0)$, then $p_0 x \neq 0$ and $q_0 x \neq 0$. If x is of type $e(i)$ for some $i \ge 1$, then $q_i x \neq 0$, and if x is of type $e(i)$ for some $i \le -1$, then $p_i x \neq 0$.*

(B) *Now suppose that the path p_i is nontrivial and $x = e(i)x$ for some $i \ge 1$ is an element of M with the property that $p_i x$ is a nonzero element of $\operatorname{soc} M \cap q_{i-1} M$. Then x is a top element of M. The same conclusion holds if $i \le -1$, the path q_i is nontrivial, and $q_i x$ is a nonzero element of $\operatorname{soc} M \cap p_{i+1} M$.*

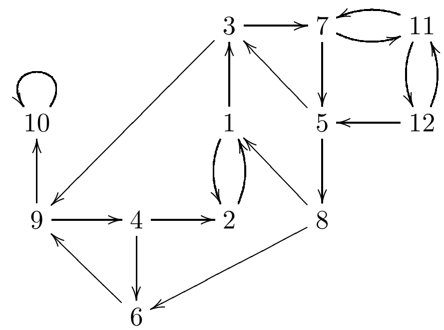
Proof. We may clearly restrict our attention to the case where M is a string or a band module. In the former case, our assertions are immediate consequences of our choices of the p_i and q_i. So suppose that M is a band module based on a primitive word v. By Observation 10(2), finite projective dimension of M forces v to be a segment of a word of finite projective dimension, whence our claims again follow from Observations 11 and Construction 12. □

We conclude this section with examples demonstrating that all theoretically possible scenarios actually occur: left and right termination of a characteristic word w, meaning that, for $i \gg 0$, the syllables p_i, q_i and p_{-i}, q_{-i} are primitive idempotents (recall that, in view of Theorem 5 and Proposition 14, this occurs precisely when the corresponding simple module $S = \Lambda e/Je$ has a $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation); onesided termination of w; non-trivial left and right periodicity,

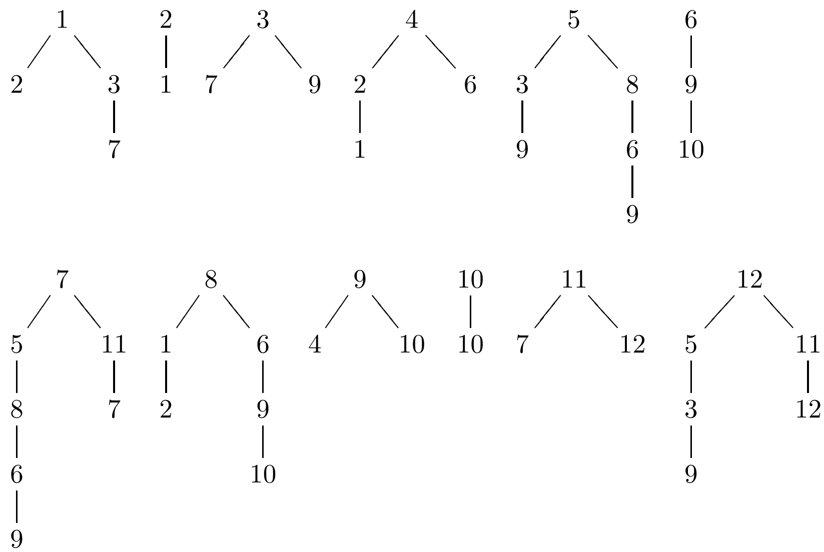
$$w = \dots uuu * * * \cdots * * * vvv \dots,$$

with primitive words u and v which are devoid of common syllables; and periodicity in the strongest possible sense, i.e., $w = \dots vvv \dots$, where v is a primitive word.

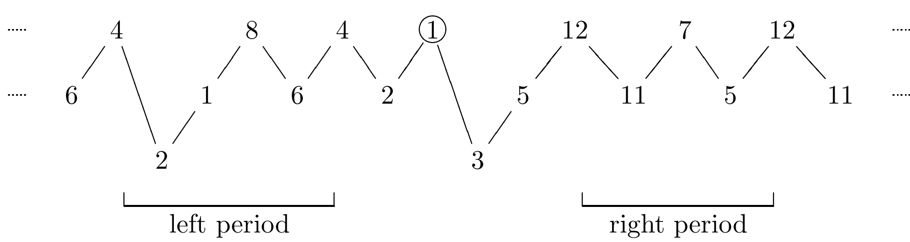
Example 17. Let Γ be the quiver

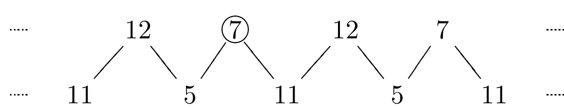


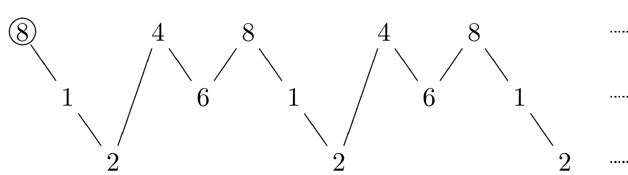
and $\Lambda = K\Gamma/I$, where the ideal $I \subseteq K\Gamma$ is chosen so that the indecomposable projective left Λ-modules have graphs



Then Λ is a string algebra with simple left modules $S_i = \Lambda e_i/Je_i$, $1 \leq i \leq 12$. One readily finds that the characteristic phantoms (in the sense of the above definition – see also Theorem 5) of S_1, S_7, and S_8 are as follows:





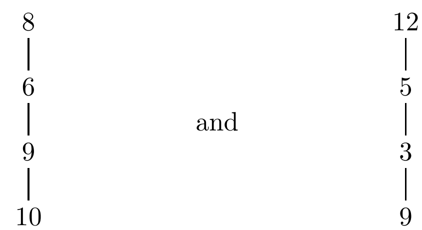


In each case, the center is circled.

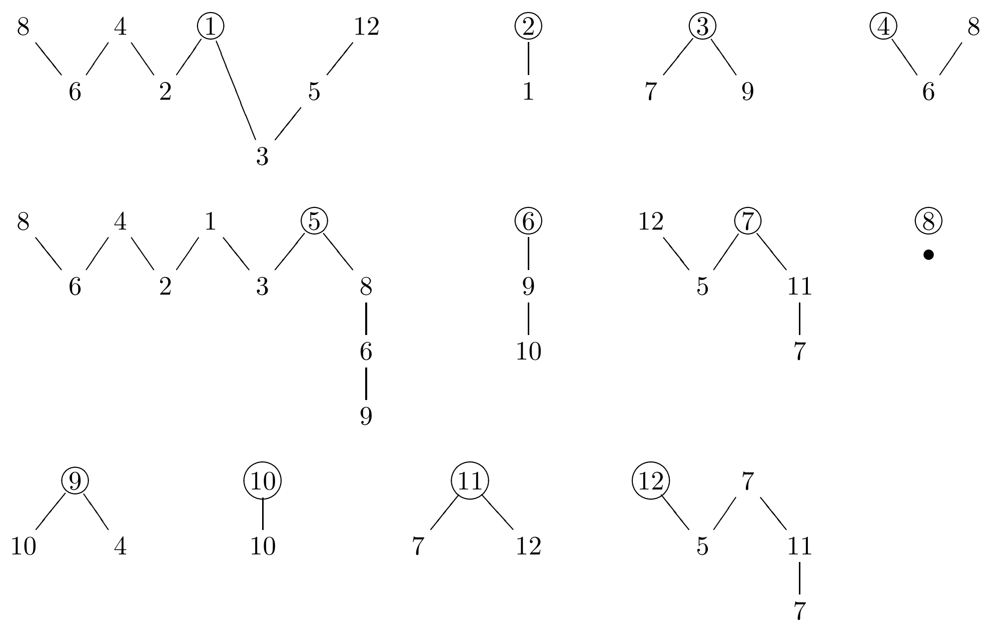
In particular, we see that the characteristic word w_1 of S_1 is two-sided infinite with distinct left and right periods, the characteristic word w_7 of S_7 is of the form $w_7 = \dots vvv \dots$, where v is a primitive word, while the characteristic word w_8 of S_8 is infinite only on one side. Note moreover that the characteristic word of S_7 coincides with that of S_{12}, while that of S_4 results from that of S_8 through deletion of the first two syllables.

In view of Theorem 5, $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ fails to be contravariantly finite in Λ-mod. The theorem, in fact, supplies more precise information: Namely, the simple modules S_2, S_3, S_6, S_9, S_{10}, and S_{11} are precisely those having $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations and, for each of the listed indices i, the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of S_i coincides with Λe_i. $\square$

Example 18. Let Γ' be the quiver with 12 vertices resulting from that of Example 17 through deletion of two arrows, those from 8 to 1 and from 12 to 11. We define the string algebra $\Lambda' = K\Gamma'/I'$, where $I' \subseteq K\Gamma'$ is chosen in such a way that the graphs of the indecomposable projective left Λ'-modules $\Lambda' e_i$, for $i \in \{1, \dots, 12\} \setminus \{8, 12\}$ coincide with those of the corresponding Λe_i-modules of Example 17, while $\Lambda' e_8$ and $\Lambda' e_{12}$ have graphs



respectively. Then the characteristic phantoms of the simple left Λ'-modules are as follows, the centers being again highlighted:

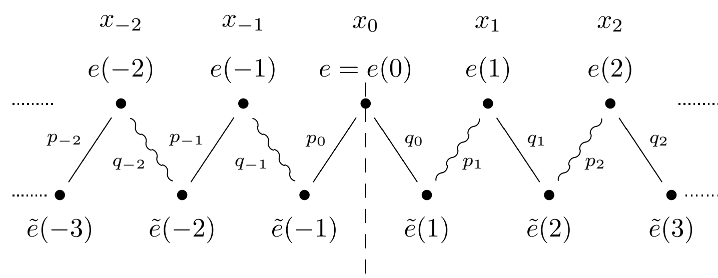


Thus Theorem 5 tells us that $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in Λ-mod, and that the displayed characteristic phantoms are the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the simple modules corresponding to the centers. $\square$

7. Proof of the main result

Our plan is to establish Theorem 5 by way of proving Proposition 14.

Throughout this section, we fix a simple left module $S = \Lambda e/Je$ with $e \in \{e_1, \dots, e_n\}$ and let $w = w(S) = (p_i^{-1} q_i)_{i \in \mathbb{Z}}$ be the characteristic word of S as described in Construction 12 of Section 6. To cope with the ambiguity arising from the fact that characteristic words are only unique up to inversion, we will, in the sequel, refer to the orientation of this fixed choice of w as *normalized*. Since all parts of the theorem involving the word w are true if the latter is trivial (recall that this occurs precisely when S has finite projective dimension), we will henceforth assume that w is nontrivial, i.e., that at least one of the paths p_0, q_0 is nontrivial. Relative to the standardized sequence of top elements $(x_i)_{i\in \mathrm{supp}(w)}$ of $\mathrm{St}(w)$, as introduced in Section 2, the string module $\mathrm{St}(w)$ has graph



Here we denote by $e(i)$ the starting point of the path q_i for $i \geq 0$, and by $\tilde{e}(i+1)$ its end point; then $e(i)$ and $\tilde{e}(i)$ are also the starting and end points of the p_i for $i \geq 1$, respectively; moreover, $e(0) = e$. The principal left segment of w is labeled similarly, as shown in the graph of $\mathrm{St}(w)$. Contrasting our usual convention, we have marked some of the edges by wiggly, as opposed to straight, lines to emphasize the following difference in roles: the straight edges indicate paths chosen as short as possible without forfeiting finite projective dimension of $\mathrm{St}(w)$, whereas the paths represented by wiggly edges are chosen as long as possible under this restriction. As in the previous sections, $\varphi : \mathrm{St}(w) \to S$ denotes the canonical map which sends x_0 to $e + Je$, while sending the other x_i to zero.

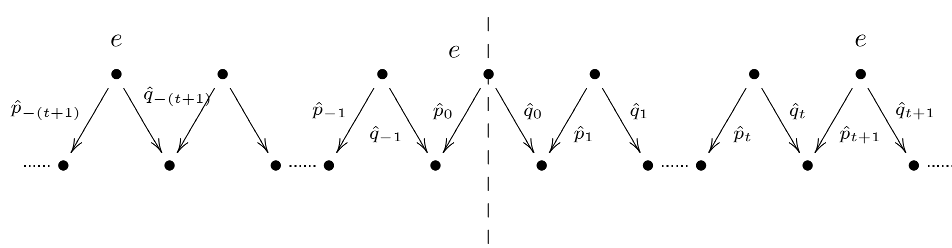
We smooth the road towards a proof of Theorem 5 with a final definition and a few auxiliary facts.

Definition 19. Two nontrivial centered words $\hat{w} = (\hat{p}_i^{-1}\hat{q}_i)_{i\in\mathbb{Z}}$ and $\tilde{w} = (\tilde{p}_i^{-1}\tilde{q}_i)_{i\in\mathbb{Z}}$ are said to *have the same orientation* if they are centered in the same primitive idempotent and either $\hat{p}_0$ and $\tilde{p}_0$ have the same first arrow, or else $\hat{q}_0$ and $\tilde{q}_0$ have the same first arrow. (Note that, if all of the paths $\hat{p}_0$, $\tilde{p}_0$, $\hat{q}_0$, $\tilde{q}_0$ are nontrivial, the condition that $\hat{w}$ and $\tilde{w}$ have the same orientation is equivalent to the requirement that $\hat{p}_0$ and $\tilde{p}_0$, as well as $\hat{q}_0$ and $\tilde{q}_0$, share first arrows.)

Let S and $w = (p_i^{-1}q_i)$ be as fixed at the beginning of this section. If $\hat{w} = (\hat{p}_i^{-1}\hat{q}_i)$ has the same orientation as w, we call $\hat{q}_i$ (resp. $\hat{p}_i$) a *right discontinuity* of $\hat{w}$ relative to w in case $i \geq 0$ and $\hat{q}_i \neq q_i$ (resp., $i \geq 1$ and $\hat{p}_i \neq p_i$); *left discontinuities* are defined symmetrically. Both types are also briefly called discontinuities of $\hat{w}$.

In the sequel, all discontinuities will be relative to the fixed characteristic word w of $S = \Lambda e/Je$.

Lemma 20. *Let $\hat{v} = \hat{p}_0^{-1}\hat{q}_0 \dots \hat{p}_t^{-1}\hat{q}_t$ be a primitive word with $t \geq 0$ and all of the listed syllables nontrivial; moreover, suppose that the joint starting point of $\hat{p}$ and $\hat{q}$ equals e. Expand $\hat{v}$ to a twosided infinite word $\hat{w} = \dots \hat{v}\hat{v}\hat{v} \dots$ centered in e as illustrated by the following graph*



where $\hat{p}_i = \hat{p}_j$ and $\hat{q}_i = \hat{q}_j$ whenever i is congruent to j modulo $t+1$. Finally, suppose that $\hat{w}$ has the same orientation as w.

If $\hat{w}$ has a right discontinuity, then the first right discontinuity of $\hat{w}$ is among the paths $\hat{q}_0, \dots, \hat{q}_t$, $\hat{p}_1, \dots, \hat{p}_{t+1}$. Similarly, if $\hat{w}$ has a left discontinuity, then the first such is among $\hat{p}_0, \dots, \hat{p}_{-t}$, $\hat{q}_{-1}, \dots, \hat{q}_{-(t+1)}$.

Now suppose that $\hat{w}$ is a word of finite projective dimension. If $\hat{w}$ has a right discontinuity, and the first right discontinuity is $\hat{q}_i$ for some i, then the path $\hat{q}_i$ contains the path q_i as a proper right subpath; if, on the *other hand, the first right discontinuity of* $\hat{w}$ *is* $\hat{p}_i$*, the latter path is contained in the path* p_i *as a proper left subpath. In case of existence, the first left discontinuity of what is subject to mirror-symmetric conditions.*

Proof. To verify the first assertion, suppose that $\hat{q}_i = q_i$ for $0 \le i \le t$ and $\hat{p}_i = p_i$ for $1 \le i \le t+1$. Then the paths q_0 and p_{t+1} starting in e are both nontrivial, and the first arrow of p_{t+1} differs from the first arrow of $q_0 = \hat{q}_0 = \hat{q}_{t+1}$. By Construction 12, this implies that $q_{t+1} = q_0$, i.e., $\hat{q}_{t+1} = q_{t+1}$, and further that $p_{t+2} = p_1 = \hat{p}_1 = \hat{p}_{t+2}$, $q_{t+2} = q_1 = \hat{q}_1 = \hat{q}_{t+2}$, and so forth, meaning that the principal right segments of w and $\hat{w}$ coincide. The counterpart dealing with the principal left segment of $\hat{w}$ is symmetric.

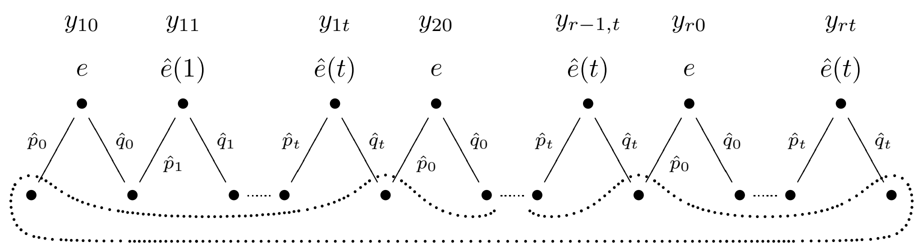
The final assertions, concerning first right and left discontinuities in case $\hat{w}$ is a word of finite projective dimension, are immediate consequence of the construction of w (Section 6, Construction 12). □

In the proof of the next lemma, it will turn out handy that every homomorphism from a pseudo-band module $\mathrm{Bd}(v^r, \phi)$ to S factors through an 'expanded' pseudo-band module $\mathrm{Bd}(v^{2r}, \psi)$.

Remark 21. *Since K is an infinite field, any pseudo-band module* $\mathrm{Bd}(v^r, \phi)$ *is contained as a direct summand in a pseudo-band module* $\mathrm{Bd}(v^s, \psi)$*, for any integer* $s \ge r$*.*

Lemma 22. *Let* $\hat{v}$ *and* $\hat{w}$ *be as in the blanket hypothesis of Lemma 20 and retain all of the notation introduced there. Moreover, suppose that* $\hat{w}$ *is a word of finite projective dimension having both right and left discontinuities.*

Then, given any pseudo-band module $B = \mathrm{Bd}(\hat{v}^r, \phi)$*, say with graph*



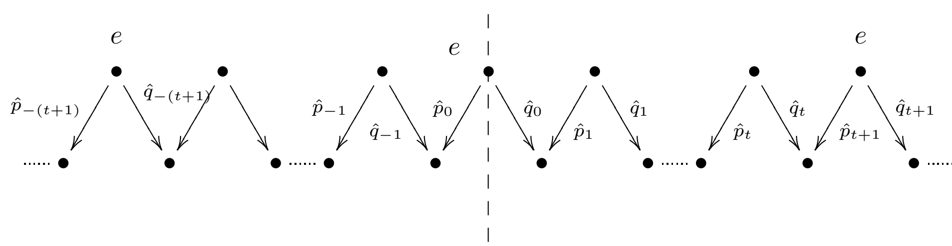
relative to a standardized sequence $y_{10}, \dots, y_{rt}$ *of top elements, the homomorphism* $f : B \to S$ *sending* y_{10} *to* $e + Je$ *and the other* y_{ij} *to zero factors through the canonical map* $\varphi : \mathrm{St}(w) \to S$*.*

Proof. We start by formalizing the graphical information provided: $\hat{q}_j y_{ij} = \hat{p}_{j+1} y_{i,j+1}$ for $j < t$, $\hat{q}_t y_{it} = \hat{p}_{i+1,0} y_{i+1,0}$ when $i < r$, and $\hat{q}_t y_{rt} = \sum_{i=1}^r c_i \hat{p}_0 y_{i0}$, where

$$\begin{pmatrix} 0 & \cdots & 0 & c_1 \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & \vdots \\ 0 & \cdots & 1 & c_r \end{pmatrix}$$

is the Frobenius companion matrix of the cyclic automorphism ϕ.

To facilitate visualization, we once more give the graph of the word $\hat{w} = \dots \hat{v}\hat{v}\hat{v} \dots$,



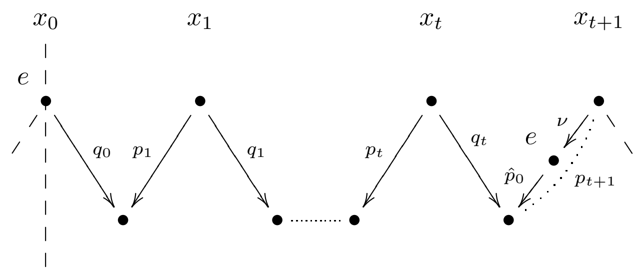
where again $\hat{p}_i = \hat{p}_j$ and $\hat{q}_i = \hat{q}_j$ whenever $i \equiv j \pmod{t+1}$. The word $\hat{w}$ having the same orientation as w, Lemma 20 guarantees that the first right discontinuity of $\hat{w}$ is among the paths q_i, p_j with $0 \le i \le t$ and $1 \le j \le t+1$, and the first left discontinuity of $\hat{w}$ is among the paths p_i, q_j with $0 \ge i \ge -t$ and $1 \ge j \ge -(t+1)$. In factoring the homomorphism f through φ, we will separately deal with the cases where neither of the paths $\hat{p}_0$, $\hat{q}_0$ is a discontinuity of $\hat{w}$, where one of them is, and where both of them are discontinuities.

The last-mentioned case is immediate: Namely, we define $g \in \mathrm{Hom}_\Lambda(B, \mathrm{St}(w))$ by setting $g(y_{10}) = x_0$ and $g(y_{ij}) = 0$ for $(i,j) \neq (1,0)$. This is legitimate, since, by Lemma 20, the paths $\hat{p}_0$ and $\hat{q}_0$ contain p_0 and q_0 as proper right subpaths, respectively; hence $\hat{p}_0 x_0 = \hat{q}_0 x_0 = 0$, where $(x_i)_{i\in\mathbb{Z}}$ is the standardized sequence of top elements of $\mathrm{St}(w)$ displayed at the beginning of the section.

Case A. One of $\hat{p}_0$, $\hat{q}_0$ is a discontinuity of $\hat{w}$, but not the other; for symmetry reasons, it is harmless to assume that $\hat{p}_0$ is a (left) discontinuity. Lemma 20 tells us that $\hat{p}_0$ contains p_0 as a proper right subpath, and again we infer $\hat{p}_0 x_0 = 0$.

Subcase A.1. $\hat{p}_{t+1}$ is the first right discontinuity of $\hat{w}$.

In this case the segment of the word w relevant to our construction has the form

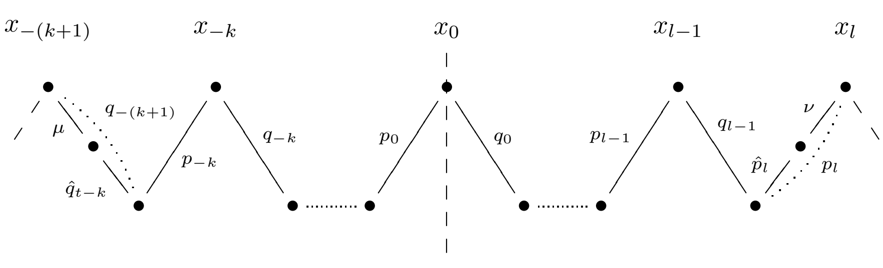


where $p_{t+1} = \hat{p}_{t+1}\nu = \hat{p}_0\nu$ for a nontrivial path ν, $q_i = \hat{q}_i$ for $0 \leq i \leq t$ and $p_i = \hat{p}_i$ for $1 \leq i \leq t$. By Remark 21, we may assume that $r \geq 2$. Define $g \in \mathrm{Hom}_\Lambda(B, \mathrm{St}(w))$ as follows, keeping in mind that $c_1 \neq 0$ because ϕ is an automorphism of K^r: Namely, let $g(y_{10}) = x_0 - (c_2/c_1)\nu x_{t+1}$, $g(y_{1j}) = x_j$ for $1 \leq j \leq t$, $g(y_{20}) = \nu x_{t+1}$, and $g(y_{ij}) = 0$ for $2 \leq i \leq r$ and $1 \leq j \leq t$. Indeed, in view of the equality $\hat{q}_0 \nu x_{t+1} = 0$, it is routine to check that $\hat{q}_j g(y_{ij}) = \hat{p}_{j+1} g(y_{i,j+1})$ for $i \geq 1$ and $j \leq t-1$, $\hat{q}_t g(y_{it}) = \hat{p}_0 g(y_{i+1,0})$ for $i \leq r-1$, and $\hat{q}_t g(y_{rt}) = \sum_{i=1}^r c_i \hat{p}_0 g(y_{i0})$ under these assignments. Moreover, we clearly have $\varphi g = f$.

Subcase A.2. The first right discontinuity of $\hat{w}$ is $\hat{p}_l$ or $\hat{q}_l$ with $1 \leq l \leq t$.

In that case, we have $p_l = \hat{p}_l \nu$, where ν is a path of length ≥ 0, and $\hat{q}_l = \sigma q_l$ with $\mathrm{length}(\sigma) \geq 0$, where either $\mathrm{length}(\nu) > 0$ or $\mathrm{length}(\sigma) > 0$, by Lemma 20. In either case we obtain $\hat{q}_l \nu x_l = 0$, and the following assignments give rise to a well-defined homomorphism $g \in \mathrm{Hom}_\Lambda(B, \mathrm{St}(w))$ with $\varphi g = f$. Namely, $g(y_{1j}) = x_j$ for $0 \leq j < l$, $g(y_{1l}) = \nu x_l$, $g(y_{1j}) = 0$ for $l+1 \leq j \leq t$ (this range being empty if $l = t$), and $g(y_{ij}) = 0$ for $i \geq 2$ and all j.

Case B. Neither $\hat{p}_0$ nor $\hat{q}_0$ is a discontinuity of $\hat{w}$. Let $k \in \{0, \dots t\}$ be such that the first left discontinuity of $\hat{w}$ is either $\hat{p}_{-(k+1)}$ or $\hat{q}_{-(k+1)}$, and $l \in \{1, \dots, t+1\}$ such that the first right discontinuity of $\hat{w}$ is either $\hat{p}_l$ or $\hat{q}_l$. Lemma 20 guarantees that either $\hat{p}_{-(k+1)} = \hat{p}_{t-k}$ contains $p_{-(k+1)}$ as a proper right subpath, or else $\hat{q}_{-(k+1)} = \hat{q}_{t-k}$ is a proper left subpath of $q_{-(k+1)}$. In either case we can write the latter path in the form $q_{-(k+1)} = \hat{q}_{t-k}\mu$, where μ is a path of length ≥ 0 satisfying the equalities $\hat{p}_{-(k+1)}\mu x_{-(k+1)} = \hat{p}_{t-k}\mu x_{-(k+1)} = 0$. Analogously, $p_l = \hat{p}_l \nu$, where ν is a path of length ≥ 0 such that $\hat{q}_l \nu x_l = 0$. The segment of the characteristic word w which is decisive for our construction is depicted in the following diagram:



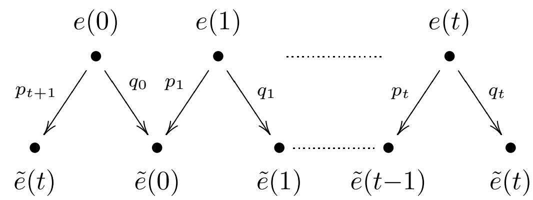
Here $p_i = \hat{p}_i$ for $-k \leq i \leq l-1$ and $q_i = \hat{q}_i$ for $-k \leq i \leq l-1$. Moreover, keep in mind that $\hat{p}_i = \hat{p}_j$ and $\hat{q}_i = \hat{q}_j$ whenever $i \equiv j \pmod{t+1}$. Remark 21 permits us to assume $r \geq 3$. Referring to the above graph of B and distinguishing between the cases where $l \leq t$ and $l = t+1$, we can thus define a map $g : \mathrm{Bd}(v^r, \phi) \to \mathrm{St}(w)$ as follows:

If $l \leq t$, we set $g(y_{10}) = x_0$, $g(y_{1j}) = x_j$ for $1 \leq j \leq l-1$, $g(y_{1l}) = \nu x_l$, $g(y_{1j}) = 0$ for $l+1 \leq j \leq t$, $g(y_{ij}) = 0$ for $2 \leq i \leq r-1$ and all j, $g(y_{rj}) = 0$ for $0 \leq j \leq t-k-1$ (note that this latter range is empty if $k = t$), $g(y_{r,t-k}) = c_1\mu x_{-(k+1)}$, and $g(y_{rj}) = c_1 x_{-t+j-1}$ for $t-k+1 \le j \le t$ (this range being empty for $k = 0$). The verification of the fact that g extends to a well-defined homomorphism $B \to \mathrm{St}(w)$ is a bit tedious – various possibilities for k need to be considered separately – and we leave it to the reader to fill in the pertinent computations.

In case $l = t+1$, we can either play the situation back to one of the previously considered cases by relabeling, or else define g directly as follows: $g(y_{10}) = x_0 - (c_2/c_1)\nu x_{t+1}$, $g(y_{1j}) = x_j$ for $1 \le j \le t$, and $g(y_{20}) = \nu x_{t+1}$; for the pairs (i,j) lexicographically larger than $(2,0)$, we keep the above specifications of $g(y_{ij})$. Again one checks that these definitions give rise to a map $g \in \mathrm{Hom}_\Lambda(B, \mathrm{St}(w))$.

Since clearly $\varphi g = f$ in either case, the proof of the lemma is complete. □

Lemma 23. *Let $t \ge 0$, and suppose that the syllables $q_0, \dots, q_t, p_1^{-1}, \dots, p_{t+1}^{-1}$ of the characteristic word $w = w(S)$ are nontrivial, yielding a primitive word $v = p_{t+1}^{-1} q_0 p_1^{-1} q_1 \dots p_t^{-1} q_t$ with graph*



Moreover, suppose that $e(0) = e$ and that the principal right segment w_{right} of w is periodic of the form $w_{right} = (q_0 p_1^{-1} \dots q_t p_{t+1}^{-1})(q_0 p_1^{-1} \dots q_t p_{t+1}^{-1}) \dots$; in other words,

$$p_{t+1}^{-1} w_{right} = vvv \dots .$$

Then the direct sums

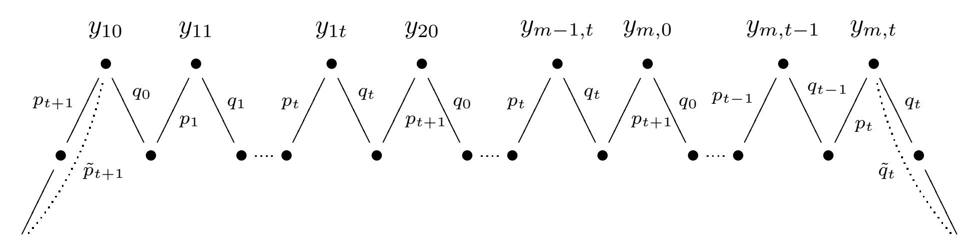
$$\left(\bigoplus_{i=0}^{t} \Lambda e(i)/Je(i)\right)^{(\mathbb{N})} \quad \text{and} \quad \left(\bigoplus_{i=0}^{t} \Lambda \tilde{e}(i)/J\tilde{e}(i)\right)^{(\mathbb{N})}$$

are $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-phantoms and $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$-phantoms of S. In particular, S has neither a $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$- nor an $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$-approximation.

Moreover, any effective $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$- or $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$-phantom of S has a copy of $\left(\bigoplus_{i=0}^{t} \Lambda e(i)/Je(i)\right)^{(\mathbb{N})}$ in its top and a copy of $\left(\bigoplus_{i=0}^{t} \Lambda \tilde{e}(i)/J\tilde{e}(i)\right)^{(\mathbb{N})}$ in its socle.

Proof. We will simultaneously verify that the above semisimple modules are $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$- and $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$-phantoms of S, as all of our test maps will have sources in $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$.

We set $\tilde{p}_{t+1} = \mathfrak{p}_{t+1} p_{t+1}$ and $\tilde{q}_t = \mathfrak{q}_t q_t$, where $\mathfrak{p}_{t+1}$ and $\mathfrak{q}_t$ are as in Proposition 1; the path p_{t+1} being nontrivial, this means that $\mathfrak{p}_{t+1}$ is the longest path such that $\tilde{p}_{t+1}$ is a path in $K\Gamma \setminus I$; the path $\tilde{q}_t$ has an analogous description. Moreover, we consider, for each positive integer $m \ge 2$, the word $u_m = \tilde{p}_{t+1}^{-1} q_0 \dots p_t^{-1} q_t v^{m-2} p_{t+1}^{-1} q_0 \dots p_t^{-1} \tilde{q}_t$. Then u_m is a word of finite projective dimension and $\mathrm{St}(u_m)$ has graph



relative to a standardized sequence of top elements y_{ij}. We equip the set of pairs (i,j) for $1 \le i \le m$ and $0 \le j \le t$ with the lexicographic order and let $f : \mathrm{St}(u_m) \to S$ be the homomorphism with $f(y_{10})) = e + Je$ and $f(y_{ij}) = 0$ for $(i,j) > (1,0)$. Next, we will check that any module $A \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ with the property that f factors through a map $\rho \in \mathrm{Hom}_\Lambda(A, S)$ contains $\left(\Lambda\tilde{e}(j)/J\tilde{e}(j)\right)^{m-1}$ in its socle and $\left(\Lambda e(j)/Je(j)\right)^{m-1}$ in its top, for $j \in \{0, \dots, t\}$. This will prove all of our claims.

For that purpose, we let $h \in \mathrm{Hom}_\Lambda(\mathrm{St}(u_m), A)$ be such that $\rho h = f$. In a first step, we prove, more precisely, that the elements $q_j h(y_{ij})$, $1 \le i \le m-1$, of A are K-linearly independent for all j. Clearly, these elements belong to the socle of A since the $q_j y_{ij}$'s belong to the socle of $\mathrm{St}(u_m)$ for $i \le m-1$. Assume, to the contrary of our claim, that the $q_j h(y_{ij})$'s, $i \le m-1$, are linearly dependent, and let the pair (k,l) be (lexicographically) minimal with the property that there exists an equality $\sum_{i=k}^{m-1} a_i q_l h(y_{il}) = 0$ for scalars $a_i \in K$ with $a_k \neq 0$. Set $y = \sum_{i=k}^{m-1} a_i y_{il} \in \mathrm{St}(u_m)$. First we observe that $(k,l) \neq (1,0)$, for otherwise we would have $\rho h(y) = a_k e + Je \neq 0$, which would make $h(y)$ a top element of A, and hence guarantee that $q_l h(y) \neq 0$ by Proposition 16(A); but the latter is inconsistent with our choice of y, and therefore we conclude $(k,l) \neq (1,0)$. This permits us to define an element $z \in \mathrm{St}(u_m)$ as follows: If $l = 0$, we have $k \ge 2$, which legitimizes the definition $z = \sum_{i=k}^{m-1} a_i y_{i-1,l}$; if, on the other hand, $l \ge 1$, we set $z = \sum_{i=k}^{m-1} a_i y_{i,l-1}$. We note that, in either case, the nonzero scalar a_k accompanies an element y_{ij} with $(i,j) < (k,l)$. Hence, in case $l = 0$, the minimal choice of (k,l) ensures that $p_{t+1} h(y) = q_t h(z) = \sum_{i=k}^{m-1} a_i q_t h(y_{i-1,t}) \neq 0$, while, in case $l \ge 1$, that choice yields $p_l h(y) = q_{l-1} h(z) = \sum_{i=k}^{m-1} a_i q_{l-1} h(y_{i,l-1}) \neq 0$. In other words, $p_{t+1} h(y)$ is a nonzero element in $\operatorname{soc} A \cap q_t A$ in the first case, and $p_l h(y)$ is a nonzero element in $\operatorname{soc} A \cap q_{l-1} A$ in the second. In either case, our hypothesis combines with part (B) of Proposition 16 to show that $h(y)$ is a top element of type $e(l)$ of A, whence our assumption that $q_l h(y)$ be zero contradicts part (A) of Proposition 16. We thus conclude that $\big(\Lambda \tilde{e}(j)/J\tilde{e}(j)\big)^{m-1}$ is contained in the socle of A as claimed.

We have shown that, for each $j \in \{0, \dots, t\}$, all linear combinations $\sum_{i=1}^{m-1} a_i q_j h(y_{ij})$ with $(a_1, \dots, a_{m-1}) \neq 0$ are nonzero elements of the socle of A, and invoking again part (B) of Proposition 16 on the model of the preceding paragraph, we infer that all linear combinations $\sum_{i=1}^{m-1} a_i h(y_{ij})$ with $(a_1, \dots, a_{m-1}) \neq 0$ are top elements of A. This means that the elements $h(y_{2k}), \dots, h(y_{mk})$ are linearly independent modulo JA and thus yields the required containment of $\big(\Lambda e(j)/Je(j)\big)^{m-1}$ in A/JA.

In conclusion, arbitrarily high finite powers of $\bigoplus_{i=0}^t \Lambda e(i)/Je(i)$ and $\bigoplus_{i=0}^t \Lambda \tilde{e}(i)/J\tilde{e}(i)$ are $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-phantoms, as well as $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$-phantoms of S, and hence so are their direct limits $\big(\bigoplus_{i=0}^t \Lambda e(i)/Je(i)\big)^{(\mathbb{N})}$ and $\big(\bigoplus_{i=0}^t \Lambda \tilde{e}(i)/J\tilde{e}(i)\big)^{(\mathbb{N})}$. $\square$

Of course the mirror image of Lemma 23 relative to the central axis of w is also true, since replacing w by its inverse is harmless. The argument we gave for the lemma actually proves a little more than we stated in our conclusion: Namely, if u is a segment of v such that $\mathrm{St}(u)$ has square-free socle (this is for instance true for any syllable u of v), then $\big(\mathrm{St}(u)\big)^{(\mathbb{N})}$ is a $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-phantom of S. The part of the lemma that addresses $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$ will be superseded by the stronger assertion of Theorem 5 that $\mathrm{St}(w)$ is an $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$-phantom of S.

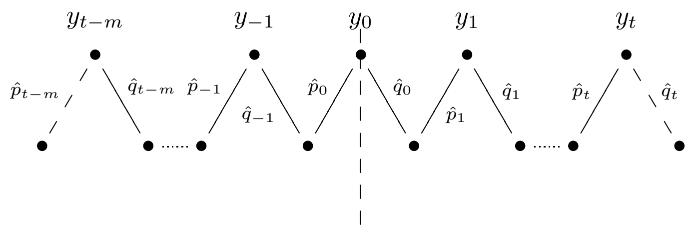
Proof of Theorem 5 via Proposition 14. Let e be one of the primitive idempotents $e_1, \dots, e_n$, and again denote the characteristic word of $S = \Lambda e/Je$ by $w = (p_i^{-1} q_i)_{i \in \mathbb{Z}}$.

(I) The word w is centered at e by construction. For left and right periodicity of w, we refer to the final paragraph of Construction 12. To justify the upper bound on the number of steps required to explicitly determine w from Γ and a set of paths generating I, we recall that the principal right segment of w is completely determined, once we have constructed its first $4n+1$ syllables $q_0, p_1^{-1}, q_1, p_2^{-1}, \dots, q_{2n}$. The algorithm of [**19**] for determining the projective dimensions of Λ-modules which are either uniserial or have graphs of the form 'V', moreover, allows us to find each successive pair $q_i p_i^{-1}$ of syllables of w in $\le 2(\dim_K \Lambda)^3$ steps.

(II) In light of Proposition 1, finiteness of the projective dimension of $\mathrm{St}(w)$ is an immediate consequence of the construction.

To verify the second part of (II), let once more $\varphi : \mathrm{St}(w) \to S = \Lambda e/Je$ be the canonical map of the centered word w. In reference to the graph of $\mathrm{St}(w)$ at the beginning of Section 7, we thus have $\varphi(x_i) = \delta_{i0}(e + Je)$ for $i \in \mathbb{Z}$. We first show that every homomorphism $M \to S$, where M is an object of $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$, factors through φ; this will prove effectiveness, once we have shown that $\mathrm{St}(w)$ is an $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$-phantom of S. It is clearly harmless to assume M to be indecomposable; in other words, we focus on the situation where M is a string module of finite projective dimension, based on a finite word $\hat{w} = \hat{p}_0^{-1} \hat{q}_0 \dots \hat{p}_m^{-1} \hat{q}_m$ with $m \ge 0$, such that the string module $M = \mathrm{St}(\hat{w})$ has a standardized sequence $y_0, y_1, \dots, y_m$ of $m+1$ top elements; see Section 2 for our conventions. Then, clearly, those homomorphisms which send any given top element y_i of type e to $e + Je$ and all the other y_j to zero constitute a K-basis of

$\mathrm{Hom}_\Lambda(\mathrm{St}(\hat{w}), S)$, whence we may assume that our map $f \in \mathrm{Hom}_\Lambda(\mathrm{St}(\hat{w}), S)$ is of the latter ilk; so suppose that there exists $i \in \{1, \dots, m\}$ such that $y_i = ey_i$, and let $f \in \mathrm{Hom}_\Lambda(\mathrm{St}(\hat{w}), S)$ be as described. We relabel the syllables of the word $\hat{w}$ if necessary, to center it in the idempotent e corresponding to y_i, and to ensure that $\hat{w}$ has the same orientation as w. If the standardized sequence of top elements y_j is re-indexed accordingly, the graph of $\mathrm{St}(\hat{w})$ takes on the form



(†)

for some $t \geq 0$. It is clearly innocuous to assume that $\hat{p}_t$ is nontrivial.

We now construct a map $g : \mathrm{St}(\hat{w}) \to \mathrm{St}(w)$ by starting with the assignment $g(y_0) = x_0$. Next, we inductively define the images $g(y_j)$ for $1 \leq j \leq t$ in such a way that $g(y_0), \dots, g(y_t)$ satisfy all of the relations tying the $\hat{p}_j y_j$ and $\hat{q}_j y_j$ for $0 \leq j \leq t$ together; and finally, we do the same for $t - m \leq j \leq -1$. Since the relations of $\mathrm{St}(\hat{w})$ can be generated by relations involving at most two consecutive y_i, this will ensure that our assignments induce a well-defined homomorphism $g \in \mathrm{Hom}_\Lambda(\mathrm{St}(\hat{w}), St(w))$.

If none of the nontrivial paths $\hat{q}_i$, $\hat{p}_{i+1}$ for $i \geq 0$ is a right discontinuity of $\hat{w}$, the assignments $g(y_i) = x_i$ for $1 \leq i \leq t$ satisfy our requirements. Next suppose that the first right discontinuity of $\hat{w}$ is some nontrivial path $\hat{q}_k$ with $k \geq 0$. Then the definitions $g(y_i) = x_i$ for $1 \leq i \leq k$ and $g(y_i) = 0$ for $i > k$ are as required, for Lemma 20 tells us that q_k is a proper right subpath of $\hat{q}_k$, whence $\hat{q}_k g(y_k) = 0$. If, finally, $\hat{w}$ has a first right discontinuity of the form $\hat{p}_k$ for some $k > 0$, then $p_k = \hat{p}_k \nu$ for a nontrivial path ν, again by Lemma 20. In that case the assignments $g(y_i) = x_i$ for $i < k$, $g(y_k) = \nu x_k$, and $g(y_i) = 0$ for $i > k$ satisfy our demands; indeed, if $\hat{q}_k$ is trivial, then $k = t$, and otherwise $\hat{q}_k g(y_k) = 0$ in view of the nontriviality of ν.

Assignments $g(y_j)$ for $t - m \leq j \leq -1$ which are compatible with the pertinent relations are made symmetrically. This procedure clearly leads to a homomorphism g with $\varphi g = f$.

To show that $\mathrm{St}(w)$ is an $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$-phantom of $S = \Lambda e / Je$, we consider, for each nonnegative integer k, the centered word

$$u_k = \tilde{p}_{-(k+1)}^{-1} q_{-(k+1)} p_{-k}^{-1} q_{-k} \dots p_0^{-1} q_0 \dots p_k^{-1} q_k p_{k+1}^{-1} \tilde{q}_{k+1};$$

here $\tilde{p}_{-(k+1)} = \mathfrak{p}_{-(k+1)} p_{-(k+1)}$ and $\tilde{q}_{k+1} = \mathfrak{q}_{k+1} q_{k+1}$ with $\mathfrak{p}_{-(k+1)}$ and $\mathfrak{q}_{k+1}$ chosen as in Proposition 1, that is, $\tilde{p}_{-(k+1)}$ and $\tilde{q}_{k+1}$ are the longest paths in $K\Gamma \setminus I$ containing $p_{-(k+1)}$ and q_{k+1} as right subpaths. By construction of $w = (p_i^{-1} q_i)_{i \in \mathbb{Z}}$ and Proposition 1, the u_k are words of finite projective dimension. As usual, we fix a standardized sequence of top elements $y_{-(k+1)}, \dots, y_0, \dots, y_{k+1}$ of $\mathrm{St}(u_k)$, and focus on the canonical map $\psi_k : \mathrm{St}(u_k) \to S$, i.e., $\psi_k(y_i) = \delta_{i0}(e + Je)$. We claim that any module $A \in \mathcal{S}^{<\infty}(\Lambda\text{-mod})$ with the property that ψ_k factors through some map in $\mathrm{Hom}_\Lambda(A, S)$ contains the string module $\mathrm{St}(p_{-k}^{-1} q_{-k} \dots p_k^{-1} q_k)$ as a submodule. Once established, this claim will entail that $\mathrm{St}(p_{-k}^{-1} q_{-k} \dots p_k^{-1} q_k)$ is an $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$-phantom of S for each k, whence so is the obvious direct limit

$$\varinjlim_{k \in \mathbb{N}} \mathrm{St}(p_{-k}^{-1} q_{-k} \dots p_k^{-1} q_k) = \mathrm{St}(w).$$

Thus, in order to prove our claim, we suppose that ψ_k factors through $A \in \mathcal{S}^{<\infty}(\Lambda\text{-mod})$ and write $A = \bigoplus_{i=1}^r A_i$, where each A_i is a finite dimensional string module of finite projective dimension and $\pi_i : A \to A_i$ is the corresponding canonical projection. That ψ_k factors through a map in $\mathrm{Hom}(A, S)$ clearly implies the existence of a map $g \in \mathrm{Hom}_\Lambda(\mathrm{St}(u_k), A)$ and an index l such that $\pi_l g(y_0)$ is a top element of A_l. Suppose that $A_l = \mathrm{St}(\hat{w})$, where $\hat{w} = \hat{p}_0^{-1} \hat{q}_0 \dots \hat{p}_m^{-1} \hat{q}_m$ is a word of finite projective dimension such that $m \geq 0$ and $\mathrm{St}(\hat{w})$ has a standardized sequence of $m + 1$ top elements, say $z_0, \dots, z_m$.

By [**10**, Theorem], $\mathrm{Hom}_\Lambda(\mathrm{St}(u_k), \mathrm{St}(\hat{w}))$ is generated, as a K-space, by maps

$$h = h[\rho, \mathfrak{u}_k, \widehat{\mathfrak{w}}]$$

of the following ilk: $\mathfrak{u}_k$ and $\widehat{\mathfrak{w}}$ are subgraphs of the graphs of the words u_k and $\hat{w}$, respectively (see Section 2 for our conventions), the first closed under arrows whose endpoints belong to $\mathfrak{u}_k$, the second closed under arrows whose starting points belong to $\widehat{\mathfrak{w}}$; moreover, ρ denotes an isomorphism $\mathfrak{u}_k \to \widehat{\mathfrak{w}}$ of directed graphs, sending any arrow in $\mathfrak{u}_k$ to an arrow in $\widehat{\mathfrak{w}}$ that carries the same label, such that the homomorphism h is induced by ρ (in the only meaningful way). Note that the image of h is $\mathrm{St}(\widehat{\mathfrak{w}})$, the latter being a submodule of $\mathrm{St}(\hat{w})$ by the closure condition imposed on $\widehat{\mathfrak{w}}$. So the fact that there exists a map in $\mathrm{Hom}_\Lambda(\mathrm{St}(u_k), \mathrm{St}(\hat{w}))$ sending y_0 to a top element of $\mathrm{St}(\hat{w})$ ensures the existence of a triple $[\rho, \mathfrak{u}_k, \widehat{\mathfrak{w}}]$, as described, together with an index $i \in \{0, \dots, m\}$, satisfying the following requirements: $\widehat{\mathfrak{w}}$ includes that vertex in the graph of $\hat{w}$ which corresponds to the top element z_i of $\mathrm{St}(\hat{w})$ – call that vertex z (it is the joint starting vertex of the paths $\hat{p}_i$ and $\hat{q}_i$ in the graph of $\hat{w}$); moreover, $\mathfrak{u}_k$ includes that vertex in the graph of u_k which corresponds to the top element y_0 of $\mathrm{St}(u_k)$ (namely, the joint starting vertex of p_0 and q_0 in the graph of u_k); and, finally, ρ sends this latter vertex to z.

In light of the preceding discussion, it suffices to prove that $\widehat{\mathfrak{w}}$ contains a subgraph isomorphic to that of the word $p_{-k}^{-1} q_k \dots p_k^{-1} q_k$. As in the effectiveness proof above, we adjust the labeling of the syllables of $\hat{w}$ and the standardized sequence of top elements of $\mathrm{St}(\hat{w})$ so that $\hat{w}$ becomes a word which is centered at $i = 0$, and the graph of $\mathrm{St}(\hat{w})$ has the form (†), displayed at the outset of our proof of (II), for a suitable nonnegative integer t. Moreover, it is clearly harmless to assume that $\hat{w}$ has the same orientation as w. Then $\hat{p}_0$ contains p_0 as a right subpath, and $\hat{q}_0$ contains q_0 as a right subpath by Proposition 16(A), because $\hat{w}$ is a word of finite projective dimension. Since the graph $\widehat{\mathfrak{w}}$ contains the paths $\hat{p}_0$ and $\hat{q}_0$ due to its closure property, we deduce that $\hat{p}_0 = p_0$ and $\hat{q}_0 = q_0$. Part (B) of that same proposition now tells us that the paths $\hat{p}_{-1}$ and $\hat{p}_1$ are contained in p_{-1} and p_1 as left subpaths, respectively. If p_{-1} and p_1 are trivial, we are done. So we assume that p_1 is nontrivial and write $p_1 = \hat{p}_1 \nu$, where ν is a path of length ≥ 0. Then $\hat{q}_0 z_0 = q_0 z_0 = \hat{p}_1 \nu z_1$ is a nonzero element of $\mathrm{soc}\,\mathrm{St}(\hat{w})$, whence Proposition 16(B) guarantees that νz_1 is a top element of $\mathrm{St}(\hat{w})$. We infer that ν is trivial and that the vertex of $\hat{w}$ corresponding to z_1 also belongs to $\widehat{\mathfrak{w}}$. Consequently, the preceding argument can be duplicated to show $\hat{q}_1 = q_1$, and next, in case p_2 is nontrivial, the equalities $\hat{p}_2 = p_2$ and $\hat{q}_2 = q_2$. An obvious induction on $i \in \{1, \dots, k\}$ yields $\hat{p}_i = p_i$ and $\hat{q}_i = q_i$, whenever p_i is nontrivial, and an analogous argument applies to the paths $\hat{p}_i$ and $\hat{q}_i$ with negative indices $i \in \{-k, \dots, -1\}$. This shows that the graph of the word $p_{-k}^{-1} q_{-k} \dots p_k^{-1} q_k$ is indeed a subgraph of $\widehat{\mathfrak{w}}$, thus proving our claim and finishing the proof of part (II) of Theorem 5.

(III) The first assertion is an immediate consequence of the equivalences. To prove the equivalences, we fix k and write $S_k = S = \Lambda e / J e$. Moreover, we let $w_k = w = (p_i^{-1} q_i)_{i \in \mathbb{Z}}$ be the characteristic word of S and $\varphi : \mathrm{St}(w) \to S$ the canonical homomorphism defined by $\varphi(x_i) = \delta_{i0}(e + Je)$.

'(i) $\Longrightarrow$ (iii)'. Suppose that w is finite. We need to ascertain that every homomorphism $f \in \mathrm{Hom}_\Lambda(M, S)$, where M is an indecomposable object of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$, factors through φ. Once we know that φ is a $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of S, minimality will be automatic, because $\mathrm{St}(w)$ is known to be indecomposable (see [**8**]). In case M is a string module of finite projective dimension, the required factorization property follows from the fact that φ is an *effective* $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$-phantom of S, which was established in part (II).

Now we focus on a band module $M = \mathrm{Bd}(\hat{v}^r, \phi)$ of finite projective dimension, where $\hat{v} = \hat{p}_0^{-1} \hat{q}_0 \dots \hat{p}_t^{-1} \hat{q}_t$ is a primitive word with all of the listed syllables nontrivial, and again denote by $y_{10}, \dots, y_{1t}$, $y_{20}, \dots, y_{2t}$, $y_{r0}, \dots, y_{rt}$ a standardized sequence of top elements of M. It clearly suffices to show that any map $f \in \mathrm{Hom}_\Lambda(M, S)$ which sends precisely one of the top elements y_{ij} of type e to $e + Je$ and the others to zero factors through φ (if none of the y_{ij} is of type e, our requirement is void). So let us assume that $f : M \to S$ is a map of the described ilk. Moreover, it is harmless to adjust the setup so that $f(y_{10}) = e + Je$ and $f(y_{ij}) = 0$ for $(i, j) \neq (1, 0)$. In this situation, M has the same graph as the pseudo-band module B in the statement of Lemma 22.

As in Lemmas 20 and 22, we denote by $\hat{w}$ the twosided infinite word $\dots \hat{v}\hat{v}\hat{v} \dots$, which we again assume to be centered at e and have the same orientation as w. Clearly, $\hat{w}$ has right and left discontinuities, since the characteristic word w of S is finite by hypothesis, whereas $\hat{w}$ is twosided infinite. In light of Proposition 1, moreover, $\hat{w}$ is a word of finite projective dimension, and so Lemma 22 applies to guarantee that f factors through φ as required.

The implications '(iii) $\Longrightarrow$ (v) $\Longrightarrow$ (iv)' and '(iii) $\Longrightarrow$ (ii)' are obvious.

'(iv) $\Longrightarrow$ (i)' follows from the fact that $\mathrm{St}(w)$ is an $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$-phantom of S by part (II).

'(ii) $\Longrightarrow$ (i)'. Suppose $w = (p_i^{-1} q_i)_{i \in \mathbb{Z}}$ is infinite; w.l.o.g., the principal right segment of w is infinite and so, in particular, the syllable q_0 is nontrivial.

First assume that there exists an index $j > 0$ such that $q_j = q_0$, and let $t \geq 1$ be minimal with the property that $q_t = q_0$. Then the principal right segment w_{right} of w has the form

$$w_{\text{right}} = (q_0 p_1^{-1} \dots q_t p_{t+1}^{-1})(q_0 p_1^{-1} \dots q_t p_{t+1}^{-1})(q_0 p_1^{-1} \dots q_t p_{t+1}^{-1}) \dots$$

by Construction 12; in other words, if $v = p_{t+1}^{-1} q_0 \dots p_t^{-1} q_t$ is a primitive word such that $p_{t+1}^{-1} w_{\text{right}} = \dots vvv \dots$, Lemma 23 tells us that S fails to have a $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation. In case p_0 is nontrivial and there exists an index $j < 0$ with $p_j = p_0$, the situation is symmetric – just flip the characteristic word w about its central axis.

Now suppose that, for all $j > 0$, we have $q_j \neq q_0$, and p_0 is either trivial, or else $p_j \neq p_0$ for all $j < 0$. Assume that, to the contrary of our claim, S has a $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation. Our aim is to infer that then S has an $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$-approximation as well; but this is incompatible with the already established implication '(iv) $\Longrightarrow$ (i)'. To that end, let A be any $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of S and B a band module occurring as a direct summand of A, say $A = B \oplus C$. We will construct another $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of S which has the form $\bigoplus_{\text{finite}} \operatorname{St}(u_j) \oplus C$, for suitable words u_j of finite projective dimension. In this way, we can eliminate all band module summands from A in favor of direct sums of string modules, to arrive at another $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of S, this one belonging to $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$. This will then give us the desired contradiction.

In order to replace the summand B of A by a direct sum of string modules as indicated, write $B = Bd(\hat{v}^r, \phi)$, where $\hat{v} = \hat{p}_0^{-1} \hat{q}_0 \dots \hat{p}_t^{-1} \hat{q}_t$ is a primitive word, $r \geq 1$. For a graph of B, we refer to the graph of the pseudo-band module of the same name in the statement of Lemma 22; again, we let $y_{10}, \dots, y_{1t}, \dots, y_{r0}, \dots, y_{rt}$ be the corresponding standardized sequence of top elements of B. As in the argument for '(i) $\Longrightarrow$ (iii)', it suffices to prove the following: Given any top element $y \in \{y_{10}, \dots, y_{rt}\}$ of type e of B, the map $f : B \to S$, sending y to the residue class $e + Je$ and the other top elements y_{ij} to zero, factors through a map $\operatorname{St}(u) \to S$, where u is a suitable word of finite projective dimension. Moreover, it is harmless to assume that y is the top element corresponding to the joint starting vertex of the paths $\hat{p}$ and $\hat{q}$. Once more, we center the infinite periodic word $\hat{w} = \dots \hat{v}\hat{v}\hat{v} \dots$ in some occurrence of this vertex, and assume (clearly still without losing generality) that $\hat{w}$ has the same orientation as w. Since B has finite projective dimension, so does the word $\hat{w}$, by Proposition 1. The non-periodicity conditions we imposed on w, moreover, force left and right discontinuities on $\hat{w}$. In light of Lemma 22, our test map f thus factors through the canonical homomorphism $\varphi : \operatorname{St}(w) \to S$. Say $f = \varphi g$, for a suitable map $g \in \operatorname{Hom}_\Lambda(B, \operatorname{St}(w))$. Let $w' = (p_i^{-1} q_i)_{-m \leq i \leq m}$ be a finite segment of w such that $\operatorname{St}(w')$ is a submodule of $\operatorname{St}(w)$ containing the image of g. Moreover, let u be a finite word of finite projective dimension which in turn contains w' as a segment; such a word u exists by Observations 10. Then $f = \varphi h$, where $h : B \to \operatorname{St}(u)$ denotes the map resulting from g through restriction of the range, which finishes the argument we have laid out.

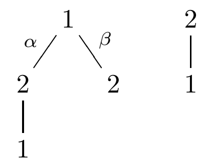
To prove the supplementary statement, let w be finite. Suppose that the K-dimension of $\operatorname{St}(w)/J\operatorname{St}(w)$ exceeds $4n$, where n is the number of vertices of Γ. Then either p_{2n} or q_{-2n} is nontrivial. We may assume the former path to be nontrivial, the other case leading to a symmetric situation. The string module $\operatorname{St}(q_0 p_1^{-1} \dots q_{2n-1} p_{2n}^{-1})$ then has $2n+1$ top elements x_i, which are K-linearly independent modulo the radical. At least three of these are normed by the same primitive idempotent. Hence at least two of the latter, say x_k and x_l for suitable $k < l$, reside atop paths p_k, p_l starting in the same arrow. In this situation, Construction 12 yields $q_k = q_l$, $p_{k+1} = p_{l+1}$, etc. (consult Observations 11), and consequently all the p_i with positive index are nontrivial. But this makes the word w infinite, thus contradicting our hypothesis.

The proof of Theorem 5 is thus complete. □

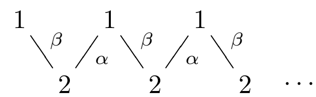
8. Concluding remarks

Theorem 5 and Proposition 14 tell us that, if $w = w(S)$ is the characteristic word of the simple module $S \in \Lambda\text{-mod}$, the string module $\operatorname{St}(w)$ is an effective $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$-phantom of S. On the other hand, $\operatorname{St}(w)$ is not an effective $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-phantom of S in general. The first known example of a finite dimensional algebra Λ for which $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ fails to be contravariantly finite, presented in [**25**], already demonstrates this.

Example 23. Let Λ be the string algebra whose two indecomposable projective left modules have the following graphs:

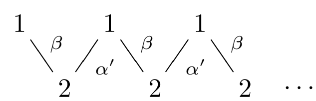


Using the techniques we developed, we see that the characteristic phantom $\mathrm{St}(w_1)$ of S_1 has graph



This infinite dimensional module is known to be both an $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$- and a $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-phantom of S_1 (see [**18**]), but, while it is effective as an $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$-phantom by Theorem 5, it is not effective as a $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-phantom. Indeed, for the band module $M = \Lambda e_1/\Lambda(\alpha - \beta) \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$, the canonical epimorphism $f : M \to S_1$ clearly fails to factor through $\mathrm{St}(w_1)$.

Furthermore, we note that the string module $\mathrm{St}(w_1)$, unique relative to the given coordinatization of Λ, does depend on the coordinates. If we adopt the new arrows $\alpha' = \alpha - \beta$ and $\beta' = \beta$ from the vertex 1 to the vertex 2, the characteristic phantom $\mathrm{St}(w')$ of S_1 with respect to the new coordinatization of Λ has graph

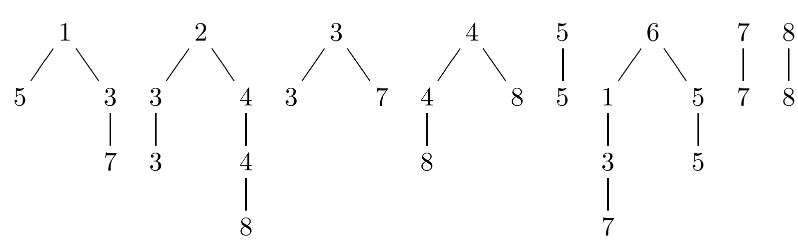


It is readily seen that $\mathrm{St}(w_1)$ is not isomorphic to $\mathrm{St}(w_1')$. □

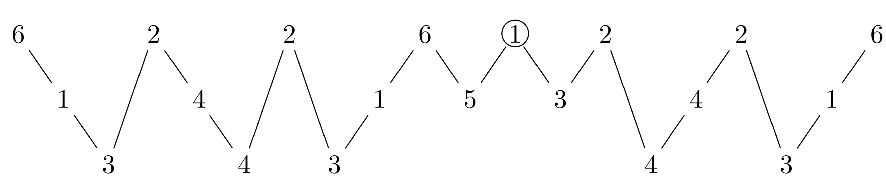
In Part II, we will address the problem of supplementing the effective $\mathcal{S}^{<\infty}(\Lambda\text{-mod})$-phantoms φ_i : $\mathrm{St}(w_i) \to S_i$ to effective $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-phantoms. It will turn out that all information required to construct such 'completions' of the $\mathrm{St}(w_i)$ is stored in the characteristic words w_i.

Concerning the shape of the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the S_i in case the word $w = w(S)$ is finite, Theorem 5 guarantees that the top of the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation $\mathrm{St}(w)$ of S has K-dimension at most $4n$, where n is the number of distinct simple Λ-modules. This bound stems from the fact that the multiplicity of any simple module T in $\mathrm{St}(w)/J\,\mathrm{St}(w)$ is bounded above by 4 (see Construction 12). While the bound on $\dim_K\big(\mathrm{St}(w)/J\,\mathrm{St}(w)\big)$ can be tightened with some additional effort, the bound on multiplicities in $\mathrm{St}(w)/J\,\mathrm{St}(w)$ is sharp.

Example 24. Let Λ be the string algebra whose indecomposable projective left modules have the following graphs

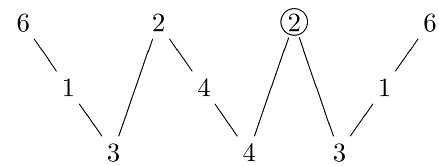


Then $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in Λ-mod, as can readily be ascertained with the aid of Theorem 5, and the graph of the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of S_1 is



□

This final example also exhibits the necessity of recording the center of the characteristic word w of S, if one aims at pinning down the correct homomorphism $\varphi : \mathrm{St}(w) \to S$, through which all homomorphisms in $\mathrm{Hom}_\Lambda\big(\mathcal{S}^{<\infty}(\Lambda\text{-mod}), S\big)$ will factor. Indeed, the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of S_2 is the string module with graph



where again the center is highlighted.

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