

STACKS OF ALGEBRAS AND THEIR HOMOLOGY

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Dedicated to Raymundo Bautista and Roberto Martínez-Villa on the occasion of their sixtieth birthdays

ABSTRACT. For any increasing function $f : \mathbb{N} \rightarrow \mathbb{N}_{\geq 2}$ which takes only finitely many distinct values, a connected finite dimensional algebra Λ is constructed, with the property that $\text{fin dim}_n \Lambda = f(n)$ for all n ; here $\text{fin dim}_n \Lambda$ is the n -generated finitistic dimension of Λ . The stacking technique developed for this construction of homological examples permits strong control over the higher syzygies of Λ -modules in terms of the algebras serving as layers.

1. INTRODUCTION AND BACKGROUND

The purpose of this paper is twofold. One of our objectives is to introduce a technique of ‘stacking’ finite dimensional algebras on top of one another so that, on one hand, the homology of the resulting algebra can be controlled in terms of the layers, while, on the other hand, this homology differs qualitatively from that of the building blocks. Our second, principal, goal is to apply such stacks towards realizing new homological phenomena.

Given a finite dimensional algebra Λ over a field K and $n \in \mathbb{N}$, we denote by $\text{fin dim}_n \Lambda$ the supremum of the finite projective dimensions attained on left Λ -modules with ‘top multiplicities $\leq n$ ’; in other words, if J denotes the Jacobson radical of Λ , we are focusing on those left Λ -modules M of finite projective dimension for which the multiplicities of the simple summands of M/JM are bounded above by n . Since, over a basic algebra, this condition just means that M can be generated by $\leq n$ elements, we refer to $\text{fin dim}_n \Lambda$ as the (left) n -generated *finitistic dimension* of Λ . While it is still open whether the little finitistic dimension,

$$\text{fin dim } \Lambda = \sup_{n \in \mathbb{N}} \text{fin dim}_n \Lambda,$$

is always finite – the question goes back to Bass’s 1960 paper [1] – it is well known that $\text{fin dim}_n \Lambda < \infty$ for all n ([7, Proposition 10.33] and [8]). This puts a spotlight on jumps $\text{fin dim}_n \Lambda < \text{fin dim}_m \Lambda$ for $n < m$. Indeed, producing a counterexample to finiteness of

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the little finitistic dimension would amount to constructing a finite dimensional algebra Λ together with an infinite sequence $n_1 < n_2 < n_3 < \dots$ of positive integers such that

$$\text{fin dim}_{n_k} \Lambda < \text{fin dim}_{n_{k+1}} \Lambda$$

for all k . While so far there has been hardly any insight into the mechanism of such jumps – the first illustrations were based on monomial algebras, where only a single jump is possible [6] – we now use our stacking technique to systematically create examples of ‘compounded jumps of arbitrary size’. Namely, for any increasing function $f : \mathbb{N} \rightarrow \mathbb{N}_{\geq 2}$ which takes only finitely many distinct values, we exhibit a connected finite dimensional algebra Λ with vanishing radical cube such that $\text{fin dim}_n \Lambda = f(n)$ for all $n \in \mathbb{N}$ (Section 4). The recursion underlying these successively ‘stacked’ examples pinpoints a typical combinatorial pattern giving rise to skips in the finitistic dimensions attained on modules of bounded length.

In this connection, we note that arbitrary finite gaps $\text{Fin dim } \Lambda - \text{fin dim } \Lambda$ between the little and big finitistic dimensions are already known to be realizable in the finite dimensional setting; here the big finitistic dimension, $\text{Fin dim } \Lambda$, is the supremum of all finite projective dimensions attained on *arbitrary* left Λ -modules. Gaps of 1 were first obtained for monomial algebras in [5], where, however, they cannot exceed 1; see [4]. Arbitrary jumps were produced by Rickard (unpublished), who showed that the big and little finitistic dimensions are additive on tensor products, and by Smalø [9], who constructed simpler examples via iterated one-point extensions of an algebra Λ_0 with $\text{Fin dim } \Lambda_0 - \text{fin dim } \Lambda_0 = 1$.

The stacking technique presented in Section 3 below is governed by less restrictive rules than one-point extensions: Any one-point extension of an algebra Λ_0 over a field K results from stacking the one-dimensional K -algebra on top of Λ_0 , but stacking an algebra Λ_1 on top of an algebra Λ_0 will not even lead to an *iterated* one-point extension of Λ_0 in general. On the other hand, this process always results in a triangular matrix algebra

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ M & \Lambda_0 \end{pmatrix}$$

for a suitable Λ_0 - Λ_1 -bimodule M . Hence initial upper and lower bounds on $\text{Fin dim } \Lambda$ in terms of Λ_0 and Λ_1 are available: Indeed, Fossum, Griffith and Reiten show in [3, Corollary 4.21] that

$$\text{Fin dim } \Lambda_0 \leq \text{Fin dim } \Lambda \leq \text{Fin dim } \Lambda_0 + \text{Fin dim } \Lambda_1 + 1;$$

the little finitistic dimensions are subject to analogous inequalities, as can easily be seen by the same method. Our stacks are considerably more specialized than the triangular matrix construction, so as to allow for tighter control of syzygies over the new algebra.

Since the algebras we target in our examples are stacks of monomial algebras, i.e., of algebras of the form $K\mathcal{Q}/I$, based on a field K , a quiver \mathcal{Q} and an admissible ideal I that can be generated by paths in $K\mathcal{Q}$, we include a brief review of the homology of monomial algebras at the end of this section for easy reference. These algebras are homologically well-understood. In particular, all second syzygies of their modules are direct sums of cyclic left ideals recruited from a finite collection (see [5] and Theorem 1 below); moreover, the

projective dimensions of these cyclic left ideals can easily be computed (for an algorithm, see [4]). The idea of our applications is to ‘stack’ the complexity observed in first syzygies over monomial algebras, while benefiting from the simplicity of second syzygies.

In Section 2, we will briefly discuss the graphs which we use to communicate certain types of modules in an intuitive format.

Throughout, Λ will be a split basic finite dimensional algebra over an arbitrary field K . The category of all left Λ -modules will be denoted by $\Lambda\text{-Mod}$, and $\Lambda\text{-mod}$ will be the full subcategory having as objects the finitely generated modules. For simplicity, we will identify Λ with a path algebra modulo relations, $K\mathcal{Q}/I$, where \mathcal{Q} is a quiver and $I \subseteq K\mathcal{Q}$ an admissible ideal in the path algebra. If p and q are paths in \mathcal{Q} , then pq will stand for ‘ q followed by p ’. Moreover, we will identify the set \mathcal{Q}_0 of vertices of \mathcal{Q} with a full set of primitive idempotents of Λ , loosely referred to as ‘the’ primitive idempotents of Λ .

Given any (left) Λ -module M , we call $x \in M$ a *top element* of M in case $x \in M \setminus JM$ and $x = ex$ for some primitive idempotent e ; in this situation we also say that x is a *top element of type e* of M . For $e \in \mathcal{Q}_0$, the simple module $\Lambda e/Je$ will be denoted by $S(e)$.

All of the finitistic dimensions, $\text{fin dim}_n \Lambda$, $\text{fin dim } \Lambda$, and $\text{Fin dim } \Lambda$, depend on the side. However, there is no need to weigh down our notation with left-right qualifiers, since we will consistently deal with *left* modules.

For the remainder of the introductory section, we assume Λ to be a monomial algebra. This means that the set of paths in $K\mathcal{Q} \setminus I$ gives rise to a K -linearly independent family of residue classes in Λ , which will be called the *nontrivial paths* in Λ ; thus the nontrivial paths in Λ form a basis for Λ over K . Clearly, it makes sense to speak of the *length* of a nontrivial path in Λ . The paths in the following set will be called the *critical paths*:

$$\mathcal{P} = \{p \in \Lambda \mid p \text{ is a nontrivial path of positive length,} \\ \text{starting in a non-source of } \mathcal{Q}, \text{ with } \text{p dim}_\Lambda \Lambda p < \infty\}.$$

We use this set to define an invariant \mathbf{s} as follows:

$$\mathbf{s} = \begin{cases} -1 & \text{if } \mathcal{P} = \emptyset \\ \max\{\text{p dim}_\Lambda \Lambda p \mid p \in \mathcal{P}\} & \text{otherwise.} \end{cases}$$

The set \mathcal{P} , as well as the number \mathbf{s} , can be readily obtained from the graphs of the indecomposable left Λ -modules.

On one hand, the smallest in our gamut of finitistic dimensions, $\text{fin dim}_1 \Lambda$, is trivially bounded below by $\mathbf{s} + 1$. On the other hand, the following theorem (see [5]) shows that all finitistic dimensions are bounded above by $\mathbf{s} + 2$. A generalization of the first part, together with a slick argument, can be found in [2].

Theorem 1. *Let M be a submodule of a projective left Λ -module, and $E(M)$ the set of those primitive idempotents e of Λ which do not annihilate M/JM .*

(1) *The syzygy $\Omega_\Lambda^1(M)$ of M is isomorphic to a direct sum of principal left ideals of Λ , each generated by a nontrivial path of positive length starting in a vertex in $E(M)$.*

In particular, all second syzygies of Λ -modules are direct sums of cyclic left ideals Λp , p a path, and all finitistic dimensions of Λ fall into the interval $[\mathbf{s} + 1, \mathbf{s} + 2]$.

(2) Given any nontrivial path q of positive length in Λ , the following statements are equivalent:

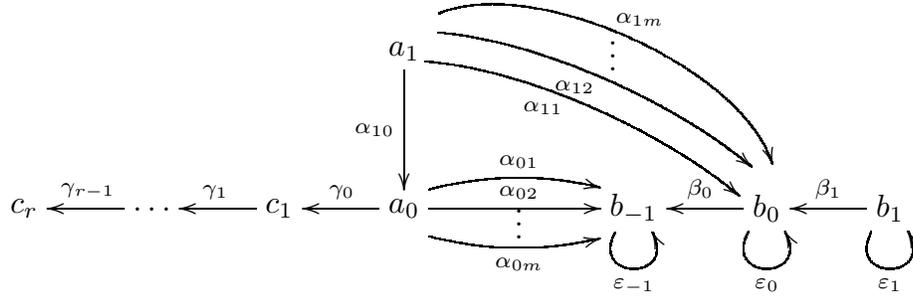
- (i) Λq is isomorphic to a direct summand of $\Omega_{\Lambda}^1(M)$.
- (ii) There exists a path of positive length in Λ , say αp where α is an arrow, together with a top element x of M such that $\Lambda q \cong \Lambda \alpha p$ and $p x \notin J^{\text{length}(p)+1} M$, while $\alpha p x = 0$.

From the first part of Theorem 1 we glean that, over a monomial algebra Λ , there is at most one positive integer n with the property that $\text{fin dim}_n \Lambda < \text{fin dim}_{n+1} \Lambda$, and these two dimensions differ by at most 1.

2. GRAPHS OF MODULES

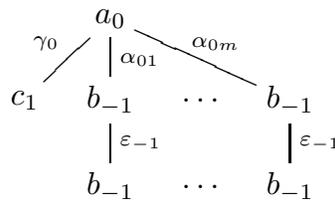
The modules arising in our examples can be represented by *layered* undirected graphs of a format which is intuitively suggestive. We use the conventions from [4, Section 5] and [6, Section 2]; but for the reader's convenience, we briefly review the very simple special cases needed here.

Let $\Lambda = KQ/I$ be a path algebra modulo relations – it will reappear as Λ_1 in Section 4 – with quiver Q as follows.



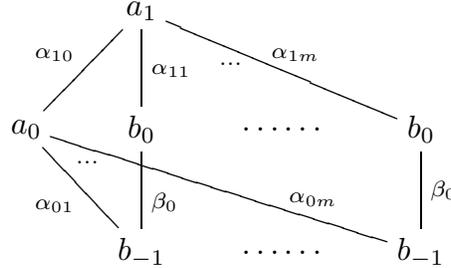
A generating set for the ideal I of relations can be communicated by way of graphs of the indecomposable projective left Λ -modules, as presented in Part I of the proof of Theorem 10. We give a few samples to explain how to interpret these graphs. While, in general, our graphs are to be read relative to a given sequence of top elements x_i of the considered module M (generating M modulo JM and linearly independent modulo JM), when $M = \Lambda e$ is indecomposable projective, we tacitly assume the choice of top element to be $x = e$.

Thus, presenting $M = \Lambda a_0$ by way of the layered graph



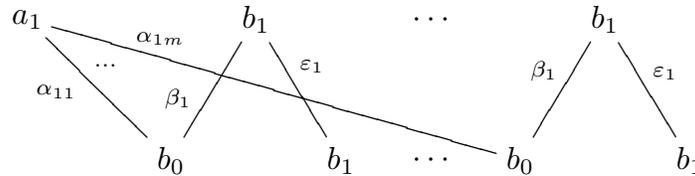
holds the following information: The left ideal $Ia_0 \subseteq K\mathcal{Q}$ is generated by $\gamma_1\gamma_0$ and $\varepsilon_{-1}^2\alpha_{0i}$ for $1 \leq i \leq m$. Equivalently, $M/JM \cong S(a_0)$, $JM/J^2M \cong S(c_1) \oplus S(b_{-1})^m$ and $J^2M \cong S(b_{-1})^m$, while $J^3M = 0$.

That $M = \Lambda a_1$ has graph



tells us that $M/JM \cong S(a_1)$, $JM/J^2M \cong S(a_0) \oplus S(b_0)^m$, $J^2M \cong S(b_{-1})^m$, and $J^3M = 0$. In particular, this implies $\gamma_0\alpha_{10} = \varepsilon_0\alpha_{1i} = \varepsilon_{-1}\alpha_{0i}\alpha_{10} = \varepsilon_{-1}\beta_0\alpha_{1i} = 0$ for $1 \leq i \leq m$. Moreover, the graph communicates the existence of nonzero scalars k_1, \dots, k_m such that $\alpha_{0i}\alpha_{10} = k_i\beta_0\alpha_{1i}$ for $1 \leq i \leq m$. (In Section 4, we will make the automatic choice $k_i = 1$, whenever we encounter non-monomial relations of this type.)

We give a final example, this time of a non-projective Λ -module N . That N has graph



relative to a sequence x_0, x_1, \dots, x_m of top elements (generating N and K -linearly independent modulo JN) means: $N/JN \cong S(a_1) \oplus S(b_1)^m$ with x_0 generating a copy of $S(a_1)$ modulo JN and each x_i for $i \geq 1$ generating a copy of $S(b_1)$ modulo JN ; in other words, x_0 is a top element of type a_1 , while x_1, \dots, x_m are top elements of type b_1 . Moreover, we glean $JN/J^2N \cong S(b_0)^m \oplus S(b_1)^m$, showing $\alpha_{10}x_0 = 0$, and $J^2N = 0$, which entails

$$\beta_0\beta_1x_i = \varepsilon_0\beta_1x_i = \varepsilon_1^2x_i = \beta_1\varepsilon_1x_i = 0 \quad \text{for } 1 \leq i \leq m.$$

Finally, the graph holds the information that $\alpha_{1i}x_0 = k_i\beta_1x_i$ for suitable scalars $k_i \in K^*$.

A final convention: A layered graph of a module M is said to be a *tree* if the underlying unlayered graph is a tree. The graph of N above is an example.

3. STACKS OF ALGEBRAS

We return to the situation where $\Lambda = K\mathcal{Q}/I$ is an arbitrary finite dimensional path algebra modulo relations. For ease of notation, we write E for the set of primitive idempotents of Λ (= vertices of \mathcal{Q}).

Definition 2.

(1) A *stacking partition* of Λ is a disjoint partition $E = E' \cup E''$ satisfying the following two conditions:

(a) Every arrow of \mathcal{Q} that starts in E' also ends in E' , i.e., $E'' \cdot (K\mathcal{Q}) \cdot E' = 0$.

(b) Suppose α is an arrow in \mathcal{Q} which starts in E'' and ends in E' , and let β be any arrow. Then $\alpha\beta \notin I$ forces β to start in a source of \mathcal{Q} .

(2) Given a stacking partition of Λ as under (1), we let \mathbf{e}' (resp., \mathbf{e}'') be the sum of all idempotents in E' (resp., E''), and set $\Lambda' = \mathbf{e}'\Lambda\mathbf{e}'$, $\Lambda'' = \mathbf{e}''\Lambda\mathbf{e}''$.

If both \mathbf{e}' and \mathbf{e}'' are nonzero, we call Λ a *2-stack*, or say that Λ is obtained by *stacking* Λ'' on top of Λ' .

(3) Suppose Δ_i for $0 \leq i \leq d$ are finite dimensional algebras. We call Λ a $(d+1)$ -*stack of* $\Delta_0, \dots, \Delta_d$ if there exist algebras $\Lambda_0, \dots, \Lambda_d$ with $\Lambda_0 = \Delta_0$ and $\Lambda_d = \Lambda$, such that Λ_i is a 2-stack with Δ_i stacked on top of Λ_{i-1} for $i \geq 1$.

A more suggestive rendering of a 2-stack Λ obtained by stacking $\Lambda'' = K\mathcal{Q}''/I''$ on top of $\Lambda' = K\mathcal{Q}'/I'$ is as follows: $\Lambda = K\mathcal{Q}/I$, where \mathcal{Q} is a quiver of the form

$$\begin{array}{ccc} \boxed{\mathcal{Q}''} & & \\ \alpha_1 \downarrow & \cdots & \downarrow \alpha_m \\ \boxed{\mathcal{Q}'} & & \end{array}$$

Here $\alpha_1, \dots, \alpha_m$ are ‘new’ arrows in \mathcal{Q} , and I is an admissible ideal containing the paths $\alpha_i\beta$ for all arrows β in \mathcal{Q}'' starting in non-sources such that, moreover, $I \cap K\mathcal{Q}' = I'$ and $I \cap K\mathcal{Q}'' = I''$. A 3-stack of algebras $\Delta_0, \Delta_1, \Delta_2$ can thus be roughly visualized in the form

$$\begin{array}{ccc} \boxed{\Delta_2} & & \\ \Downarrow & \cdots & \Downarrow \\ \boxed{\Delta_1} & & \\ \Downarrow & \cdots & \Downarrow \\ \boxed{\Delta_0} & & \end{array}$$

which motivates the terminology ‘stacking’.

In case Λ is given in terms of a quiver and a generating set for I , the problem of recognizing stacking partitions of E can often be resolved by mere inspection of the data. In general, E will have many different stacking partitions, a fact that can be used to

advantage in obtaining a maximum of homological information about Λ ; this is witnessed by our applications. To motivate condition (b) in the definition of a stacking partition, we point to the following obvious fact: Whenever M is a submodule of the radical of a projective module, we have $e(M/JM) = 0$ for all sources e of Λ ; in other words, the set $E(M)$ of Theorem 1 consists of non-sources in this situation.

Moreover, we will see that condition (b) ensures that, on the level of second syzygies, one obtains good separation of the ‘contributions’ from the layers of a 2-stack. This condition aims specifically at applications involving monomial algebras as building blocks, since their second syzygies become structurally transparent. If only the third or higher syzygies over the algebras one wishes to stack are known to have good properties, it is advantageous to relax our key definition as follows: For $c \in \mathbb{N}$, a partition $E = E' \cup E''$ is a *stacking partition of complexity c* in case (a) holds and condition (b) is relaxed as follows: Given any arrow $\alpha \in E' \cdot (K\mathcal{Q}) \cdot E''$, the product $\alpha\beta$ belongs to I for all arrows β starting in the endpoint of a path of length c in \mathcal{Q} . In this sense, our stacking partitions are of complexity 1.

For the remainder of this section, we assume that $E = E' \cup E''$ is a stacking partition of Λ with both E' and E'' nonempty. We retain the notation \mathbf{e}' , \mathbf{e}'' , Λ' , and Λ'' from part (2) of the definition. Moreover, we let \mathcal{Q}' and \mathcal{Q}'' be the full subquivers of \mathcal{Q} with vertex sets E' and E'' , respectively. Clearly,

$$\Lambda' \cong K\mathcal{Q}'/I \cap K\mathcal{Q}' \quad \text{and} \quad \Lambda'' \cong K\mathcal{Q}''/I \cap K\mathcal{Q}'',$$

and the Jacobson radicals of these algebras are $J' = \mathbf{e}'J\mathbf{e}'$ and $J'' = \mathbf{e}''J\mathbf{e}''$, respectively. On the side, we note that, whenever \mathcal{Q}'' contains a loop, Λ does not result from iterated one-point extensions of Λ' .

Observe that, for any $N \in \Lambda\text{-Mod}$, the Λ' -component $\mathbf{e}'N$ is a Λ -submodule of N . The analogous statement for $\mathbf{e}''N$ is obviously false; it even fails for first syzygies of Λ -modules. In general, we only have a K -vectorspace decomposition $\Omega^1(N) = \mathbf{e}'\Omega^1(N) \oplus \mathbf{e}''\Omega^1(N)$. However, on the level of second syzygies in $\Lambda\text{-Mod}$, we obtain nice splittings into Λ' and Λ'' -components. This is the pivotal point in relating the homological properties of stacks to those of their building blocks.

Proposition 3. *If X is a second syzygy of a left Λ -module N , then both $\mathbf{e}'X$ and $\mathbf{e}''X$ are Λ -submodules of X , and thus the K -vector space decomposition*

$$X = \mathbf{e}'X \oplus \mathbf{e}''X$$

is a Λ -direct sum.

In particular, $\text{p dim}_\Lambda N < \infty \iff \text{p dim}_\Lambda \mathbf{e}'X < \infty$ and $\text{p dim}_\Lambda \mathbf{e}''X < \infty$.

Proof. We only need to show that $\mathbf{e}''X$ is a Λ -submodule of X . By hypothesis, X is the kernel of a map $f : P \rightarrow M$, where M is a submodule of the radical of a projective module. This entails $e(M/JM) = e(P/JP) = 0$ for all sources e of \mathcal{Q} . Consequently, condition (b) of a stacking partition implies the following: Whenever α is an arrow in \mathcal{Q} and $x \in \mathbf{e}''X \subseteq P$, we have $\alpha x = \mathbf{e}''\alpha x$. Indeed, x is a linear combination of elements

$p_i x_i$, where the x_i are top elements of P and the p_i paths of positive lengths starting in non-sources and ending in E'' ; hence each product αp_i is either zero in Λ or else a path ending in E'' . \square

This focuses the discussion on the question of how the projective dimensions of the components $\mathbf{e}'X$ and $\mathbf{e}''X$, viewed as Λ' - and Λ'' -modules respectively, relate to their Λ -projective dimensions. For $\mathbf{e}'X$ this is obvious – we will nevertheless record it – for $\mathbf{e}''X$ it is a far more intricate problem.

The following straight-forward lemma only uses the fact that $\mathbf{e}''\Lambda\mathbf{e}' = 0$.

Lemma 4.

(1) *If N is any left Λ -module, the minimal projective resolution of $\mathbf{e}'N$ over Λ' coincides with the minimal projective resolution of $\mathbf{e}'N$ over Λ , i.e., $\Omega_{\Lambda'}^i(\mathbf{e}'N) = \Omega_{\Lambda}^i(\mathbf{e}'N)$ for all i . (Note, however, that, given a projective Λ -module Q , its Λ' -component $\mathbf{e}'Q$ need not be projective as a Λ - or, equivalently, as a Λ' -module.)*

(2) *Given any left Λ -module N and a Λ -projective cover $g : Q \rightarrow N$ with kernel M , the restriction $\mathbf{e}''g : \mathbf{e}''Q \rightarrow \mathbf{e}''N$ is a Λ'' -projective cover of $\mathbf{e}''N$ with kernel $\mathbf{e}''M$.*

In particular, $\Omega_{\Lambda''}^i(\mathbf{e}''N) = \mathbf{e}''\Omega_{\Lambda}^i(N)$ for all $i \geq 0$.

Proof. The first claim is obvious. For part (2), we observe that the algebra Λ'' , viewed as a left module over itself, has the following decomposition into left ideals, by the definition of a stacking partition: Namely $\Lambda'' = \bigoplus_{e \in E''} e\Lambda\mathbf{e}'' = \bigoplus_{e \in E''} e\Lambda = \bigoplus_{e \in E'', f \in E} e\Lambda f$. That $\mathbf{e}''J = \mathbf{e}''J\mathbf{e}''$ and hence $\mathbf{e}''M \subseteq J''(\mathbf{e}''Q)$, has similar reasons. \square

Noting that Proposition 3 carries over to direct summands of second syzygies in Λ -Mod, we derive the following consequence.

Proposition 5.

(1) *For any left Λ -module N ,*

$$\text{p dim}_{\Lambda} \mathbf{e}'N = \text{p dim}_{\Lambda'} \mathbf{e}'N.$$

(2) *Suppose that X is any Λ -direct summand of a second syzygy in Λ -Mod. Then*

(a) $\text{p dim}_{\Lambda} \mathbf{e}''X \geq \text{p dim}_{\Lambda''} \mathbf{e}''X$.

(b) *If $\text{p dim}_{\Lambda} \mathbf{e}''X < \infty$, then*

$$\text{p dim}_{\Lambda} \mathbf{e}''X \leq \text{p dim}_{\Lambda''} \mathbf{e}''X + \mathbf{t} + 1,$$

where $\mathbf{t} = \max\{\text{p dim}_{\Lambda'} \mathbf{e}'\Lambda e \mid e \in E'' \setminus \{\text{sources of } \mathcal{Q}''\}\}$ and $\text{p dim}_{\Lambda'} \mathbf{e}'\Lambda e < \infty\}$ in case the relevant set is nonempty, and $\mathbf{t} = -1$ otherwise.

(c) *If $\text{p dim}_{\Lambda} \mathbf{e}''X = \infty$, then either $\text{p dim}_{\Lambda''} \mathbf{e}''X = \infty$, or else there exists an idempotent $e \in E''$ such that $\text{p dim}_{\Lambda'} \mathbf{e}'\Lambda e = \infty$.*

Proof. (2) From Proposition 3 we know that $\mathbf{e}''X$ is a Λ -submodule of X . Thus (a) is an immediate consequence of Lemma 4.

For part (b), suppose $\text{p dim}_{\Lambda} \mathbf{e}''X < \infty$. It is, moreover, harmless to assume that $\text{p dim}_{\Lambda''} \mathbf{e}''X < \infty$. If $f_0 : P_0 \rightarrow \mathbf{e}''X$ is a Λ -projective cover, then clearly $e(P_0/JP_0) = 0$

for all sources e of \mathcal{Q} , and hence the simple summands of P_0/JP_0 correspond to idempotents in $E'' \setminus \{\text{sources of } \mathcal{Q}''\}$. Using once more Lemma 4 and Proposition 3, we further see that $\Omega_{\Lambda''}^1(\mathbf{e}''X) = \mathbf{e}''\Omega_{\Lambda}^1(\mathbf{e}''X)$ is a Λ -submodule of $\Omega_{\Lambda}^1(\mathbf{e}''X)$ and, in view of $\mathbf{e}'P_0 \subseteq \ker(f_0)$, we obtain a Λ -direct decomposition

$$\Omega_{\Lambda}^1(\mathbf{e}''X) = \mathbf{e}'P_0 \oplus \Omega_{\Lambda''}^1(\mathbf{e}''X).$$

Our hypothesis that $\text{p dim}_{\Lambda} \mathbf{e}''X$ be finite ensures that $\text{p dim}_{\Lambda'} \mathbf{e}'P_0 \leq \mathbf{t}$. So, if $\text{p dim}_{\Lambda''} \mathbf{e}''X = 0$, the desired inequality follows. Otherwise, we repeat the preceding argument with $\Omega_{\Lambda}^1(\mathbf{e}''X)$ instead of X , to obtain

$$\Omega_{\Lambda}^2(\mathbf{e}''X) = \mathbf{e}'P_1 \oplus \Omega_{\Lambda''}^2(\mathbf{e}''X),$$

where P_1 is a Λ -projective cover of $\mathbf{e}''\Omega_{\Lambda}^1(\mathbf{e}''X)$. (This is a legitimate move, because $\mathbf{e}''X$ is a Λ -direct summand of X , and hence $\Omega_{\Lambda}^1(\mathbf{e}''X)$ again satisfies the blanket hypothesis of (2).) Thus our claim is also true in case $\text{p dim}_{\Lambda''} \mathbf{e}''X = 1$. An obvious induction now completes the argument.

Part (c) is obtained analogously. \square

It is now easy to deduce that the little finitistic dimensions of stacks are governed by the inequalities mentioned in the introduction. Since

$$\Lambda = \begin{pmatrix} \Lambda'' & 0 \\ \mathbf{e}'\Lambda\mathbf{e}'' & \Lambda' \end{pmatrix},$$

these inequalities actually hold in far greater generality, as the methods of [3, Corollary 4.21] show. For the reader's convenience, we include an argument for our special case.

Corollary 6. *The left little finitistic dimension of Λ satisfies the inequalities*

$$\text{fin dim } \Lambda' \leq \text{fin dim } \Lambda \leq \text{fin dim } \Lambda' + \text{fin dim } \Lambda'' + 1.$$

Moreover,

$$\text{fin dim}_n \Lambda' \leq \text{fin dim}_n \Lambda$$

for all $n \in \mathbb{N}$.

Proof. The final set of inequalities is obvious, as is the fact that $\text{fin dim } \Lambda' \leq \text{fin dim } \Lambda$. To check the upper bound on $\text{fin dim } \Lambda$, let $N \in \Lambda\text{-mod}$ with $\text{p dim}_{\Lambda} N < \infty$. Repeated use of Lemma 4 yields $\Omega_{\Lambda''}^2(\mathbf{e}''N) = \Omega_{\Lambda''}^1(\mathbf{e}''\Omega_{\Lambda}^1(N)) = \mathbf{e}''\Omega_{\Lambda}^2(N) = \mathbf{e}''X$ if $X = \Omega_{\Lambda}^2(N)$. By induction,

$$\Omega_{\Lambda''}^k(\mathbf{e}''N) = \mathbf{e}''\Omega_{\Lambda}^k(N) = \mathbf{e}''\Omega_{\Lambda}^{k-2}(X)$$

for all $k \geq 2$. In particular, $\mathbf{e}''N$ has finite projective dimension over Λ'' , which shows $\text{p dim}_{\Lambda''} \mathbf{e}''N \leq \text{fin dim } \Lambda''$. We first deal with the case where $\text{fin dim } \Lambda'' = 0$. In this situation, $\text{p dim}_{\Lambda''} \mathbf{e}''N = 0$, whence $\mathbf{e}''\Omega_{\Lambda}^1(N) = 0$ by Lemma 4; this makes $\Omega_{\Lambda}^1(N)$ a Λ' -module with $\text{p dim}_{\Lambda} \Omega_{\Lambda}^1(N) = \text{p dim}_{\Lambda'} \Omega_{\Lambda}^1(N)$ and thus ensures $\text{fin dim } \Lambda \leq \text{fin dim } \Lambda' + 1$.

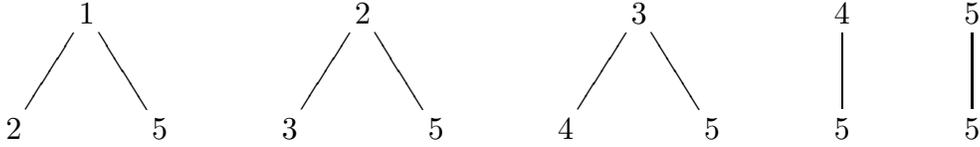
Now suppose that $\text{fin dim } \Lambda'' \geq 1$. Without loss of generality, we may assume that $\text{p dim}_{\Lambda''} \mathbf{e}''N \geq 2$, for otherwise $\mathbf{e}''X$ would vanish and $\text{p dim}_{\Lambda} N$ would be bounded above by the sum $2 + \text{p dim}_{\Lambda} \mathbf{e}'X = 2 + \text{p dim}_{\Lambda'} \mathbf{e}'X$, which is at most $\text{fin dim } \Lambda' + 2$. Hence $\text{p dim}_{\Lambda} N = 2 + \max\{\text{p dim}_{\Lambda} \mathbf{e}'X, \text{p dim}_{\Lambda} \mathbf{e}''X\}$. Now $\text{p dim}_{\Lambda} \mathbf{e}'X = \text{p dim}_{\Lambda'} \mathbf{e}'X$ is bounded above by $\text{fin dim } \Lambda'$, and $\text{p dim}_{\Lambda} \mathbf{e}''X \leq \text{p dim}_{\Lambda''} \mathbf{e}''X + \mathbf{t} + 1$ by Proposition 5. The final term equals $(\text{p dim}_{\Lambda''} \mathbf{e}''N - 2) + \mathbf{t} + 1$ which is in turn bounded from above by $(\text{fin dim } \Lambda'' - 2) + \text{fin dim } \Lambda' + 1$. Consequently,

$$\text{p dim}_{\Lambda} N \leq \text{fin dim } \Lambda' + \text{fin dim } \Lambda'' + 1$$

as required. \square

In general, $\text{fin dim } \Lambda''$ is not a lower bound for $\text{fin dim } \Lambda$ – examples to the contrary are ubiquitous.

Example 7. Suppose that Λ is a monomial algebra such that the indecomposable projectives in Λ -mod have the following graphs:



Then the vertex sets $E' = \{5\}$ and $E'' = E \setminus E'$ define a stacking partition of Λ , with $\text{fin dim } \Lambda = \text{fin dim } \Lambda' = 0$, and $\text{gl dim } \Lambda'' = 3$.

Already the case where the underlying graph of \mathcal{Q} is a Dynkin diagram of type A_n yields examples showing the upper bound on $\text{fin dim } \Lambda$ of Corollary 6 to be optimal, even when all algebras involved have finite global dimensions.

Example 8. Suppose that \mathcal{Q} has underlying graph A_5 with arrows $i \rightarrow i + 1$, and set $\Lambda = K\mathcal{Q}/I$, where I is generated by all paths of length 2. Consider the stacking partition $E' = \{4, 5\}$ and $E'' = \{1, 2, 3\}$ of Λ , and observe, that $\text{gl dim } \Lambda' = 1$, $\text{gl dim } \Lambda'' = 2$, and $\text{gl dim } \Lambda = \text{gl dim } \Lambda' + \text{gl dim } \Lambda'' + 1$.

We add a few specialized comments addressing stacks of monomial algebras for use in the next section. Suppose that $\Lambda'' = K\mathcal{Q}''/(K\mathcal{Q}'' \cap I)$ is a monomial algebra, and, as in Theorem 1, consider the set of critical paths of Λ'' :

$$\mathcal{P}'' = \{p \in \Lambda'' \mid p \text{ is a nontrivial path of positive length, starting in a non-source of } \mathcal{Q}'', \text{ with } \text{p dim}_{\Lambda''} \Lambda''p < \infty\}.$$

Moreover, let \mathbf{s}'' be the supremum of the projective dimensions attained on this set, as in Theorem 1. Since, given any $N \in \Lambda$ -Mod, the Λ'' -module $\mathbf{e}''N$ has second syzygy $\Omega_{\Lambda''}^2(\mathbf{e}''N) \cong \mathbf{e}''\Omega_{\Lambda}^2(N)$, that theorem shows $\mathbf{e}''\Omega_{\Lambda}^2(N)$ to split into a direct sum of cyclic

modules isomorphic to left ideals $\Lambda''q$ for suitable paths q in Λ'' . Clearly, the $\Lambda''q$ are even left ideals of Λ , i.e. $\Lambda''q = \Lambda q$, since the eligible paths q start in non-sources. Yet, in general, the Λ -projective dimensions of the $\Lambda''q$ will still exceed their Λ'' -projective dimensions; in particular, the former may be infinite while the latter are finite (see Example 7 above). In the construction we are targeting, this problem does not arise, however, since all of the end points of the paths in \mathcal{P}'' are ‘homogeneous’ in the following sense: A vertex $e \in E''$ is called *homogeneous* in case all arrows of \mathcal{Q} starting in e end in E'' . If all paths in \mathcal{P}'' end in homogeneous vertices, then clearly $\text{p dim}_\Lambda \Lambda''p = \text{p dim}_{\Lambda''} \Lambda''p$ for all $p \in \mathcal{P}''$. Combining these considerations with the preceding results, we obtain:

Corollary 9. *Let Λ'' be a monomial algebra. Retaining the above notation, suppose that all end points of the paths in \mathcal{P}'' are homogeneous. If N is any left Λ -module of finite projective dimension, then the Λ'' -projective dimension of the syzygy $\Omega_{\Lambda''}^2(\mathbf{e}''N)$ coincides with its Λ -projective dimension.*

In particular: If N is finitely generated and $\text{p dim}_\Lambda N > \text{fin dim } \Lambda' + 2$, then $\text{p dim}_{\Lambda''} \mathbf{e}''N = \text{p dim}_\Lambda N$.

Proof. Proposition 3 yields a Λ -direct decomposition

$$\Omega_\Lambda^2(N) = \mathbf{e}'\Omega_\Lambda^2(N) \oplus \mathbf{e}''\Omega_\Lambda^2(N),$$

which shows $\Omega_{\Lambda''}^2(\mathbf{e}''N) = \mathbf{e}''\Omega_\Lambda^2(N)$ to have finite projective dimension as a Λ -module. By Proposition 5, we infer that $\Omega_{\Lambda''}^2(\mathbf{e}''N)$ has finite projective dimension also as a Λ'' -module, and consequently Theorem 1 shows this syzygy to decompose in the form

$$\Omega_{\Lambda''}^2(\mathbf{e}''N) = \bigoplus_{i \in I} \Lambda''p_i,$$

where the p_i are paths in \mathcal{P}'' . Using our homogeneity hypothesis, we thus conclude $\text{p dim}_{\Lambda''} \Omega_{\Lambda''}^2(\mathbf{e}''N) = \text{p dim}_\Lambda \Omega_{\Lambda''}^2(\mathbf{e}''N)$ as required. For the final assertion, one just has to keep in mind that the Λ -projective dimension of $\mathbf{e}'\Omega_\Lambda^2(N)$ equals the Λ' -projective dimension (Proposition 5(1)). \square

4. THE KEY EXAMPLES

Theorem 10. *Given any increasing function $f : \mathbb{N} \rightarrow \mathbb{N}_{\geq 2}$ that takes only finitely many distinct values, there exists a connected finite dimensional algebra $\Lambda = K\mathcal{Q}/I$ with vanishing radical cube such that*

$$\text{fin dim}_n \Lambda = f(n) \text{ for } n \in \mathbb{N}.$$

Moreover, if $d = \max f - \min f$, then Λ can be constructed as a $(\lceil d/2 \rceil + 1)$ -stack of monomial algebras with the additional property that, for each $n \in \mathbb{N}$, $\text{fin dim}_n \Lambda$ is attained on an n -generated module of Loewy length at most 2 having a tree graph.

Our proof consists of a recursive construction technique involving d stages, where d is as in the theorem. To avoid overly cumbersome notation, we will only deal with two types of step functions, the first exhibiting a single jump of size s , the other involving two jumps of sizes s and t , respectively. The general recursive pattern is clear from our constructions.

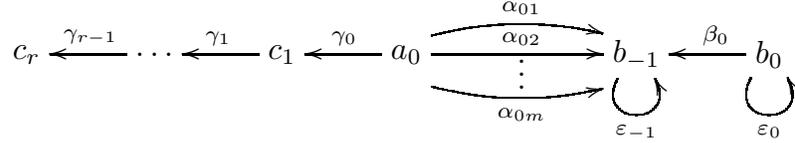
Part I of the proof of Theorem 10. Fix $m, r, s \in \mathbb{N}$ with $m, r \geq 2$, and assume that $f(k) = r$ for $k \leq m - 1$, while $f(k) = r + s$ for $k \geq m$.

We start by constructing a sequence $\Lambda_0, \Lambda_1, \dots, \Lambda_s$ of finite dimensional algebras, where Λ_0 is a monomial algebra and each Λ_ℓ for $\ell \geq 1$ results from stacking another monomial algebra on top of $\Lambda_{\ell-1}$. Then we will show that, for $0 \leq \ell \leq s$,

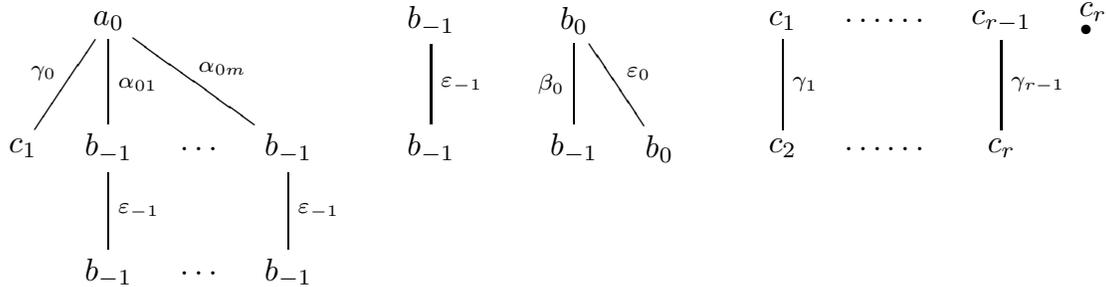
$$(\dagger) \quad \text{fin dim}_k \Lambda_\ell = r \quad \text{for } k \leq m - 1 \quad \text{and} \quad \text{fin dim}_k \Lambda_\ell = r + \ell \quad \text{for } k \geq m.$$

Note that, for the present choice of f , the difference $\max f - \min f$ equals s . When we assemble the building blocks for our argument under the ‘final claim of Part I’, we will explain how the algebra $\Lambda = \Lambda_s$ can be obtained through a more economical stacking of only $\lceil s/2 \rceil + 1$ monomial layers.

Step 0. We base Λ_0 on the following quiver $\mathcal{Q}^{(0)}$:



The algebra Λ_0 results from $K\mathcal{Q}^{(0)}$ by factoring out the ideal generated by the relations ε_i^2 for $i = -1, 0$, as well as $\varepsilon_{-1}\beta_0, \beta_0\varepsilon_0$ and $\gamma_j\gamma_{j-1}$ for $1 \leq j \leq r-1$. Thus the indecomposable projective left Λ_0 -modules $\Lambda_0 a_0, \Lambda_0 b_i$, and $\Lambda_0 c_i$ have graphs

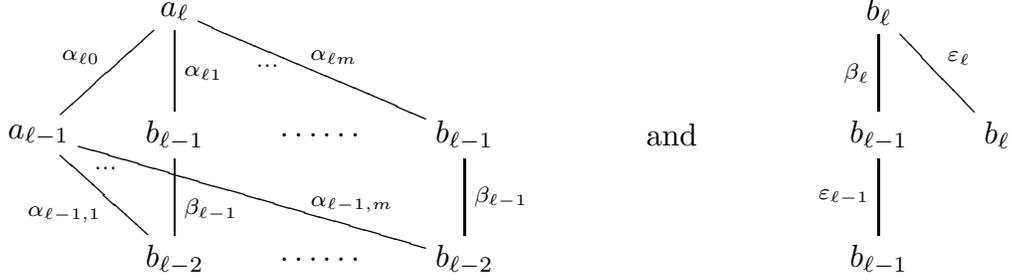


respectively.

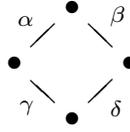
Step ℓ (for $\ell \geq 1$). The algebra Λ_ℓ is based on a quiver $\mathcal{Q}^{(\ell)}$ having vertex set $\mathcal{Q}_0^{(\ell)} = \mathcal{Q}_0^{(\ell-1)} \cup \{a_\ell, b_\ell\}$ and additional arrows

$$\alpha_{\ell 0} : a_\ell \rightarrow a_{\ell-1}, \quad \alpha_{\ell 1}, \dots, \alpha_{\ell m} : a_\ell \rightarrow b_{\ell-1}, \quad \beta_\ell : b_\ell \rightarrow b_{\ell-1}, \quad \varepsilon_\ell : b_\ell \rightarrow b_\ell.$$

The graphs of the indecomposable projective Λ_ℓ -modules $\Lambda_\ell e$, where e is a vertex of $\mathcal{Q}^{(\ell-1)}$, are the same as those of the corresponding $\Lambda_{\ell-1}$ -modules $\Lambda_{\ell-1}e$, and the ‘new’ indecomposable projective Λ_ℓ -modules $\Lambda_\ell a_\ell$ and $\Lambda_\ell b_\ell$ have graphs



respectively; here each subgraph of the form



of the graph of $\Lambda_\ell a_\ell$ is to reflect a relation $\gamma\alpha - \delta\beta$ of Λ_ℓ . (In Section 2, we explained how the graphs of $\Lambda_1 a_0$ and $\Lambda_1 a_1$ are to be interpreted.)

Our proof for the asserted values of the finitistic dimensions of the Λ_ℓ (see \dagger) above will be deduced from the following sequence of lemmas. As a prerequisite for applying the results of Section 3, we only need the obvious fact that each Λ_ℓ is an ℓ -stack of monomial algebras; if $E = \mathcal{Q}_0^{(\ell)}$ is the vertex set of $\mathcal{Q}^{(\ell)}$, we use the stacking partition $E' = \mathcal{Q}_0^{(\ell-1)}$ and $E'' = \{a_\ell, b_\ell\}$ of E . In particular, this means that Λ_k -mod embeds into Λ_ℓ -mod for $k < \ell$; that, for any Λ_k -module X , the Λ_ℓ -projective cover of X coincides with the Λ_k -projective cover (Lemma 4), and hence that $\text{p dim}_{\Lambda_k} X = \text{p dim}_{\Lambda_\ell} X$. Consequently, there is no need to emphasize the base algebra in dealing with projective covers and computing projective dimensions. We will denote the Jacobson radical of Λ_ℓ by J_ℓ . Note that $J_k X = J_\ell X$ for all Λ_k -modules X by Section 3.

Lemma 11. *Let $\ell \geq 0$, and suppose that N is a $\Lambda_{\ell+1}$ -module of finite projective dimension. Then $b_{\ell+1}\Omega^1(N) = 0$; in particular, $\Omega^1(N)$ is a Λ_ℓ -module and $\text{fin dim } \Lambda_{\ell+1} \leq 1 + \text{fin dim } \Lambda_\ell$.*

Proof. If Q is a projective cover of N , write $Q = Q_1 \oplus Q_2$, where Q_2 is a direct sum of copies of $\Lambda_{\ell+1}b_{\ell+1}$ and Q_1 has no direct summands in common with Q_2 . Then $b_{\ell+1}JQ = b_{\ell+1}JQ_2 = \varepsilon_{\ell+1}Q_2$ is a direct sum of copies of $S(b_{\ell+1})$ and a $\Lambda_{\ell+1}$ -direct summand of JQ . Since $\text{p dim } S(b_{\ell+1}) = \infty$, this implies $b_{\ell+1}\Omega^1(N) = 0$. Using $\alpha_{\ell+1}J_{\ell+1} = 0$, we infer that $\Omega^1(N)$ is indeed a Λ_ℓ -module. The final claim is thus obvious. \square

Lemma 12. *Let $\ell \geq 0$, and suppose N is a $\Lambda_{\ell+1}$ -module of finite projective dimension containing a top element y of type $a_{\ell+1}$ such that $\alpha_{\ell+1,0}y = 0$. Then N/JN contains the simple module $S(b_{\ell+1})$ with multiplicity at least m .*

Proof. Let $\psi : Q = \Lambda_{\ell+1}\tilde{y} \oplus \tilde{Q} \rightarrow N$ be a projective cover, where \tilde{y} is a top element of type $a_{\ell+1}$ of Q , such that $\psi(\tilde{y}) = y$. Moreover, set $M = \ker(\psi)$, i.e., $M \cong \Omega^1(N)$. Then M is

a Λ_ℓ -module by Lemma 11, and $x = \alpha_{\ell+1,0}\tilde{y}$ is a top element of type a_ℓ of M . Finally, let $\phi : P = \Lambda\tilde{x} \oplus \tilde{P} \rightarrow M$ be a projective cover of M , where $\tilde{x} \in P$ is a top element of type a_ℓ with $\phi(\tilde{x}) = x$.

We focus on the case $\ell \geq 1$ in the sequel, the argument for $\ell = 0$ being analogous, modulo small adjustments. (Due to the difference in make-up of $\Lambda_\ell a_\ell$ for $\ell \geq 1$ and $\ell = 0$, in the latter case, the element $\varepsilon_{-1}\alpha_{0i}\tilde{x}$ takes over the role played by $\beta_{\ell-1}\alpha_{\ell i}\tilde{x}$ below.)

Fix $i \in \{1, \dots, m\}$. Since $x = a_\ell x$ belongs to $J_{\ell+1}\tilde{y} \cong J_{\ell+1}a_{\ell+1}$, we see that $\alpha_{\ell i}x$ is nonzero and generates a copy of $S(b_{\ell-1})$ in the socle of M , whereas $\beta_{\ell-1}\alpha_{\ell i}x$ is zero. Consequently, $\alpha_{\ell i}\tilde{x} \notin \ker(\phi) = \Omega^2(N)$, whereas $\beta_{\ell-1}\alpha_{\ell i}\tilde{x} \in \text{soc ker}(\phi) \setminus \{0\}$. If $\beta_{\ell-1}\alpha_{\ell i}\tilde{x}$ were a top element of $\ker(\phi)$, necessarily of type $b_{\ell-2}$, this element would generate a direct summand isomorphic to $S(b_{\ell-2})$ in $\ker(\phi) = \Omega^2(N)$, clearly an impossibility. Therefore, $\beta_{\ell-1}\alpha_{\ell i}\tilde{x}$ lies in $J_\ell \ker(\phi) = J_{\ell-1} \ker(\phi)$; in fact, $\beta_{\ell-1}\alpha_{\ell i}\tilde{x}$ belongs to $J_{\ell-1} \ker(\phi) \cap \Lambda_{\ell-1}\alpha_{\ell i}\tilde{x}$ and therefore also to $J_{\ell-1}\pi(\ker(\phi)) \subseteq J_{\ell-1}J_\ell\tilde{x}$, where $\pi : P \rightarrow \Lambda\tilde{x}$ is the projection along \tilde{P} . Inspection of the graphs of the indecomposable projective Λ_ℓ -modules thus reveals the existence of a top element z_i of $J_{\ell-1}P$ with the following properties: (1) $\alpha_{\ell i}\tilde{x} - z_i$ is a top element of $\ker(\phi)$ of type $b_{\ell-1}$, and (2) $\beta_{\ell-1}(\alpha_{\ell i}\tilde{x} - z_i) = \beta_{\ell-1}\alpha_{\ell i}\tilde{x}$. By (1), the top element z_i of $J_{\ell-1}P$ is of type $b_{\ell-1}$. From (2) we infer that $z_i = \beta_\ell \tilde{x}_i$ for some top element \tilde{x}_i of type b_ℓ of P . Thus x and $x_i = \phi(\tilde{x}_i)$ are top elements of M having types a_ℓ and b_ℓ , respectively, with $\alpha_{\ell i}x = \beta_\ell x_i$. In particular, $\beta_\ell x_i$ belongs to $M \cap \Lambda_{\ell+1}\tilde{y} \subseteq \Lambda_{\ell+1}a_{\ell+1}$. So, if $\sigma : Q \rightarrow \Lambda_{\ell+1}\tilde{y}$ is the projection along \tilde{Q} , then $\alpha_{\ell i}x = \beta_\ell \sigma(x_i)$, and the structure of $\Lambda_{\ell+1}a_{\ell+1}$ guarantees $\sigma(x_i) = \alpha_{\ell+1,i}\tilde{y}$. From $\beta_\ell(x_i - \sigma(x_i)) = 0$ we moreover glean $x_i - \sigma(x_i) = \beta_{\ell+1}\tilde{y}_i$, where \tilde{y}_i is either zero or a top element of type $b_{\ell+1}$ of Q .

Next we check that $\tilde{y}_1, \dots, \tilde{y}_m$ are actually top elements of Q which are K -linearly independent modulo JQ : Indeed, the $x_i = \sigma(x_i) + \beta_{\ell+1}\tilde{y}_i$ for $1 \leq i \leq m$ are linearly independent top elements of type b_ℓ of M , since the multiples $\beta_\ell x_i$ are linearly independent by construction. We infer that the elements

$$\varepsilon_\ell x_i = \varepsilon_\ell \beta_{\ell+1} \tilde{y}_i, \quad 1 \leq i \leq m$$

generate m independent copies of $S(b_\ell)$ in the socle of M ; for otherwise we would obtain $b_\ell \Omega^1(M) \neq 0$, contradicting Lemma 11. This forces $\tilde{y}_1, \dots, \tilde{y}_m$ to be linearly independent modulo JQ and thus gives rise to m top elements $y_i = \psi(\tilde{y}_i)$ of N which are linearly independent modulo JN . \square

Lemma 13. *Let $\ell \geq 1$, and suppose M is a Λ_ℓ -module of Loewy length 2 with $a_\ell M = 0$, but $b_\ell M \neq 0$. Then $\text{p dim } M = \infty$.*

Proof. By hypothesis, the projective cover of M does not contain a summand $\Lambda_\ell a_\ell$, but does contain a copy of $\Lambda_\ell b_\ell$. Inspection of the graphs of $\Lambda_\ell b_\ell$ and $\Lambda_\ell b_{\ell-1}$ thus makes it clear that $\Omega^1(M)$ either has a direct summand isomorphic to $S(b_{\ell-1})$, or else a direct

summand isomorphic to the module $X_{\ell-1}$ with graph $\begin{array}{c} b_{\ell-1} \\ | \\ b_{\ell-1} \end{array}$. We know $S(b_{\ell-1})$ to have

infinite projective dimension and compute $\Omega^{\ell-1}(X_{\ell-1}) = X_0$. Thus $\Omega^\ell(X_{\ell-1}) \cong S(b_{\ell-1})$, and we conclude that $X_{\ell-1}$ has infinite projective dimension as well. \square

Lemma 14. *Let $\ell \geq 1$, and suppose that N is an indecomposable non-projective $\Lambda_{\ell+1}$ -module of finite projective dimension such that $a_{\ell+1}N \neq 0$ or $b_{\ell+1}N \neq 0$. Then $a_\ell\Omega^1(N) \neq 0$.*

Proof. By Lemma 13, it suffices to show that either $a_\ell\Omega^1(N) \neq 0$ or $b_\ell\Omega^1(N) \neq 0$. Let $\psi : Q \rightarrow N$ be a projective cover of N , say $Q = Q_1 \oplus Q_2 \oplus Q_3$ where Q_1 is a direct sum of copies of $\Lambda_{\ell+1}a_{\ell+1}$, Q_2 a direct sum of copies of $\Lambda_{\ell+1}b_{\ell+1}$, and Q_3 has no direct summands in common with $Q_1 \oplus Q_2$. From Lemma 11, we know $b_{\ell+1}\Omega^1(N) = 0$. Assume, to the contrary of our claim, that $a_\ell\Omega^1(N) = b_\ell\Omega^1(N) = 0$, and note that $b_\ell\Omega^1(N) = b_{\ell+1}\Omega^1(N) = 0$ implies $\Omega^1(N) = \ker(\psi) \subseteq Q_1 \oplus Q_3$, which places a direct summand isomorphic to Q_2 into N . In view of the fact that N is indecomposable nonprojective, this entails $Q_2 = 0$. Moreover, our assumption forces $\ker(\psi)$ to be contained in $b_{\ell-1}Q_1 \oplus JQ_3$; however, $\ker(\psi) \not\subseteq JQ_3$, due to indecomposability of N - indeed $Q_2 = 0$ implies $Q_1 \neq 0$ by hypothesis. Observe moreover that $b_{\ell-1}Q_1 = \beta_\ell JQ_1$ is a nonzero direct sum of copies of $S(b_{\ell-1})$. Again we deduce that $\ker(\psi)$ either contains a copy of $S(b_{\ell-1})$ or else a copy of the module $X_{\ell-1}$ as introduced in the proof of Lemma 13. But as we argued before, this contradicts finiteness of $\text{p dim } N$ since $\ell - 1 \geq 0$. \square

Lemma 15. *Let $\ell \geq 2$, and suppose that N is an indecomposable non-projective Λ_ℓ -module of finite projective dimension such that $a_\ell N \neq 0$ or $b_\ell N \neq 0$. Then N/JN contains a copy of $(S(b_\ell))^m$. In particular, N requires at least m generators.*

Proof. Invoking Lemma 14, we obtain $a_{\ell-1}\Omega^1(N) \neq 0$. Inspection of the radicals of the indecomposable projective Λ_ℓ -modules now yields a top element of type a_ℓ in N which is annihilated by $\alpha_{\ell 0}$. Therefore our assertion follows from Lemma 12. \square

Lemma 16. *All finitistic dimensions of Λ_0 are equal to r , that is,*

$$\text{fin dim}_k \Lambda_0 = \text{fin dim } \Lambda_0 = \text{Fin dim } \Lambda_0 = r \quad \text{for all } k.$$

Proof. The inequality $\text{fin dim}_1 \Lambda_0 \geq r$ is due to the fact that $\text{p dim } S(a_0) = r$. To establish the inequality $\text{Fin dim } \Lambda_0 \leq r$, one checks that the \mathbf{s} -invariant of Λ_0 (as in Theorem 1) is $\text{p dim } \Lambda_0 \gamma_1 = \text{p dim } S(c_2) = r - 2$, and then applies Theorem 1. \square

Final claim of Part I. For the present choice of the target function f , the algebra $\Lambda = \Lambda_s$ satisfies all conditions listed in Theorem 10.

Proof of the final claim. By construction, $J^3 = 0$. To see that Λ is even an $(\lceil s/2 \rceil + 1)$ -stack of monomial algebras, consider the following alternate stacking partition of the set E of vertices of Λ : Namely, $E_0 = \mathcal{Q}_0^{(0)}$, and $E_\ell = \left(\mathcal{Q}_0^{(2\ell-1)} \cup \mathcal{Q}_0^{(2\ell)} \right) \setminus E_{\ell-1}$, whenever $\ell \geq 1$ and $2\ell \leq s$. If s is odd, we define $E_{\lceil s/2 \rceil}$ to be $\{a_s, b_s\} = \mathcal{Q}_0^{(s)} \setminus E_{(s-1)/2}$.

We now establish the equalities (\dagger) preceding the construction of the Λ_ℓ .

Returning to the notation employed in the construction of the algebras Λ_ℓ , we combine Lemma 16 with the last statement of Lemma 11 to obtain, via an obvious induction on ℓ , the following family of inequalities:

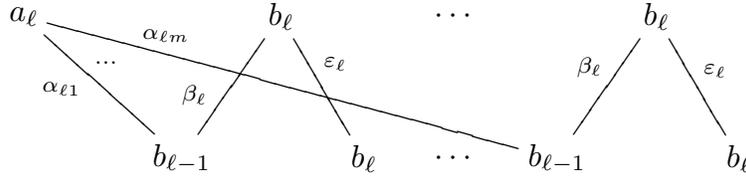
$$r = \text{fin dim}_k \Lambda_0 \leq \text{fin dim}_k \Lambda_\ell \leq \text{fin dim } \Lambda_\ell \leq \text{fin dim } \Lambda_0 + \ell = r + \ell,$$

for all $k \in \mathbb{N}$ and $\ell \leq s$.

Next we verify that $\text{findim}_m \Lambda_\ell \geq r + \ell$, which, in view of the above inequalities, will show that $\text{findim}_k \Lambda_\ell = r + \ell$ for all $k \geq m$. For each $\ell \geq 0$, we define a Λ_ℓ -module $N_\ell = P_\ell/V_\ell$ as follows:

$$P_\ell = \Lambda_\ell x_{\ell 0} \oplus \bigoplus_{i=1}^m \Lambda_\ell x_{\ell i}$$

with $x_{\ell 0} = a_\ell$ and $x_{\ell i} = b_\ell$ for $1 \leq i \leq m$, i.e., $P_\ell = \Lambda a_\ell \oplus (\Lambda b_\ell)^m$, and $V_\ell \subseteq J_\ell P_\ell$ is generated by $\gamma_0 x_{\ell 0}$ and the differences $\alpha_{\ell i} x_{\ell 0} - \beta_\ell x_{\ell i}$ for $1 \leq i \leq m$. Note that N_ℓ has Loewy length 2 and a tree graph, namely



relative to the obvious choice of top elements. Moreover, we see that, for $\ell \geq 1$, we have $\Omega^1(N_\ell) = N_{\ell-1}$, while $\Omega^1(N_0) = S(c_1) \oplus (\Lambda_0 b_{-1})^m$ has projective dimension $r - 1$ (see [4] for methodology). Hence $\text{p dim } N_\ell = r + \ell$.

So only $\text{findim}_{m-1} \Lambda_\ell \leq r$ for $1 \leq \ell \leq s$ remains to be checked. In light of Lemma 15, it suffices to verify this for $\ell = 1$. Indeed, if $\ell \geq 2$ and N is a module over Λ_ℓ , but not over $\Lambda_{\ell-1}$, then N fails to be annihilated by at least one of a_ℓ, b_ℓ .

We write $\Delta = \Lambda_1$ for ease of notation and consider another stacking partition of the vertex set E of Δ . Namely $E' = \{b_{-1}\}$ and $E'' = E \setminus E'$. With \mathbf{e}' and \mathbf{e}'' as in Section 3, we define $\Delta' = \mathbf{e}'\Delta\mathbf{e}'$ and $\Delta'' = \mathbf{e}''\Delta\mathbf{e}''$ and note that both Δ' and Δ'' are monomial algebras. It is readily seen that $\text{findim } \Delta' = 0$. Moreover, we observe that γ_0 is the only path $p \in \mathcal{P}''$ (the set of critical paths of the monomial algebra Δ'') such that the maximum \mathbf{s}'' of Theorem 1 is attained on $\Delta''p$. In particular, $\mathbf{s}'' = \text{p dim } \Delta''\gamma_0 = r - 1 \geq 1$. Finally, we note that all paths in \mathcal{P}'' end in homogeneous vertices.

If N is any Δ -module with $\text{p dim}_\Delta N = r + 1$, Corollary 9 therefore tells us that the Δ'' -projective dimension of $\Omega_{\Delta''}^2(\mathbf{e}''N)$ also equals \mathbf{s}'' . Consequently, Theorem 1 forces $\Omega_{\Delta''}^2(\mathbf{e}''N)$ to have a direct summand isomorphic to $\Delta''\gamma_0$. By the second part of Theorem 1, we further obtain a top element $y \in \mathbf{e}''N$ of type a_1 with $\alpha_{10}y = 0$. Clearly, y is also a top element of N , whence N requires at least m generators by Lemma 12. This proves $\text{findim}_{m-1} \Delta \leq r$ as required.

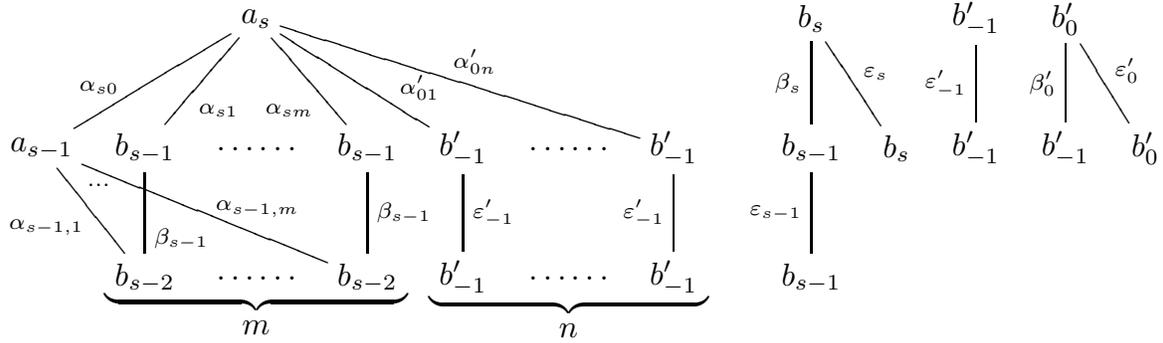
That the final condition of Theorem 10 is met, is clear from the above considerations.

Part II of the proof of Theorem 10. Fix $m, n, r, s, t \in \mathbb{N}$ with $2 \leq m < n$ and $r \geq 2$, and assume the target function f of Theorem 10 to be

$$\begin{aligned} f(k) &= r & \text{for } 1 \leq k \leq m-1, \\ f(k) &= r+s & \text{for } m \leq k \leq n-1, \\ \text{and } f(k) &= r+s+t & \text{for } k \geq n. \end{aligned}$$

For the construction of a finite dimensional algebra Λ having the finitistic dimensions prescribed by f , let $\Lambda_0, \Lambda_1, \dots, \Lambda_{s-1}$ be as in Part I. We slightly modify the final algebra Λ_s constructed before - its finitistic dimensions will remain unchanged - to ‘switch gear’ in our recursive pattern so as to smooth the road for another jump.

Step s. Given $\Lambda_0, \dots, \Lambda_{s-1}$ as before and keeping the notation pertaining to these algebras, we enlarge the quiver $\mathcal{Q}^{(s-1)}$ of Λ_{s-1} to $\mathcal{Q}^{(s)}$ as follows: We add the vertices a_s, b_s, b'_{-1}, b'_0 and supplement arrows and relations as indicated by the following graphs of the indecomposable projective Λ_s -modules:



Here we preserve our convention concerning non-monomial relations of Λ_s communicated by the graph of $\Lambda_s a_s$; namely, that each subgraph of type \tilde{A}_3 of the graph of $\Lambda_s a_s$ corresponds to a relation $\alpha_{s-1,i} \alpha_{s0} - \beta_{s-1} \alpha_{si}$ in the ideal I of relations.

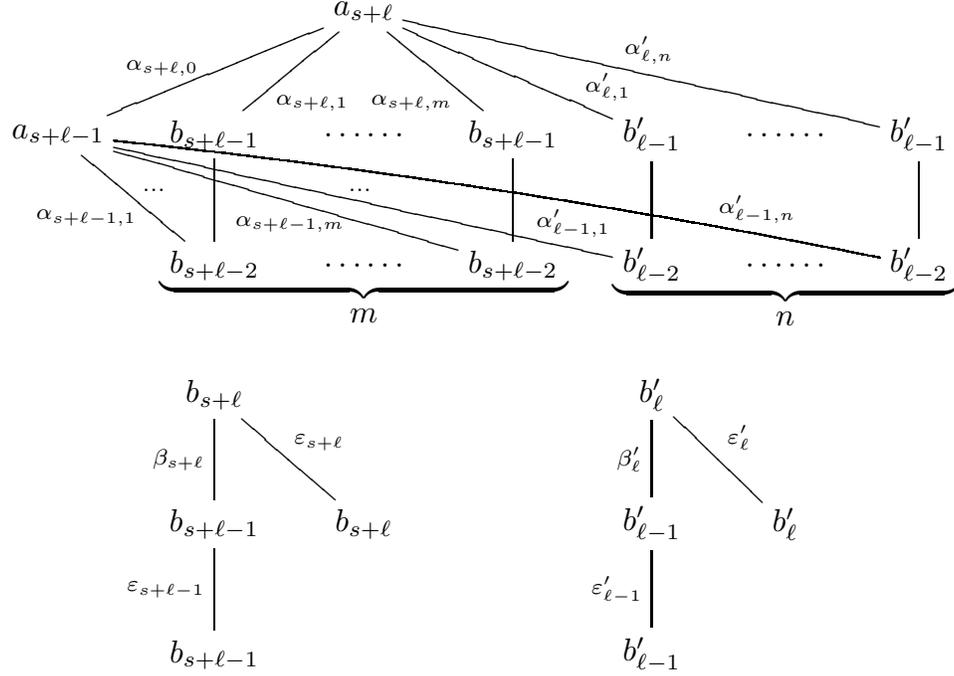
As before, $\text{fin dim}_k \Lambda_s \geq r$ for all k . To check that $\text{fin dim}_k \Lambda_s \leq r$ for $k \leq m - 1$, let N be a finitely generated indecomposable Λ_s -module of projective dimension $r + 1$. If $s = 1$, the argument given under the final claim of Step I shows N to require at least m generators. So suppose that $s \geq 2$. If we can show that $a_s N \neq 0$ or $b_s N \neq 0$, then again the reasoning of Step I provides what we need (Lemmas 14 and 15 carry over to the new format of Λ_s , with minor modifications of the arguments). So it suffices to consider the case where $a_s N = b_s N = 0$. Let \mathbf{e} be the sum of the primed primitive idempotents, that is, $\mathbf{e} = b'_{-1} + b'_0$. Our annihilation assumption ensures a direct-sum decomposition of N into Λ_s -modules N_1 and N_2 such that $\mathbf{e}N_1 = 0$ and $\mathbf{e}N_2 = N_2$. Since N is indecomposable and $\text{p dim}_{\Lambda_s} N_2 = \text{p dim}_{\mathbf{e}(\Lambda_s)\mathbf{e}} N_2 \leq \text{fin dim } \mathbf{e}(\Lambda_s)\mathbf{e} = 0$, we infer that $N = N_1$ is a Λ_{s-1} -module. In view of the known equality $\text{fin dim}_{m-1} \Lambda_{s-1} = r$, we thus again glean a minimum of m generators for N .

To see that $\text{fin dim}_k \Lambda_s = r + s$ for $k \geq m$, we can rely on the previous arguments. However, the presentation of the m -generated module $N_s = P_s/V_s$ that has the same graph as the module of that name in Part I needs to take the slightly altered structure of Λ_s into account as follows: Again,

$$P_s = \bigoplus_{i=0}^m \Lambda_s x_{si} \quad \text{with } x_{s0} = a_s \text{ and } x_{si} = b_s \text{ for } 1 \leq i \leq m,$$

but now V_s is the submodule generated by $\alpha_{s0}x_{s0}$, $\alpha_{si}x_{s0} - \beta_s x_{si}$, for $1 \leq i \leq m$, and $\alpha'_{0j}x_{s0}$ for $1 \leq j \leq n$. Clearly $\Omega^1(N_s) = N_{s-1} \oplus (\Lambda_s b'_{-1})^n$, whence $\text{p dim } N_s = r + s$.

Step $s + \ell$. for $\ell \geq 1$. We introduce three new vertices, $a_{s+\ell}$, $b_{s+\ell}$, and b'_ℓ , giving rise to indecomposable projective left $\Lambda_{s+\ell}$ -modules whose graphs revert to the mold of Part I.



Lemmas 11-15 of Part I have analogues applying to the present situation. We list them in a format that permits us to carry over the previous arguments almost verbatim.

Lemma 11'. *Let $\ell \geq 0$, and suppose that N is a $\Lambda_{s+\ell+1}$ -module of finite projective dimension. Then $b_{s+\ell+1}\Omega^1(N) = b'_{\ell+1}\Omega^1(N) = 0$; in particular, $\Omega^1(N)$ is a $\Lambda_{s+\ell}$ -module. \square*

Lemma 12'. *Let $\ell \geq 0$, and suppose N is a $\Lambda_{s+\ell+1}$ -module of finite projective dimension containing a top element y of type $a_{s+\ell+1}$ such that $\alpha_{s+\ell+1,0}y = 0$. Then N/JN contains $(S(b_{s+\ell+1}))^m$ and $(S(b'_{\ell+1}))^n$. \square*

Lemma 13'. *Let $\ell \geq 1$, and suppose M is a $\Lambda_{s+\ell}$ -module of Loewy length 2 with $a_{s+\ell}M = 0$, but $b_{s+\ell}M \neq 0$ or $b'_\ell M \neq 0$. Then $\text{p dim } M = \infty$. \square*

To prove the case $\ell = 0$ in the next lemma, note that, whenever $\text{p dim } N < \infty$ and $b'_1 N \neq 0$, the inequality $a_{s+1}N \neq 0$ is automatic; hence, the hypothesis boils down to ' $a_{s+1}N \neq 0$ or $b_{s+1}N \neq 0$ ' in that case.

Lemma 14'. *Let $\ell \geq 0$, and suppose that N is an indecomposable non-projective $\Lambda_{s+\ell+1}$ -module of finite projective dimension such that $a_{s+\ell+1}N \neq 0$ or $b_{s+\ell+1}N \neq 0$, or else $b'_{\ell+1}N \neq 0$. Then $a_{s+\ell}\Omega^1(N) \neq 0$. \square*

Lemma 15'. (a) Let $\ell \geq 1$, and suppose that N is an indecomposable non-projective $\Lambda_{s+\ell}$ -module of finite projective dimension with $a_{s+\ell}N \neq 0$ or $b_{s+\ell}N \neq 0$. Then N/JN contains a copy of $(S(b_{s+\ell}))^m$. In particular, N requires at least m generators.

(b) Let $\ell \geq 2$, and suppose that N is an indecomposable non-projective $\Lambda_{s+\ell}$ -module of finite projective dimension with $a_{s+\ell}N \neq 0$ or $b_{s+\ell}N \neq 0$ or $b'_\ell N \neq 0$. Then N/JN contains a copy of $(S(b'_\ell))^n$. In particular, N requires at least n generators. \square

Final claim of Part II. The algebra $\Lambda = \Lambda_{s+t}$ has finitistic dimensions $\text{fin dim}_k \Lambda = f(k)$ for $k \in \mathbb{N}$ and satisfies the additional conditions of Theorem 10.

Proof of the final claim. Here $d = s + t$. To see that Λ is a $(\lceil d/2 \rceil + 1)$ -stack of monomial algebras, we proceed as in Part I, by setting $E_0 = \mathcal{Q}_0^{(0)}$ and adding on the additional vertices of any pair $\Lambda_{2\ell-1}, \Lambda_{2\ell}$ for $2\ell \leq s + t$ in the following steps to move from $E_{\ell-1}$ to E_ℓ . Since every left Λ_s -module is also a $\Lambda_{s+\ell}$ -module, we see that

$$r = \text{fin dim}_k \Lambda_s \leq \text{fin dim}_k \Lambda_{s+\ell}$$

for $k \leq m - 1$, and as before we obtain

$$r + s = \text{fin dim}_k \Lambda_s \leq \text{fin dim}_k \Lambda_{s+\ell} \leq \text{fin dim } \Lambda_s + \ell = r + s + \ell$$

for $k \geq m$. So we only need to verify the following:

- (1) $\text{fin dim}_k \Lambda_{s+\ell} \leq r$ for $k \leq m - 1$ and all $\ell \geq 1$
- (2) $\text{fin dim}_k \Lambda_{s+\ell} \leq r + s$ for $m \leq k \leq n - 1$ and all $\ell \geq 1$
- (3) $\text{fin dim}_k \Lambda_{s+\ell} \geq r + s + \ell$ for $k \geq n$ and all $\ell \geq 1$.

Concerning (1): In view of Lemma 15', it suffices to prove $\text{fin dim}_{m-1} \Lambda_{s+1} \leq r$. We write $\Delta = \Lambda_{s+1}$ for convenience. Let $N \in \Delta\text{-mod}$ be indecomposable of projective dimension at least $r + 1$. If $a_{s+1}N \neq 0$ or $b_{s+1}N \neq 0$, then N requires at least m generators by Lemma 15'(a). So suppose that $a_{s+1}N = b_{s+1}N = 0$. But in that case, indecomposability of N forces b'_1 to annihilate N as well. Indeed, since $N \not\cong \Delta b'_1$, the inequality $b'_1 N \neq 0$ would place a direct summand isomorphic to a submodule X of Jb'_1 into the syzygy $\Omega^1(N)$; but this is not permissible because all such modules X have infinite projective dimension. This makes N a Λ_s -module, and guarantees that N is not $(m-1)$ -generated. The argument for (1) is thus complete.

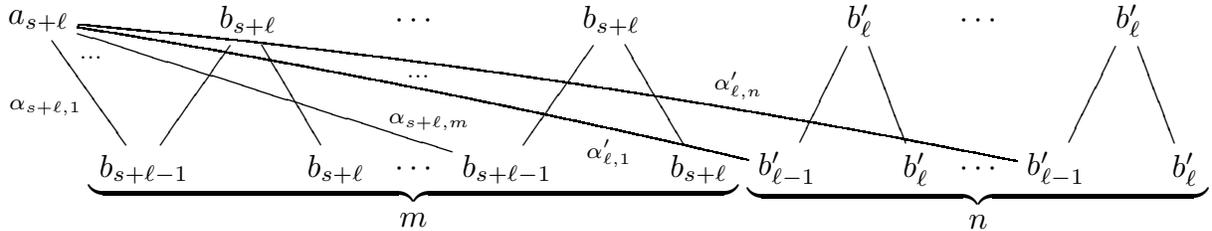
Concerning (2): Once more, Lemma 15' restricts our focus to $\Delta = \Lambda_{s+1}$. So suppose $N \in \Delta\text{-mod}$ is indecomposable with $\text{p dim } N \geq r + s + 1$; in particular, this implies $4 \leq \text{p dim } N < \infty$. We aim at an application of Lemma 12' to show that N requires at least n generators. In other words, we wish to show that the multiplicity μ of $S(a_{s+1})$ in N/JN exceeds the multiplicity ν of $S(a_s)$ in JN/J^2N . Clearly $\nu \leq \mu$. To obtain strict inequality, let \mathbf{e} be the sum of all primitive idempotents of the form b_i and b'_j in Δ , and observe that $\mathbf{e}\Delta\mathbf{e}$ is a monomial algebra of finitistic dimension ≤ 1 (Theorem 1); in particular, this ensures that neither N nor $\Omega^1(N)$ is an $\mathbf{e}\Delta\mathbf{e}$ -module. Moreover, we observe that $\text{p dim } X = \infty$ for any indecomposable $\mathbf{e}\Delta\mathbf{e}$ -module X of Loewy length at most 2, except for $X = \Delta b_{-1}, \Delta b'_{-1}, \Delta b_0$, and $\Delta b'_0$.

First assume that $\mu = 0$, meaning $a_{s+1}N = 0$. In view of the properties of $\mathbf{e}\Delta\mathbf{e}$, indecomposability of N then implies $b_{s+1}N = b'_1N = 0$, which makes N a Λ_s -module. But this contradicts $\text{findim } \Lambda_s = r + s$, and we conclude $\mu \geq 1$. Once more, we use indecomposability of N , combined with finiteness of $\text{pdim } N$, to see that $a_k(N/JN) = 0$ for all $k \leq s$. Consequently, the only simple module of the form $S(a_j)$ potentially occurring in JN/J^2N is $S(a_s)$. Since $a_k(J^2N) = 0$ for all k , equality $\nu = \mu$ would therefore make $\Omega^1(N)$ an $\mathbf{e}\Delta\mathbf{e}$ -module. This contradiction shows $\nu < \mu$, and an application of Lemma 12' completes the proof of (2).

Concerning (3): For arbitrary choice of $\ell \geq 1$, we consider the n -generated $\Lambda_{s+\ell}$ -module $N_{s+\ell} = P_{s+\ell}/V_{s+\ell}$, where

$$P_{s+\ell} = \Lambda_{s+\ell} x_{s+\ell,0} \oplus \left(\bigoplus_{i=1}^m \Lambda_{s+\ell} x_{s+\ell,i} \right) \oplus \left(\bigoplus_{i=1}^n \Lambda_{s+\ell} x'_{\ell,i} \right);$$

here $x_{s+\ell,0} = a_{s+\ell}$, $x_{s+\ell,i} = b_{s+\ell}$ for $1 \leq i \leq m$, and $x'_{\ell,i} = b'_\ell$ for $1 \leq i \leq n$. In other words, $P_{s+\ell} = \Lambda_{s+\ell} a_{s+\ell} \oplus (\Lambda_{s+\ell} b_{s+\ell})^m \oplus (\Lambda_{s+\ell} b'_\ell)^n$. Moreover $V_{s+\ell}$ is the submodule of $P_{s+\ell}$ generated by $\alpha_{s+\ell,0}x_{s+\ell,0}$, the differences $\alpha_{s+\ell,i}x_{s+\ell,0} - \beta_{s+\ell}x_{s+\ell,i}$ for $1 \leq i \leq m$, and the differences $\alpha'_{\ell,i}x_{s+\ell,0} - \beta'_\ell x'_{\ell,i}$ for $1 \leq i \leq n$. Then $N_{s+\ell}$ has the following graph relative to the listed top elements:



As in Part I, we have $\Omega^1(N_{s+\ell}) = N_{s+\ell-1}$ whenever $\ell \geq 2$; moreover, $\Omega^2(N_{s+1}) = \Omega^1(N_s)$. In light of $\text{pdim } N_s = r + s$, we conclude that $\text{pdim } N_{s+\ell} = r + s + \ell$ as desired. \square

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