

Moduli spaces of graded representations of finite dimensional algebras

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ABSTRACT. Let Λ be a basic finite dimensional algebra over an algebraically closed field, presented as a path algebra modulo relations; further, assume that Λ is graded by lengths of paths. The paper addresses the classifiability, via moduli spaces, of classes of graded Λ -modules with fixed dimension d and fixed top T . It is shown that such moduli spaces exist far more frequently than they do for ungraded modules. In the local case (i.e., when T is simple), the graded d -dimensional Λ -modules with top T always possess a fine moduli space which classifies these modules up to graded-isomorphism; moreover, this moduli space is a projective variety with a distinguished affine cover that can be constructed from quiver and relations of Λ . When T is not simple, existence of a coarse moduli space for the graded d -dimensional Λ -modules with top T forces these modules to be direct sums of local modules; under the latter condition, a finite collection of isomorphism invariants of the modules in question yields a partition into subclasses, each of which has a fine moduli space (again projective) parametrizing the corresponding graded-isomorphism classes.

1. Introduction

Let Λ be a finite dimensional algebra with radical J over an algebraically closed field K , and fix a finite dimensional semisimple (left) Λ -module T together with a positive integer d . In [4], the second author explored the existence and structure of moduli spaces classifying, up to isomorphism, those d -dimensional (left) representations M of Λ whose tops M/JM equal T under identification of isomorphic semisimple modules. The vehicle for tackling this classification problem is a projective variety, \mathfrak{Grass}_d^T , parametrizing the d -dimensional modules with top T ; see Section 2.

The general goal driving such investigations is to demonstrate that, even over a wild algebra Λ , major portions of the representation theory may behave tamely,

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being accessible to classification in a quite stringent sense. The idea of *moduli* goes back to Riemann’s 1857 classification of nonsingular projective curves of fixed genus in terms of continuous structure determining invariants; it was made precise by Mumford in the 1960’s. In rough terms, adapted to our representation-theoretic context, it amounts to the following: First one introduces a concept of *family*, which pins down what it means that a collection of finite dimensional representations be parametrized “continuously” by the points of a variety X . Given this prerequisite, a fine or coarse moduli space for a class of finite dimensional representations of Λ is a variety that continuously and bijectively parametrizes the isomorphism classes of the considered representations in a fashion satisfying a certain *coarse* or *fine* universal property. In slightly more precise language, a *fine moduli space* – the crucial concept in this paper – is the parametrizing variety of a distinguished family satisfying the postulate that any other family of representations recruited from the given class be uniquely “induced” by the distinguished one. Existence provided, both fine and coarse moduli spaces are unique up to canonical isomorphism due to the pertinent universality conditions.

Here we focus on a *graded* basic finite dimensional algebra Λ and address two problems closely related to the one mentioned at the outset: (1) That of deciding classifiability of the *graded* d -dimensional left Λ -modules M with fixed top T , up to graded-isomorphism; and, more restrictively, classifiability of those graded candidates M which have fixed *radical layering* $\mathbb{S}(M) = (J^l M / J^{l+1} M)_{l \geq 0}$. In either case, “classifiability” stands for existence of a fine or coarse moduli space. (2) In case existence of a moduli space is secured, the problem of determining the structure of this space and of constructing a universal family for the considered class of representations.

Our base field K being algebraically closed, we may assume without loss of generality that Λ is a path algebra modulo relations, meaning that $\Lambda = KQ/I$ for a quiver Q and an admissible ideal I in the path algebra KQ . In fact, we specialize to the situation where Λ is *graded by lengths of paths*, meaning that I is a homogeneous ideal with respect to the natural grading of KQ through path lengths. We will start by showing that the set of those points in the mentioned variety \mathbf{Grass}_d^T , which correspond to the graded d -dimensional modules with top T that are generated in degree zero, form a closed – and hence projective – subvariety of \mathbf{Grass}_d^T , denoted by $\mathbf{Gr-Grass}_d^T$. More strongly, we will verify the following: Suppose $\mathbb{S} = (\mathbb{S}_0, \mathbb{S}_1, \dots, \mathbb{S}_L)$ is a sequence of semisimple modules with $\mathbb{S}_0 = T$, where $J^{L+1} = 0$ and the dimensions of the \mathbb{S}_l add up to d . Then the following subset $\mathbf{Gr-Grass}(\mathbb{S})$ of \mathbf{Grass}_d^T is closed: Namely, the set of those points in $\mathbf{Gr-Grass}_d^T$ which correspond to the graded modules M with $\mathbb{S}(M) = \mathbb{S}$. In alternate terms, $\mathbf{Gr-Grass}(\mathbb{S})$ is a projective variety parametrizing the d -dimensional graded modules generated in degree zero with radical layering \mathbb{S} . Closedness of these subvarieties entails, in particular, that each $\mathbf{Gr-Grass}(\mathbb{S})$ is a union of irreducible components of $\mathbf{Gr-Grass}_d^T$. This is the first crucial difference between the graded and ungraded settings. Indeed, by contrast, the subvariety $\mathbf{Grass}(\mathbb{S})$ consisting of *all* points in \mathbf{Grass}_d^T corresponding to (not necessarily graded) modules with radical layering \mathbb{S} fails to be closed in \mathbf{Grass}_d^T in general.

Naturally, the *graded* d -dimensional representations with top T possess a fine/coarse moduli space whenever all d -dimensional representations with top T do. On

the other hand, not too surprisingly, existence of a moduli space is a far more frequent event in the graded than in the ungraded situation, as a grading accounts for increased rigidity. What is surprising is the extent of this discrepancy: For instance, given a simple module T with projective cover P , the d -dimensional top- T modules have a fine (equivalently, a coarse) moduli space classifying them up to isomorphism precisely when every submodule C of JP having codimension d in P is invariant under endomorphisms of P ; the latter requirement imposes strong restrictions on the underlying triple (Λ, T, d) ; see [4, Corollary 4.5]. However, when one narrows one's view to graded representations under graded-isomorphism, existence of a fine moduli space is automatic for a simple top T :

THEOREM A. *If Λ is path-length-graded and T a simple left Λ -module, then, for any positive integer d , the graded d -dimensional Λ -modules with top T possess a fine moduli space, classifying their graded-isomorphism classes. This moduli space equals Gr-Grass_d^T .*

Calling a module *local* if it has a simple top, we will more generally prove the following:

THEOREM B. *Suppose that Λ is path-length-graded, $T \in \Lambda\text{-mod}$ any semisimple Λ -module, and \mathbb{S} a sequence of semisimple Λ -modules as above. Moreover, let $\mathcal{C}(T)$ (resp. $\mathcal{C}(\mathbb{S})$) be the class of all graded d -dimensional Λ -modules with top T (resp. with radical layering \mathbb{S}). Then the following are true:*

- *If there is a coarse moduli space classifying the graded-isomorphism classes in $\mathcal{C}(T)$ (resp. $\mathcal{C}(\mathbb{S})$), then every object in $\mathcal{C}(T)$ (resp. $\mathcal{C}(\mathbb{S})$) is a direct sum of local modules.*
- *Conversely, if $\mathcal{C}(T)$ (resp. $\mathcal{C}(\mathbb{S})$) consists of direct sums of local modules, then $\mathcal{C}(T)$ (resp. $\mathcal{C}(\mathbb{S})$) can be partitioned into finitely many subclasses, each of which has a fine moduli space.*

All moduli spaces arising in the latter case are projective.

In parallel with the ungraded situation, each of the varieties Gr-Grass_d^T possesses a distinguished affine cover, accessible from quiver and relations of Λ , which provides the key to analyses of concrete examples.

This leaves the question of which projective varieties occur among the irreducible components of fine moduli spaces for graded modules with fixed dimension and top. We use examples of Hille in [2], which are in turn based on a construction technique introduced by the second author in [3], to show that every irreducible projective variety arises as an irreducible component of such a space.

Our approach to moduli problems for representations is fundamentally different from that of King in [7], where the targeted modules are those that are semistable with respect to a given additive function $\Theta : K_0(\Lambda\text{-mod}) \rightarrow \mathbb{R}$. King's definition of semistability allows for the adaptation of techniques developed by Mumford with the aim of classifying vector bundles. On one hand, in King's approach (coarse) moduli spaces for Θ -semistable representations are guaranteed to exist. On the other hand, in general these classes of modules are hard to assess in size and to describe in more manageable terms, while their classification through moduli spaces is a priori only up to an equivalence relation considerably coarser than isomorphism.

Concerning strategy: Evidently, every local graded module is generated in a single degree, which, for purposes of classification, we may assume to be zero. As for the general case, we will show that classifiability up to graded-isomorphism

(through a moduli space) of the graded d -dimensional modules with fixed top T , generated in mixed degrees, forces these graded objects to be direct sums of local graded submodules. We are thus led back to a situation in which restriction to graded modules generated in degree zero is harmless. The proof of this reduction step requires an extra layer of technicalities likely to obscure the underlying ideas; we will therefore defer it to an appendix (Section 6). In Sections 2–5, we will only consider graded modules generated in degree zero.

In Section 2, we will provide prerequisites; in particular, we will introduce the varieties Gr-Grass_d^T and $\text{Gr-Grass}(\mathbb{S})$ and verify their projectivity. In Section 3, we will prepare for proofs of the main results by introducing the mentioned affine cover of the variety Gr-Grass_d^T and by constructing a pivotal family of graded modules with top T ; this family will turn out to be the universal one (see Section 2 for a definition) in case a fine moduli space exists. Section 4 contains proofs of upgraded versions of Theorems A and B, the latter restricted to graded modules generated in degree zero. Section 5 is devoted to examples. The appendix, finally, will remove the restriction concerning degree-zero generating sets from the results for nonlocal modules.

2. Further terminology and Background

We will be fairly complete in setting up our conventions, even fairly standard ones, for the convenience of the reader whose expertise lies at the periphery of the subject.

Let Λ be a basic finite dimensional algebra over an algebraically closed field K . Without loss of generality, we assume Λ to be a path algebra modulo relations, that is, $\Lambda = KQ/I$ for a quiver Q and an admissible ideal I in the path algebra KQ .

Gradings. Throughout, we suppose Λ to be graded in terms of path lengths, meaning that I is homogeneous with respect to the length-grading of KQ . Denoting by J the Jacobson radical of Λ , we let L be maximal with $J^L \neq 0$. Then the grading of Λ takes on the form $\Lambda = \bigoplus_{0 \leq l \leq L} \Lambda_l$, where $\Lambda_l \cong J^l/J^{l+1}$ is the homogeneous component of degree l of Λ . The vertices e_1, \dots, e_n of Q will be identified with the primitive idempotents of Λ corresponding to the paths of length zero, that is, the e_i will also stand for the I -residues of the paths of length 0 in Λ_0 . The factor modules $S_i = \Lambda e_i / J e_i$ then form an irredundant set of representatives for the simple left Λ -modules; unless we explicitly state otherwise, we consider the S_i – and hence all semisimple modules – as homogeneous modules in degree 0, systematically identifying isomorphic semisimple modules. Clearly, the grading of any indecomposable projective module Λe_i inherited from that of Λ yields a graded local module which is generated in degree zero. Whenever $P = \bigoplus_{1 \leq i \leq n} (\Lambda e_i)^{t_i}$, we let $P = \bigoplus_{0 \leq l \leq L} P_l$ be the resulting decomposition into homogeneous subspaces. Given two graded modules M, M' , we call a morphism $f : M \rightarrow M'$ *homogeneous of degree s* in case $f(M_l) \subseteq M'_{l+s}$ for all l ; the attribute “homogeneous” by itself stands for “homogeneous of degree zero”. Whenever there is an isomorphism $M \rightarrow M'$ which is homogeneous of some degree s , we call M and M' *graded-isomorphic*; so, in particular, two graded modules generated in degree zero are graded-isomorphic if and only if they are isomorphic by way of a homogeneous map.

Paths in Λ and top elements of modules. We will observe the following conventions: The product pq of two paths p and q in KQ stands for “first q , then p ”; in

particular, pq is zero unless the end point of q coincides with the starting point of p . In this spirit, we call a path p_1 a *right subpath* of p if $p = p_2p_1$ for some path p_2 . A *path in Λ* will be an element of the form $p + I$, where p is a path in $KQ \setminus I$, whence Λ_l consists of the K -linear combinations of paths of length l in Λ ; in light of homogeneity of the ideal I , the length of a path in Λ is an unambiguous quantity. We will suppress the residue notation, provided there is no risk of confusion, and gloss over the distinction between the left Λ -structure of a module $M \in \Lambda\text{-mod}$ and its induced KQ -structure. A homogeneous element x of a graded module M generated in degree zero will be called a *top element* of M if $x \in M_0 \setminus JM$ and x is normed by some e_i , meaning that $x = e_ix$. If M is ungraded, we waive the homogeneity condition imposed on x . Any sequence x_1, \dots, x_m of top elements of M generating M and linearly independent modulo JM will be called a *full sequence of top elements of M* .

Background on moduli problems. We refer to [8], but recall the definition of a fine moduli space in the context of representations. Our concept of a family of Λ -modules is that introduced by King in [7]: Namely, a *family of d -dimensional Λ -modules parametrized by an algebraic variety X* is a pair (Δ, δ) , where Δ is a (geometric) vector bundle of rank d over X and $\delta : \Lambda \rightarrow \text{End}(\Delta)$ a K -algebra homomorphism. We are interested in *families of graded d -dimensional representations of Λ* , which means that the left Λ -multiplication induced by δ on each fibre Δ_x of Δ yields a graded module. In addition, we want the Δ_x to be “continuously graded” as x traces X , meaning that all fibres are generated in the same sets of degrees (see Section 6). Primarily (Sections 2–5), we will be interested in families of Λ -modules all of which are generated in degree zero. Our notion of *equivalence of families of graded modules* parametrized by some variety X is the coarsest possible to separate graded-isomorphism classes: Namely, $(\Delta^1, \delta^1) \sim (\Delta^2, \delta^2)$ precisely when, for each $x \in X$, the fibre of Δ^1 above x is graded-isomorphic as a left Λ -module to the fibre of Δ^2 above x . As is common, given a family (Δ, δ) parametrized by X and a morphism $\tau : Y \rightarrow X$ of varieties, the *induced family* $\tau^*(\Delta, \delta)$ over Y is the pullback of (Δ, δ) along τ . The family $\tau^*(\Delta, \delta)$ is again a family of graded modules if (Δ, δ) is. In this setup, a variety Z is a *fine moduli space* for the (families of) graded d -dimensional modules with top T if there exists a family (Γ, γ) of such modules parametrized by Z , which has the property that an arbitrary family parametrized by some variety X is equivalent to a family $\tau^*(\Gamma, \gamma)$ induced via a unique morphism $\tau : X \rightarrow Z$. Accordingly, Γ is then called the *universal family*. In particular, the requirements on Γ entail that every d -dimensional module with top T be isomorphic to precisely one fibre of the bundle Γ . More commonly (but equivalently), a fine moduli space is defined in terms of representability of the contravariant functor from the category of varieties to the category of sets, assigning to any variety X the set of equivalence classes of families parametrized by X . For details, as well as for the concept of a *coarse moduli space*, see [8, pp. 23, 24].

Ungraded Grassmannians of modules with top T . We fix a natural number d , a semisimple module T , say $T = \bigoplus_{1 \leq i \leq n} S_i^{t_i}$, and denote by $P = \bigoplus_{1 \leq i \leq n} (\Lambda e_i)^{t_i}$ its projective cover. The module P comes equipped with the obvious grading under which it is obviously generated in degree zero. It will be convenient to write P in the form

$$P = \bigoplus_{1 \leq r \leq t} \Lambda z_r,$$

where $\mathbf{t} = \sum_i t_i$ and z_1, \dots, z_t is a sequence of top elements (all of which are homogeneous of degree zero); in particular, $z_r = e(r)z_r$ for suitable idempotents $e(r)$ in $\{e_1, \dots, e_n\}$. A natural choice of such top elements consists of the primitive idempotents e_i themselves, each with multiplicity t_i , distinguished by their ‘slots’ in the above decomposition of P ; to distinguish these slots, we will write $\mathbf{e}(r)$ for the element of the direct sum $\bigoplus_{1 \leq i \leq n} (\Lambda e_i)^{t_i}$ which has entries zero outside the slot labeled r and carries $e(r)$ in the r -th slot.

The ungraded Grassmannian of d -dimensional left Λ -modules with top T was defined in [4] as follows: If $\mathfrak{Gr}(d', JP)$ is the classical Grassmannian of d' -dimensional subspaces of the K -space JP , where $d' = \dim_K P - d$, then

$$\mathbf{Grass}_d^T = \{C \in \mathfrak{Gr}(d', JP) \mid C \text{ is a } \Lambda\text{-submodule of } JP\},$$

a closed subvariety of $\mathfrak{Gr}(d', JP)$ and consequently projective. It is accompanied by a natural surjection

$$\mathbf{Grass}_d^T \longrightarrow \{\text{isomorphism classes of } d\text{-dimensional modules with top } T\},$$

sending C to the class of P/C . Clearly, the fibres of this map coincide with the orbits of the natural $\text{Aut}_\Lambda(P)$ -action on \mathbf{Grass}_d^T .

Recall moreover that by the radical layering of a module M we mean the sequence

$$\mathbb{S}(M) = (M/JM, JM/J^2M, \dots, J^{L-1}M/J^L M, J^L M).$$

Since we identify semisimple modules with their isomorphism classes, we are, in effect, dealing with a matrix of discrete invariants of M keeping count of the multiplicities of the simple modules in the various semisimple layers $J^l M/J^{l+1}M$ of M . Correspondingly, we consider the following action-stable locally closed subvarieties of \mathbf{Grass}_d^T , which clearly cover the latter variety, namely

$$\mathbf{Grass}(\mathbb{S}) = \{C \in \mathbf{Grass}_d^T \mid \mathbb{S}(P/C) = \mathbb{S}\}$$

for any d -dimensional semisimple sequence \mathbb{S} with top T ; by this we mean any sequence $(\mathbb{S}_0, \dots, \mathbb{S}_L)$ of semisimple modules such that $\mathbb{S}_0 = T$ with each \mathbb{S}_l embedding into $J^l P/J^{l+1}P$, and $\sum_{0 \leq l \leq L} \dim \mathbb{S}_l = d$.

Graded Grassmannians of modules with top T . In keeping with our goal of classifying graded modules, we restrict our focus to a closed subvariety of \mathbf{Grass}_d^T paired with an action of the group of homogeneous automorphisms of P as follows: Let M be a d -dimensional graded module generated in degree zero with top T , the latter being homogeneous of degree zero. Clearly M has a graded projective cover $\pi : P \rightarrow M$, where $P = \bigoplus_l P_l$ is equipped with the natural grading and π is homogeneous. Thus we obtain a graded projective module P generated in degree zero such that, up to isomorphism, M equals P/C , where C is a homogeneous submodule of P . If $M' = P/C'$ is another such module, then M and M' are graded isomorphic if and only if there exists a homogeneous automorphism $f : P \rightarrow P$ with the property that $f(C) = C'$. We glean that classifying the d -dimensional graded modules with radical layering \mathbb{S} which are generated in degree zero (up to graded-isomorphism) boils down to classifying the homogeneous points $C \in \mathbf{Grass}(\mathbb{S})$.

To this end, we define $\text{Gr-Grass}(\mathbb{S})$ to be the set of all those points $C \in \mathbf{Grass}(\mathbb{S})$ which are homogeneous submodules of JP . To see that $\text{Gr-Grass}(\mathbb{S})$ is projective, let d_{li} be the multiplicity of S_i in \mathbb{S}_l and P_{li} the K -subspace of P_l generated by all pz_r with $1 \leq r \leq \mathbf{t}$ such that p is a path (of length l) ending in e_i . Moreover, we

denote the classical Grassmannian of all $(\dim P_i - d_i)$ -dimensional subspaces of P_i by $\text{Gr-Grass}(\mathbb{S})_{i_i}$. Then $e_i(P_l/C_l) \cong e_i \mathbb{S}_l$ for C as above, whence

$$\text{Gr-Grass}(\mathbb{S}) = \{C \in \text{Grass}(\mathbb{S}) \mid C = \bigoplus_{1 \leq l \leq L, 1 \leq i \leq n} C_{li} \text{ with } C_{li} \in \text{Gr-Grass}(\mathbb{S})_{i_i}\}.$$

This subvariety of $\text{Grass}(\mathbb{S})$ is isomorphic to a closed subvariety of the product $\prod_{l,i} \text{Gr-Grass}(\mathbb{S})_{i_i}$, and hence it is projective and closed in Grass_d^T . If, analogously, we define Gr-Grass_d^T to be the set of all points in Grass_d^T which are homogeneous submodules of JP , then

$$\text{Gr-Grass}_d^T = \bigcup_{\mathbb{S}} \text{Gr-Grass}(\mathbb{S})$$

is a finite union of closed subvarieties of Grass_d^T , and consequently Gr-Grass_d^T is projective as well.

By $\text{Gr-Aut}_\Lambda(P)$ we denote the subgroup of $\text{Aut}_\Lambda(P)$ consisting of the homogeneous automorphisms of P . Clearly, $\text{Gr-Aut}_\Lambda(P)$ acts morphically on the $\text{Gr-Grass}(\mathbb{S})$ and Gr-Grass_d^T , and by the above comments, the orbits are in one-to-one correspondence with the isomorphism classes of graded d -dimensional modules (generated in degree zero) having radical layering \mathbb{S} or top T , respectively. Note that $\text{Gr-Aut}_\Lambda(P) \cong \text{Aut}_\Lambda(T) \cong \prod_{1 \leq i \leq n} \text{GL}_{t_i}$. Moreover, we note that, given an isomorphism $\bar{f} : P/C \rightarrow P/C'$ induced by some $f \in \text{Gr-Aut}_\Lambda(P)$, the ‘‘distinguished’’ sequence $z_r + C$ of top elements of P/C is mapped to a sequence of top elements of P/C' , the $f(z_r) + C'$ being again homogeneous of degree zero.

Before we proceed, we give a pedantic reformulation of the moduli problem tackled in Sections 3–5.

MODULI PROBLEM FOR d -DIMENSIONAL GRADED MODULES GENERATED IN DEGREE ZERO, WITH FIXED TOP T . For any point $C \in \text{Gr-Grass}_d^T$, the module $M = P/C$ has a natural grading, namely $M = \bigoplus_{0 \leq l \leq L} M_l$, where M_l is the canonical image of P_l modulo C_l ; in particular, M_0 is the canonical image of $\bigoplus_{1 \leq r \leq t} K z_r$, M_1 the image of $\bigoplus_{1 \leq r \leq t} \sum_{\alpha \text{ arrow}} K \alpha z_r$, etc. Clearly, M is generated in degree zero under this grading. As we saw, classifying the d -dimensional graded modules with top T (resp., with radical layering \mathbb{S}) which are generated in degree zero amounts to classifying the modules of the form P/C with $C \in \text{Gr-Grass}_d^T$ (resp., with $C \in \text{Gr-Grass}(\mathbb{S})$) up to graded-isomorphism.

Recall that a family of d -dimensional graded modules with fixed top T generated in degree zero is a family (Δ, δ) of modules each fibre of which has the listed properties. Two such families, (Δ^1, δ^1) and (Δ^2, δ^2) , parametrized by the same variety X , are equivalent if each Δ_x^1 is graded-isomorphic to Δ_x^2 .

Our primary questions are: When is there a family that is universal for the families of equivalence classes of graded d -dimensional modules generated in degree zero with top T ; in other words, when does our classification problem admit a fine moduli space? When does a coarse moduli space exist? In case of existence, what can be said about the geometry of such moduli spaces?

3. The distinguished affine cover of Gr-Grass_d^T and a pivotal family of graded modules

The concepts introduced and discussed in this section will only be relevant to proofs and examples; they will not appear in the statements of the main results.

The affine cover of Gr-Grass_d^T which we will present in the sequel will provide the basis for the construction of a family of graded modules which will be instrumental in solving the problems posed at the end of Section 2. This family will coincide with the universal family in case a fine moduli space exists. Moreover, being computable from quiver and relations of Λ , this cover is the principal resource for the analysis of examples. We will briefly recall some concepts and facts provided in [6], which will serve as our point of departure.

DEFINITIONS 3.1.

(1) An (*abstract*) d -dimensional skeleton with top T is any sequence $\sigma = (\sigma^{(1)}, \dots, \sigma^{(t)})$ of sets $\sigma^{(r)} \subseteq KQe(r) \setminus I$ – repetitions allowed – with $\sum_{1 \leq r \leq t} |\sigma^{(r)}| = d$, such that each $\sigma^{(r)}$ is a nonempty set of paths starting in the vertex $e(r)$ which is closed under right subpaths; the latter means that $pq \in \sigma^{(r)}$ implies $q \in \sigma^{(r)}$. (In particular, this condition guarantees that each $\sigma^{(r)}$ contains the path $e(r)$ of length zero.)

To distinguish between occurrences of the same path p in the intersection of two sets $\sigma^{(r)}$ and $\sigma^{(s)}$ with $r \neq s$, but $e(r) = e(s)$, we tag the elements of $\sigma^{(r)}$ with the superscript r and write $\sigma^{(r)}$ as $\{p^{(r)} \mid p \in \sigma^{(r)}\}$. This notational device makes it unambiguous to treat σ as a disjoint union $\bigsqcup_{1 \leq r \leq t} \sigma^{(r)}$.

It is natural to view a skeleton σ as a disjoint union of \mathfrak{t} tree graphs, each represented by some $\sigma^{(r)}$, having “root” $e(r)$ and edges labeled by the arrows making up the paths in $\sigma^{(r)}$.

(2) Given a d -dimensional skeleton σ with top T , a σ -critical pair is a pair $(\alpha, p^{(r)})$ with $1 \leq r \leq \mathfrak{t}$, where α is an arrow and $p^{(r)}$ a path in $\sigma^{(r)}$ such that $\alpha p^{(r)}$ is a path of length at most L in KQ which does not belong to $\sigma^{(r)}$. Moreover, for any such σ -critical pair, $\sigma(\alpha, p^{(r)})$ denotes the set of all paths $q \in \sigma$ (i.e., $q = q^{(s)} \in \sigma^{(s)}$ for some s) with $\text{length}(q) \geq \text{length}(\alpha p^{(r)})$ and $\text{end}(q^{(s)}) = \text{end}(\alpha)$.

(3) Let $M \cong P/C$ with $C \in \text{Grass}_d^T$ and let σ be a d -dimensional skeleton with top T ; by $\sigma_l^{(r)}$ we denote the set of paths of length l in $\sigma^{(r)}$. We say that M has skeleton σ relative to the presentation $\bigoplus_{1 \leq r \leq \mathfrak{t}} \Lambda z_r$ of P in case the following holds: For each $0 \leq l \leq L$, the union $\bigcup_{1 \leq r \leq \mathfrak{t}} \sigma_l^{(r)}(z_r + C)$ induces a basis for $J^l M / J^{l+1} M$. We call σ a skeleton of M if σ is a skeleton of M relative to some presentation of P .

(4) Finally, we set

$$\text{Grass}(\sigma) = \{C \in \text{Grass}_d^T \mid \sigma \text{ is a skeleton of } M \text{ relative to } P = \bigoplus_{1 \leq r \leq \mathfrak{t}} \Lambda z_r\}$$

and

$$\text{Gr-Grass}(\sigma) = \text{Grass}(\sigma) \cap \text{Gr-Grass}_d^T.$$

Whereas the concept of a skeleton of P/C relative to the presentation $P = \bigoplus_{1 \leq r \leq \mathfrak{t}} \Lambda z_r$ is tied to the sequence $(z_r)_{r \leq \mathfrak{t}}$ of top elements, the set of *all* skeleta

of M is an isomorphism invariant of M . In fact an abstract d -dimensional skeleton σ with top T is a skeleton of M if and only if

- (\bullet) there exists a sequence of top elements m_1, \dots, m_t of M such that, for each l ,

$$\bigcup_{1 \leq r \leq t} \sigma_l^{(r)} m_r \quad \text{induces a basis for } J^l M / J^{l+1} M.$$

In particular, M is represented by *some* point in $\mathbf{Grass}(\sigma)$ if and only if condition (\bullet) holds. (Yet, not all points in $\mathbf{Grass}(\mathbb{S}(M))$ representing M will belong to $\mathbf{Grass}(\sigma)$ in general.)

Clearly, each d -dimensional Λ -module M with top T has at least one skeleton relative to our fixed presentation $P = \bigoplus_{1 \leq r \leq t} \Lambda z_r$. In particular, \mathbf{Grass}_d^T is covered by the locally closed subvarieties $\mathbf{Grass}(\sigma)$. More precisely, if \mathbb{S} is a d -dimensional semisimple sequence, then $\mathbf{Grass}(\mathbb{S})$ is the union of those charts $\mathbf{Grass}(\sigma)$ which have non-empty intersection with $\mathbf{Grass}(\mathbb{S})$. In fact, $\mathbf{Grass}(\mathbb{S})$ is the union of those subvarieties $\mathbf{Grass}(\sigma)$ corresponding to the skeleta σ compatible with $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ in the following sense: For each $l \leq L$ and $i \leq n$, the number of paths in $\bigsqcup_{r \leq t} \sigma_l^{(r)}$ that end in the vertex e_i coincides with the multiplicity of the simple module S_i in \mathbb{S}_l .

That C be a point in $\mathbf{Grass}(\sigma)$ obviously entails the existence of unique scalars $c_{\alpha p^{(r)}, q^{(s)}}$ with the property that

$$\alpha p^{(r)}(z_r + C) = \sum_{q^{(s)} \in \sigma(\alpha, p^{(r)})} c_{\alpha p^{(r)}, q^{(s)}} q^{(s)}(z_s + C),$$

whenever $(\alpha, p^{(r)})$ is a σ -critical pair. Conversely, the isomorphism type of $M = P/C$ is completely determined by the family of these scalars. Thus we obtain a bijection

$$\psi : \mathbf{Grass}(\sigma) \rightarrow \mathbb{A}^N, \quad C \mapsto c = (c_{\alpha p^{(r)}, q^{(s)}})_{(\alpha, p^{(r)}) \text{ } \sigma\text{-critical, } q^{(s)} \in \sigma(\alpha, p^{(r)})}$$

where N is the *disjoint* union of the $\sigma(\alpha, p^{(r)})$. The latter sets of paths are not a priori disjoint; in the index set N we force disjointness by indexing the elements of $\sigma(\alpha, p^{(r)})$ by the pertinent critical pair $(\alpha, p^{(r)})$. The following result from [6] summarizes the properties of the cover $(\mathbf{Grass}(\sigma))_\sigma$ which will be relevant here.

KNOWN FACTS 3.2 CONCERNING THE UNGRADED SETTING. *For every d -dimensional skeleton σ with top T , the set $\mathbf{Grass}(\sigma)$ is a locally closed affine subvariety of \mathbf{Grass}_d^T which is isomorphic to a closed subvariety of \mathbb{A}^N by way of the map $\psi : \mathbf{Grass}(\sigma) \rightarrow \mathbb{A}^N$ described above. Moreover, given any d -dimensional semisimple sequence \mathbb{S} with top T , the varieties $\mathbf{Grass}(\sigma)$, where σ traces the skeleta compatible with \mathbb{S} , form an open affine cover of $\mathbf{Grass}(\mathbb{S})$. (On the other hand, the $\mathbf{Grass}(\sigma)$ fail to be open in \mathbf{Grass}_d^T in general.) \square*

Next we adjust the statements in 3.2 to the graded scenario. The fact that the affine sets $\text{Gr-Grass}(\sigma)$ form an *open* cover of Gr-Grass_d^T in this case already anticipates the firmer grip we have on the graded setting.

FACTS 3.3 CONCERNING THE GRADED SETTING. *For every d -dimensional skeleton σ with top T , the set $\text{Gr-Grass}(\sigma)$ is an open affine subvariety of Gr-Grass_d^T which is isomorphic to a closed subvariety of \mathbb{A}^N by way of the restriction of ψ to $\text{Gr-Grass}(\sigma)$. In particular, given any d -dimensional semisimple sequence \mathbb{S} with*

top T , the varieties $\text{Gr-Grass}(\sigma)$, where σ traces the skeleta compatible with \mathbb{S} , form an open affine cover of $\text{Gr-Grass}(\mathbb{S})$.

PROOF. First we observe that $\text{Gr-Grass}(\sigma)$ coincides with the intersection $\text{Gr-Grass}_d^T \cap \text{Schu}(\sigma)$, where $\text{Schu}(\sigma)$ is the following open Schubert cell of the classical Grassmannian $\mathfrak{Gr}(d', JP)$ in which Gr-Grass_d^T is located: Namely, $\text{Schu}(\sigma)$ coincides of those subspaces $C \subseteq JP$ for which $P = C \oplus \bigoplus_{r \leq t, p^{(r)} \in \sigma^{(r)}} p^{(r)} z_r$; indeed invoke gradedness to verify that σ is a skeleton of P/C precisely when the residue classes $p^{(r)}(z_r + C)$ with $r \leq t$ and $p^{(r)} \in \sigma^{(r)}$ form a basis for P/C . Thus the $\text{Gr-Grass}(\sigma)$ are open in Grass_d^T and a fortiori in Gr-Grass_d^T .

The remainder of the claim will follow from the preceding facts if we can show that the image of $\text{Gr-Grass}(\sigma)$ under ψ is a closed subvariety of $\psi(\text{Grass}(\sigma))$ in \mathbb{A}^N . But this is clear, since a point

$$c = (c_{\alpha p^{(r)}, q^{(s)}})_{(\alpha, p^{(r)}) \text{ } \sigma\text{-critical, } q^{(s)} \in \sigma(\alpha, p^{(r)})}$$

in $\psi(\text{Grass}(\sigma))$ belongs to the image of $\text{Gr-Grass}(\sigma)$ if and only if $c_{\alpha p^{(r)}, q^{(s)}} = 0$ whenever $q^{(s)}$ is strictly longer than $\alpha p^{(r)}$. \square

One of the assets of this particular affine cover lies in its computability. Polynomials for the image of the map $\psi : \text{Grass}(\sigma) \rightarrow \mathbb{A}^N$ can easily be obtained from quiver and relations of Λ (as was shown in [4]); a computer program is being established by the authors of this paper. We will see in Section 5 that an even less labor-intensive algorithm yields polynomials for $\text{Gr-Grass}(\sigma)$. We will identify C with c whenever convenient.

The open affine cover $(\text{Gr-Grass}(\sigma))_\sigma$ of Gr-Grass_d^T allows us to tie the graded modules P/C with $C \in \text{Gr-Grass}_d^T$ into a family, which will turn out to be universal (in the sense of Section 2) in many cases.

PROPOSITION 3.4. *There exists a family (Γ, γ) of d -dimensional graded modules with top T , parametrized by Gr-Grass_d^T , such that the fibre above any point $C \in \text{Gr-Grass}_d^T$, endowed with the Λ -module structure induced by γ , is P/C with the grading inherited from P .*

PROOF. For each d -dimensional skeleton σ with top T , consider the following d -dimensional subspace

$$V_\sigma = \bigoplus_{r \leq t, p^{(r)} \in \sigma^{(r)}} K p^{(r)} z_r$$

of the K -space P . Note that the elements $z_r = e(r) z_r$ belong to each V_σ . The trivial vector bundle $\pi_\sigma : \text{Gr-Grass}(\sigma) \times V_\sigma \rightarrow \text{Gr-Grass}(\sigma)$ of rank d clearly becomes a family of graded modules when paired with the algebra homomorphism $\gamma_\sigma : \Lambda \rightarrow \text{End}(\text{Gr-Grass}(\sigma) \times V_\sigma)$ which assigns to $\lambda \in \Lambda$ the bundle endomorphism that sends any (C, x) to the pair (C, y) , where y is the unique element in V_σ with $\lambda(x + C) = y + C$ in P/C .

To glue these trivial bundles together along the intersections of the affine patches $\text{Gr-Grass}(\sigma)$ of Gr-Grass_d^T , let ρ and σ be two abstract d -dimensional skeleta with top T , and let $g_{\rho, \sigma} : \text{Gr-Grass}(\rho) \cap \text{Gr-Grass}(\sigma) \rightarrow \text{GL}_d$ be the map which sends any point C in the intersection $\text{Gr-Grass}(\rho) \cap \text{Gr-Grass}(\sigma)$ to the isomorphism $V_\rho \rightarrow V_\sigma$ that maps any $x \in V_\rho$ to the unique element $x' \in V_\sigma$ for which $x + C = x' + C$ in P/C ; note that $g_{\rho, \sigma}(C)$ is a well-defined K -space

isomorphism and $g_{\rho,\sigma}$ a morphism of varieties. Moreover, our construction entails that the $g_{\rho,\sigma}(C)$ preserve the distinguished elements $z_r + C$ and that the maps $g_{\rho,\sigma}$, where ρ and σ run through the d -dimensional skeleta with top T , satisfy the cocycle condition, $g_{\sigma,\tau}(C) \circ g_{\rho,\sigma}(C) = g_{\rho,\tau}(C)$, for all choices of $C \in \text{Gr-Grass}(\rho) \cap \text{Gr-Grass}(\sigma) \cap \text{Gr-Grass}(\tau)$. Hence we obtain a vector bundle Γ over $\text{Gr-Grass}(\sigma)$, which coincides with the above trivial bundles on the affine patches of our cover. The homomorphisms γ_σ are compatible with the gluing maps $g_{\rho,\sigma}$ and thus yield an algebra homomorphism $\gamma : \Lambda \rightarrow \text{End}(\Gamma)$ which induces the Λ -structure of P/C on the fibre above any point C and, in particular, makes the $z_r(C)$ homogeneous of degree zero. \square

We follow [8, p. 37] in the concept of a *family having the local universal property* relative to our moduli problem. The next observation serves merely as an auxiliary to the proofs of our main results.

OBSERVATION 3.5. *The family (Γ, γ) of Proposition 3.4 has the local universal property. In other words, given an arbitrary family (Δ, δ) of d -dimensional graded modules with top T which are generated in degree zero, any point x in the parametrizing variety X of Δ has a neighborhood U such that $(\Delta|_U, \delta)$ is induced from (Γ, γ) by way of some (not necessarily unique) morphism $\tau : U \rightarrow \text{Gr-Grass}_d^T$. By the latter we mean that the family $(\Delta|_U, \delta)$ is equivalent to $\tau^*(\Gamma, \gamma)$.*

PROOF. Without loss of generality $d \geq \dim T = \mathfrak{t}$.

Suppose (Δ, δ) is any family of d -dimensional graded modules with top T generated in degree zero, X being its parametrizing variety. Let $x \in X$. In showing that, when restricted to a suitable neighborhood of x , this family is induced by (Γ, γ) , it is clearly harmless to take Δ to be a trivial bundle $X \times K^d$. We fix a sequence z_1, \dots, z_t of linearly independent elements of K^d and assume that, under the Λ -structure induced by δ on any fibre Δ_y , these elements form a sequence $z_1(y), \dots, z_t(y)$ of top elements of Δ_y ; this assumption is permissible, because (Δ, δ) is equivalent to a family $X \times K^d$ with this property.

Assume that the fibre of Δ above x has skeleton σ relative to the sequence $z_1(x), \dots, z_t(x)$. Then the set Y of all $y \in X$ such that Δ_y has skeleton σ relative to $z_1(y), \dots, z_t(y)$ is an open subset of X containing x ; indeed, the condition that the elements $\delta(p^{(r)})(z_r(y))$ with $r \leq \mathfrak{t}$ and $p^{(r)} \in \sigma^{(r)}$ be linearly independent is open. We may thus further assume that $X = Y$. Thus, when verifying local universality at x , we are essentially dealing with the trivial subbundle $\text{Gr-Grass}(\sigma) \times V_\sigma$ of (Γ, γ) . Identify the points $C \in \text{Gr-Grass}(\sigma)$ with the points $c \in \mathbb{A}^N$ described in the above coordinatization of $\text{Gr-Grass}(\sigma)$, and define a map $\tau : X \rightarrow \text{Gr-Grass}(\sigma)$ as follows: For any $y \in X$, there is a unique point $C = c(y) = (c_{\alpha p^{(r)}, q^{(s)}}) \in \text{Gr-Grass}(\sigma)$ with the property

$$\alpha p^{(r)} z_r(y) = \sum_{q^{(s)} \in \sigma(\alpha, p^{(r)})} c_{\alpha p^{(r)}, q^{(s)}} q^{(s)} z_s(y)$$

for all σ -critical pairs $(\alpha, p^{(r)})$. Setting $\tau(y) = c(y)$ yields a morphism $X \rightarrow \text{Gr-Grass}(\sigma)$, since, for each $r \leq \mathfrak{t}$, the map $\delta(-, z_r) : X \rightarrow \Delta = X \times K^d$, $y \mapsto \delta(y, z_r)$, is a morphism. Thus τ meets our requirement. \square

Finally, we record an immediate consequence of the preceding observations which holds independent interest.

COROLLARY 3.6. *Each of the subvarieties $\text{Gr-Grass}(\mathbb{S})$ of Gr-Grass_d^T is open and closed in Gr-Grass_d^T and thus a union of connected components.*

PROOF. In Section 2, under the heading *Graded Grassmannians*, we saw that, given any d -dimensional semisimple sequence \mathbb{S} with top T , the set $\text{Grass}(\mathbb{S})$ is closed in Gr-Grass_d^T . On the other hand, by Facts 3.3 above, $\text{Grass}(\mathbb{S})$ is open, being a union of suitable charts $\text{Gr-Grass}(\sigma)$. \square

4. Sharpened version of Theorem A and a special case of Theorem B

In light of Observation 3.5, a well-known criterion for the existence of a coarse moduli space applies to the question of when the graded Λ -modules with fixed dimension and fixed top possess such a (weakly) universal parametrizing space.

Recall that a *categorical quotient* of Gr-Grass_d^T by the action of $\text{Gr-Aut}_\Lambda(P)$ is a variety $\text{Gr-Grass}_d^T // \text{Gr-Aut}_\Lambda(P)$ together with a morphism

$$h : \text{Gr-Grass}_d^T \rightarrow \text{Gr-Grass}_d^T // \text{Gr-Aut}_\Lambda(P)$$

which is constant on the $\text{Gr-Aut}_\Lambda(P)$ -orbits and has the property that every morphism defined on Gr-Grass_d^T and constant on these orbits factors uniquely through h .

CRITERION 4.1. (See [8, Proposition 2.13]) *The d -dimensional graded modules with top T (resp., with radical layering \mathbb{S}) generated in degree zero possess a coarse moduli space precisely when Gr-Grass_d^T (resp., $\text{Gr-Grass}(\mathbb{S})$) has a categorical quotient modulo $\text{Gr-Aut}_\Lambda(P)$ that separates $\text{Gr-Aut}_\Lambda(P)$ -orbits. In case of existence, this quotient is the coarse moduli space.*

In particular, closedness of the $\text{Gr-Aut}_\Lambda(P)$ -orbits in Gr-Grass_d^T is a necessary condition for the existence of a coarse moduli space.

PROOF. For the final assertion, invest the fact that Gr-Grass_d^T and all the $\text{Gr-Grass}(\mathbb{S})$ are closed in Gr-Grass_d^T . \square

This criterion leads to the first of our main results, a more detailed version of Theorem A. As pointed out earlier, each local graded module is graded-isomorphic to one generated in degree zero. Therefore, given any simple T , the factor modules P/C with $C \in \text{Gr-Grass}_d^T$, where P is endowed with the natural grading, are representative of all graded modules with top T , up to graded-isomorphism.

THEOREM 4.2. *Suppose that $\Lambda = KQ/I$ is a finite dimensional algebra which is graded by path lengths, and let T be a simple Λ -module. Then the projective variety Gr-Grass_d^T is a fine moduli space for the graded d -dimensional Λ -modules with top T , classifying them up to graded-isomorphism. Moreover, the family (Γ, γ) constructed in Proposition 3.4 is universal.*

A fortiori, given any d -dimensional semisimple sequence \mathbb{S} with top T , the variety $\text{Gr-Grass}(\mathbb{S})$ is a fine moduli space for the graded Λ -modules with radical layering \mathbb{S} .

In both instances, the moduli space is a projective variety.

PROOF. Since the only graded automorphisms of P are multiplications by nonzero scalars, $\text{Gr-Aut}_\Lambda(P) \cong K^*$. In particular, all homogeneous submodules of P are invariant under $\text{Gr-Aut}_\Lambda(P)$, which makes the $\text{Gr-Aut}_\Lambda(P)$ -orbits of

Gr-Grass_d^T singletons. Consequently, Gr-Grass_d^T is its own geometric (and a fortiori, categorical) quotient modulo $\text{Gr-Aut}_\Lambda(P)$. Thus Criterion 4.1 shows Gr-Grass_d^T to be a coarse moduli space for the families of graded d -dimensional modules with top T .

To see that Gr-Grass_d^T is even a fine moduli space, we will check that the family (Γ, γ) of Proposition 3.4 is universal. Let (Δ, δ) be any family of graded modules of the form P/C with $C \in \text{Gr-Grass}_d^T$, parametrized by a variety X say; again, the projective module P is endowed with its natural grading. Define a map $\tau : X \rightarrow \text{Gr-Grass}_d^T$ by sending any point $x \in X$ to the unique point $C \in \text{Gr-Grass}_d^T$ having the property that the fibre of Δ above x is graded-isomorphic to P/C . Due to the fact that Gr-Grass_d^T is already known to be a coarse moduli space for our problem, the bijection α from the set of graded-isomorphism classes of graded d -dimensional modules with top T to the variety Gr-Grass_d^T , defined by $[P/C] \mapsto C$, satisfies the conditions of 1.6' in [8, p. 24]. The first of these conditions guarantees that τ is a morphism of varieties. That $(\Delta, \delta) \sim \tau^*(\Gamma, \gamma)$ is clear, as is uniqueness of τ with this property. This proves the first claim.

Clearly, the restriction to the closed subvariety $\text{Gr-Grass}(\mathbb{S})$ of the above universal family parametrized by Gr-Grass_d^T is universal for the graded modules with radical layering \mathbb{S} .

That both Gr-Grass_d^T and $\text{Gr-Grass}(\mathbb{S})$ are projective was shown in Section 2. \square

As pointed out before, given Λ , and a semisimple sequence \mathbb{S} with simple top T , the *full* class of modules with radical layering \mathbb{S} need not have a moduli space. When it does, that moduli space need not be projective, In fact, any affine variety can be realized as a fine moduli space of the form $\text{Grass}(\mathbb{S})$; see [3, Theorem G] and [1, Corollary B].

While in the local case, that is, in the case of a simple top T , classifying arbitrary graded modules is clearly equivalent to classifying those that are generated in degree zero because every local graded module is generated in a single degree, this is a priori no longer true for graded modules with fixed, but unrestricted, top. As announced in the introduction, Section 6 will fill in the gap. There we will show that existence of a moduli space for d -dimensional graded modules with top T forces all of these modules to be direct sums of local submodules. This will reduce the general situation to the one addressed in the next theorem.

THEOREM 4.3. *Let $\Lambda = KQ/I$ be as in Theorem 4.2, and T any finite dimensional semisimple Λ -module. Moreover, let \mathbb{S} be any d -dimensional semisimple sequence with top T . In the following statements, the term “graded module” will stand for “graded module generated in degree zero”.*

(1) *If the d -dimensional graded Λ -modules with radical layering \mathbb{S} have a coarse moduli space classifying them up to graded-isomorphism, then all such modules are (as graded objects) direct sums of local modules.*

A fortiori: If the d -dimensional graded Λ -modules with top T possess a coarse moduli space, they are all direct sums of graded local components.

(2) *Conversely, suppose that all d -dimensional graded Λ -modules with radical layering \mathbb{S} (resp., with top T) are direct sums of graded local submodules. Then there exists a finite partition of the considered class of modules such that the pertinent*

disjoint subclasses have fine moduli spaces providing classification up to graded-isomorphism.

PROOF. (1) Suppose the graded d -dimensional modules with radical layering \mathbb{S} have a coarse moduli space. Criterion 4.1 then forces all $\text{Gr-Aut}_\Lambda(\mathbb{P})$ -orbits of the points $C \in \text{Gr-Grass}(\mathbb{S})$ to be closed in Gr-Grass_d^T and hence in Grass_d^T (see Section 2).

So we only need to show that, whenever P/C with $C \in \text{Gr-Grass}(\mathbb{S})$ is not a direct sum of graded local modules, the orbit $\text{Gr-Aut}_\Lambda(\mathbb{P}).C$ fails to be closed in Grass_d^T . Assume that $P/C = M \oplus N$, where M, N are graded and M is indecomposable and nonlocal. Without loss of generality $C = U \oplus V$ with $U \subseteq \bigoplus_{1 \leq r \leq u} \Lambda z_r$ and $V \subseteq \bigoplus_{u+1 \leq r \leq t} \Lambda z_r$ such that $M = (\bigoplus_{1 \leq r \leq u} \Lambda z_r)/U$ and $N = (\bigoplus_{u+1 \leq r \leq t} \Lambda z_r)/V$. In particular, $u > 1$. Given any $\tau \in K^*$, we consider the automorphism $f_\tau \in \text{Gr-Aut}_\Lambda(\mathbb{P})$ defined by $f_\tau(z_1) = \tau z_1$ and $f_\tau(z_r) = z_r$ for $r \geq 2$. The curve $K^* \rightarrow \text{Gr-Aut}_\Lambda(\mathbb{P}).C$, given by $\tau \mapsto f_\tau(C)$, has a unique extension $\mathbb{P}^1 \rightarrow \text{Gr-Aut}_\Lambda(\mathbb{P}).\bar{C}$ due to completeness of $\text{Gr-Aut}_\Lambda(\mathbb{P}).C$. We denote the value of this extension at infinity by $C' = \lim_{\tau \rightarrow \infty} f_\tau(C)$. First we note that C' is again a d' -dimensional subspace of JP . To see that C' does not belong to $\text{Gr-Aut}_\Lambda(\mathbb{P}).C$, we let $\pi_1 : P \rightarrow \Lambda z_1$ be the projection along $\bigoplus_{r \geq 2} \Lambda z_r$. Setting $\mu = \dim \pi_1(C)$, we pick elements $b_1, \dots, b_\mu \in U$ such that $\pi_1(b_1), \dots, \pi_1(b_\mu)$ form a basis for $\pi_1(C) = \pi_1(U)$. We supplement it with a basis $b_{\mu+1}, \dots, b_\nu$ for $U \cap \text{Ker}(\pi_1)$ to obtain a basis b_1, \dots, b_ν for U . Finally, we add on a basis $b_{\nu+1}, \dots, b_{d'}$ for V , which results in a basis $b_1, \dots, b_{d'}$ for C . Clearly, $b_{\mu+1}, \dots, b_{d'}$ are fixed by all f_τ , and hence belong to C' . Moreover, the following spaces are contained in C' (cf. [5, Lemma 4.7]): Namely, the one-dimensional subspaces $\lim_{\tau \rightarrow \infty} f_\tau(Kb_r)$ for r between 1 and μ . If $b_r = \sum_{1 \leq i \leq t} \lambda_{ri} z_i$ with $\lambda_{ri} \in \Lambda$, then the latter space equals

$$\lim_{\tau \rightarrow \infty} K(\lambda_{r1} z_1 + \sum_{i \geq 2} (1/\tau) \lambda_{ri} z_i) = K \lambda_{r1} z_1 = K \pi_1(b_r).$$

Since the elements $\pi_1(b_1), \dots, \pi_1(b_\mu), b_{\mu+1}, \dots, b_{d'}$ of C' are linearly independent by construction, they form a basis for C' . Consequently, $C' = C_1 \oplus C_2 \oplus C_3$, where $C_1 = \pi_1(C) = \pi_1(U)$, $C_2 = U \cap (\bigoplus_{2 \leq r \leq u} \Lambda z_r) = \sum_{\mu+1 \leq r \leq \nu} Kb_r$, and $C_3 = V$. We thus obtain the following decomposition of the Λ -module P/C' :

$$P/C' = (\Lambda z_1/C_1) \oplus \left(\left(\bigoplus_{2 \leq r \leq u} \Lambda z_r \right) / C_2 \right) \oplus \left(\left(\bigoplus_{u+1 \leq r \leq d'} \Lambda z_r \right) / C_3 \right).$$

Clearly, all three summands are non-trivial, and the third equals N . This shows that the number of indecomposable summands of P/C' exceeds that of P/C , whence $P/C \not\cong P/C'$ as required, and the proof of (1) is complete.

For (2), we assume that all graded d -dimensional modules with top T are direct sums of locals, and consider the following equivalence relation on the set of all partitions (d_1, \dots, d_t) of d with the property that $d_r \geq 1$ for all r . Namely, we call partitions (d_1, \dots, d_t) and (d'_1, \dots, d'_t) equivalent if, for every $a \in \mathbb{N}$ and $i \leq n$, the number of $r \in \{1, \dots, t\}$ with $e(r) = e_i$ and $d_r = a$ equals the number of those r for which $e(r) = i$ and $d'_r = a$. In the following, we will identify partitions of the described type with their equivalence classes. For each partition (d_1, \dots, d_t) , we thus obtain a class $\mathcal{C}(d_1, \dots, d_t)$ of modules which are direct sums $M_1 \oplus \dots \oplus M_t$, where M_r is a graded local module with top $S(r) = \Lambda e(r)/J e(r)$. In view of our

equivalence, the class of all graded d -dimensional modules is the disjoint union of the classes $\mathcal{C}(d_1, \dots, d_t)$, where (d_1, \dots, d_t) runs through the permissible partitions of d . By Theorem 4.2, the variety $\prod_{1 \leq r \leq t} \text{Gr-Grass}_{d_r}^{S(r)}$ is a fine moduli space for the graded isomorphism classes of the objects in $\mathcal{C}(d_1, \dots, d_t)$. The more restricted situation, where only the d -dimensional graded modules with radical layering \mathbb{S} are assumed to be direct sums of local submodules, is dealt with analogously. This proves (2). \square

5. Which varieties occur as fine moduli spaces of graded modules? Examples

Throughout this section, we let T be a simple Λ -module: $T = S_0$ and $P = \Lambda e_0$. As before, we assume $\Lambda = KQ/I$ to be graded by path lengths.

As we will see, the full spectrum of possibilities already occurs in the situation of local graded modules. Namely, each irreducible projective variety arises as a fine moduli space $\text{Gr-Grass}(\mathbb{S})$, where \mathbb{S} is a semisimple sequence with simple top T . If the dimension of \mathbb{S} is d , then $\text{Gr-Grass}(\mathbb{S})$ is an irreducible component of Gr-Grass_d^T in this situation, showing that arbitrary irreducible projective varieties can be realized as irreducible components of fine moduli spaces Gr-Grass_d^T (keep in mind that $\text{Gr-Grass}(\mathbb{S})$ is always a union of irreducible components of Gr-Grass_d^T). We will use a family of examples constructed by Hille in [2] which, in turn, is based on a construction in [3, proof of Theorem G]. Our arguments will illustrate the general computational method sketched below.

Our first set of examples is completely straightforward and does not require any preparation.

EXAMPLES 5.1. *Let \mathbb{S} be a semisimple sequence with simple top T . If $\mathbb{S}_l = 0$ for $l \geq 2$, then $\text{Gr-Grass}(\mathbb{S})$ is either empty or a direct product of classical Grassmannians $\mathfrak{Gr}(u, K^v)$. (Note that in Loewy length two, $\text{Gr-Grass}(\mathbb{S}) = \mathfrak{Grass}(\mathbb{S})$.)*

Conversely, every direct product of classical Grassmannians occurs as a fine moduli space of local modules of Loewy length two with fixed radical layering.

PROOF. By our blanket hypothesis, $\mathbb{S}_0 = T = S_1$. For the first statement, suppose $\mathbb{S}_1 = \bigoplus_{1 \leq i \leq n} S_i^{m_i}$ and assume that $\text{Gr-Grass}(\mathbb{S}) \neq \emptyset$. Moreover, for $1 \leq i \leq n$, let $\alpha_{i1}, \dots, \alpha_{in_i}$ be the distinct arrows from the vertex 1 to the vertex i of Q . Then $m_i \leq n_i$, and any C in $\text{Gr-Grass}(\mathbb{S})$ is a direct sum of subspaces C_i of $\bigoplus_{1 \leq j \leq n_i} K\alpha_{ij} = e_i J e_1 / e_i J^2 e_1$ of dimension $n_i - m_i$, respectively. Conversely, any such direct sum of subspaces yields a point in $\text{Gr-Grass}(\mathbb{S})$. This shows $\text{Gr-Grass}(\mathbb{S})$ to be isomorphic to the direct product $\prod_{1 \leq i \leq n} \mathfrak{Gr}(n_i - m_i, K^{n_i})$.

In light of the preceding paragraph, it is clear how to choose Q , so as to realize any given product of Grassmannians $\mathfrak{Gr}(d_i, K^{n_i})$. \square

Suppose \mathbb{S} is any d -dimensional semisimple sequence with top T and σ a skeleton compatible with \mathbb{S} . In [4], we described a method for determining the distinguished affine cover of $\mathfrak{Grass}(\mathbb{S})$ – see Section 3 – in the ungraded case. The algorithm provided there only requires a minor adjustment to yield the corresponding cover for the graded version $\text{Gr-Grass}(\mathbb{S})$; in fact, computationally, the graded variant amounts to a significant reduction of labor. We will present the procedure for obtaining polynomials defining $\text{Gr-Grass}(\sigma)$ in affine coordinates without proof, since the argument for [4, Theorem 3.14] is readily adapted to the graded situation.

Returning to the notation of Section 3, we first simplify the notation of a skeleton σ with top $T = S_1$, to reflect the fact that $\mathfrak{t} = t_1 = 1$ in this section. This means $\sigma = \sigma^{(1)}$ and allows us to drop the superscript. So, in the present case, σ is simply a set of paths in KQe_1 of cardinality d , which is closed under right subpaths. As spelled out in the proof of Facts 3.3, in dealing with graded modules, we replace the set of paths $\sigma(\alpha, p)$ for any σ -critical pair (α, p) by the following subset $\text{Gr-}\sigma(\alpha, p) \subseteq \sigma(\alpha, p)$. Namely,

$$\text{Gr-}\sigma(\alpha, p) = \{q \in \sigma(\alpha, p) \mid \text{length}(q) = \text{length}(\alpha p)\}.$$

Then, clearly, any module P/C with $C \in \text{Gr-}\mathfrak{Grass}(\sigma)$ satisfies

$$\alpha p(e_1 + C) = \sum_{q \in \text{Gr-}\sigma(\alpha, p)} c_{\alpha p, q} q(e_1 + C)$$

for unique scalars $c_{\alpha p, q}$. Modifying the notation of Section 3, we let N be the disjoint union of the sets $\text{Gr-}\sigma(\alpha, p)$, where (α, p) traces the σ -critical pairs. (Again, a priori, this union may fail to be disjoint; we make it disjoint through suitable labeling.) Then the map

$$\psi : \text{Gr-}\mathfrak{Grass}(\sigma) \rightarrow \mathbb{A}^N, \quad C \mapsto c = (c_{\alpha p, q})_{(\alpha, p) \text{ } \sigma\text{-critical, } q \in \text{Gr-}\sigma(\alpha, p)}$$

is an isomorphism of varieties. Our goal is to determine polynomials whose zero locus in \mathbb{A}^N coincides with the image of ψ . The polynomials we will construct will be in variables $X_{\alpha p, q}$, where (α, p) traces the σ -critical pairs and q the corresponding sets $\text{Gr-}\sigma(\alpha, p)$.

5.2. THE CONGRUENCE RELATION INDUCED BY σ . Keeping σ fixed, we consider the polynomial ring

$$\mathcal{A} = \mathcal{A}(\sigma) := KQ[X_{\alpha p, q} \mid (\alpha, p) \text{ } \sigma\text{-critical, } q \in \text{Gr-}\sigma(\alpha, p)]$$

over the path algebra KQ . On the ring \mathcal{A} , we consider congruence modulo the left ideal

$$\mathcal{C} = \mathcal{C}(\sigma) := \bigoplus_{2 \leq i \leq n} \mathcal{A}e_i + \sum_{(\alpha, p) \text{ } \sigma\text{-critical}} \mathcal{A}(\alpha p - \sum_{q \in \text{Gr-}\sigma(\alpha, p)} X_{\alpha p, q} q),$$

and denote this relation by $\hat{=}$.

In complete analogy with [4, proof of Proposition 3.12], one shows that the quotient \mathcal{A}/\mathcal{C} is a free left module over the commutative polynomial ring $K[X_{\alpha p, q}]$, the cosets $p + \mathcal{C}$ of the paths in σ forming a basis. This means that, for any element $z \in \mathcal{A}$, there exist unique polynomials $\tau_q(X) = \tau_q^z(X)$ in $K[X_{\alpha p, q}]$ such that

$$z \hat{=} \sum_{q \in \sigma} \tau_q(X) q.$$

We explain how to obtain the $\tau_q(X)$ in case $z = p$ is a path, this being sufficient for dealing with arbitrary elements $z \in \mathcal{A}$. If p starts in a vertex different from e_1 , set $\tau_q(X) = 0$ for all q . Now suppose that $p = pe_1$, and let p_1 be the longest right subpath of p that belongs to σ ; this path may have length zero, that is, coincide with e_1 . If $p_1 = p$, set $\tau_p(X) = 1$ and $\tau_q(X) = 0$ for $q \neq p$; otherwise, write $p = p'\alpha p_1$ for some σ -critical pair (α, p_1) and some left subpath p' of p , potentially of length zero. Then $z \hat{=} \sum_{q \in \text{Gr-}\sigma(\alpha, p)} X_{\alpha p, q} p' q$. Iterate this step for each of the

paths $p'q$ appearing in the latter sum, noting that they all have strictly longer right subpaths in σ than does p , while having the same length as p . Thus an inductive procedure will take us to the desired normal form of z under $\hat{=}$.

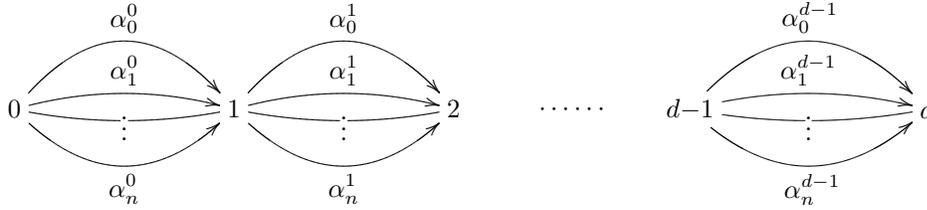
5.3 POLYNOMIALS FOR $\text{Gr-Grass}(\sigma)$. Let \mathcal{R} be any finite generating set for the left ideal $Ie_1 + \cdots + Ie_t$ of KQ – note that such a generating set always exists since all paths of lengths $\geq L + 1$ belong to I . For each $\rho \in \mathcal{R}$, let $\tau_q^\rho(X)$, $q \in \sigma$, be the unique polynomials in $K[X_{\alpha p, q}]$ with

$$\rho \hat{=} \sum_{q \in \sigma} \tau_q^\rho(X) q,$$

as guaranteed by 5.2. Then the zero locus $V(\tau_q^\rho(X) \mid \rho \in \mathcal{R}, q \in \sigma)$ in \mathbb{A}^N of these polynomials is the image of $\text{Gr-Grass}(\sigma)$ under the isomorphism $\psi : \text{Gr-Grass}(\sigma) \rightarrow \text{Im}(\psi) \subseteq \mathbb{A}^N$.

From Examples 5.1 we already know that, for $n \geq 0$, projective n -space can be realized as a fine moduli space $\text{Gr-Grass}(\mathbb{S})$ for a suitable semisimple sequence \mathbb{S} with simple top. In preparation for the examples announced at the beginning of this section, we again realize \mathbb{P}^n in the form $\text{Gr-Grass}(\mathbb{S})$, but this time in a more “ample” setting that will allow us to modify $\text{Gr-Grass}(\mathbb{S})$ by means of additional relations factored out of the pertinent path algebra KQ . Since the above method allows for verification of our claims in very elementary terms, we will include brief arguments for the convenience of the reader.

EXAMPLES 5.4. [2] Let Q be the quiver



and $\Lambda = KQ/I$, where I is generated by all differences $\alpha_i^r \alpha_j^{r-1} - \alpha_j^r \alpha_i^{r-1}$ for $0 \leq i, j \leq n$ and $1 \leq r \leq d-1$. Consider the $(d+1)$ -dimensional semisimple sequence $\mathbb{S} = (S_0, \dots, S_d)$. Again, $P = \Lambda e_0$. We will verify that $\text{Gr-Grass}(\mathbb{S}) \cong \mathbb{P}^n$ for any choice of d .

If $d = 1$, the modules with radical layering \mathbb{S} are uniserial of length 2, and by 5.1, we obtain $\text{Gr-Grass}(\mathbb{S}) \cong \mathfrak{Gr}(n, K^{n+1}) \cong \mathbb{P}^n$ as desired. The isomorphism $\mathfrak{Gr}(n, K^{n+1}) \cong \mathbb{P}^n$ being non-canonical, it will be helpful to specify a concrete incarnation, say $F : \mathbb{P}^n \rightarrow \text{Gr-Grass}(\mathbb{S})$,

$$(k_0 : k_1 : \cdots : k_n) \mapsto \sum_{0 \leq i, j \leq n} \Lambda(k_i \alpha_j^0 - k_j \alpha_i^0).$$

Note that, for $k_0 = 1$, this latter submodule of P equals $\sum_{1 \leq i \leq n} \Lambda(\alpha_i^0 - k_i \alpha_0^0)$.

Now let $d \geq 1$ be arbitrary. We ascertain that every module M with radical layering \mathbb{S} is completely determined by the factor module $M/J^2 M$ of length 2 with radical layers (S_0, S_1) . Let x be a top element of M . Due to the symmetry of our example, we may assume without loss of generality that $\alpha_0^0 x \neq 0$, meaning that

$M/J^2M \cong P/F(k_0 : k_1 : \cdots : k_n)$, with $k_0 = 1$ and suitable $k_i \in K$ for $i \geq 1$. Then it is readily checked – for details consult the following paragraph – that M has skeleton $\sigma = \{e_0, \alpha_0^0, \alpha_0^1 \alpha_0^0, \dots, p\}$, where $p = \alpha_0^{d-1} \cdots \alpha_0^0$, and M is completely determined by the scalars k_i . Indeed, $\alpha_{i_r}^r \cdots \alpha_{i_0}^0 x = k_{i_r} \cdots k_{i_0} \alpha_0^r \cdots \alpha_0^0 x$ for any $r \leq d-1$ and any choice of $i_0, \dots, i_r \in \{0, \dots, n\}$. Conversely, it is clear that every module of length 2 with radical layers (S_0, S_1) occurs as a quotient of a module M with radical layering \mathbb{S} . Thus, again, $\text{Gr-Grass}(\mathbb{S}) \cong \text{Gr-Grass}((S_0, S_1)) \cong \mathbb{P}^n$, an isomorphism being available as in the case $d = 1$.

For justification, we display the congruences determining $\text{Grass}(\sigma)$, where σ is the skeleton consisting of all right subpaths of $p = \alpha_0^{d-1} \cdots \alpha_0^0$ as above. As a left ideal, I is generated by the differences of the form $\alpha_{i'_s}^s \alpha_{i'_{s-1}}^{s-1} \cdots \alpha_{i'_r}^r - \alpha_{i_s}^s \alpha_{i_{s-1}}^{s-1} \cdots \alpha_{i_r}^r$ for $0 \leq r < s \leq d-1$, where (i'_s, \dots, i'_r) is any permutation of (i_s, \dots, i_r) . Moreover, we have (dn) σ -critical pairs, $(\alpha_i^r, \alpha_0^{r-1} \cdots \alpha_0^0)$ for $0 \leq r \leq d-1$ and $0 \leq i \leq n$, giving rise to the basic congruences $\alpha_i^r \alpha_0^{r-1} \cdots \alpha_0^0 \hat{=} X_i^r \alpha_0^{r-1} \cdots \alpha_0^0$. As a result of substituting them into any path of the form $q = \alpha_{i_r}^r \cdots \alpha_{i_0}^0$ of Λ , we obtain the following list of congruences: $\alpha_{i_r}^r \cdots \alpha_{i_0}^0 \hat{=} X_{i_r}^r \cdots X_{i_0}^0 \alpha_0^r \cdots \alpha_0^0$. The variables X_0^0, \dots, X_n^0 can be chosen freely. The relations in I thus yield $X_i^r = X_i^0$ for $0 \leq i \leq n$ and all r , in accordance with the previous paragraph.

The next examples finally show that, indeed, every irreducible projective variety takes on the form $\text{Gr-Grass}(\mathbb{S})$ for some graded algebra Λ and a suitable semisimple sequence \mathbb{S} with simple top. They will thus confirm the assertion we made at the beginning of the section.

EXAMPLES 5.5. [2] Let $V \subseteq \mathbb{P}^n$ be an irreducible projective variety, determined by homogeneous polynomials $f_1, \dots, f_s \in K[X_0, \dots, X_n]$ say. Suppose moreover that the degrees $d(1), \dots, d(s)$ of these polynomials are bounded from above by d . To realize V in the form $\text{Gr-Grass}(\mathbb{S})$, let $\Lambda = KQ/I$, where Q is the quiver of Examples 5.4. The ideal I of KQ is generated by the relations listed in 5.4, next to the following additional ones, labeled g_1, \dots, g_s , one for every polynomial f_r : Write each monomial occurring in f_r in the form $X_{i_{d(r)}} \cdots X_{i_1}$ with $i_j \in \{0, \dots, n\}$, where the order of the factors is irrelevant, and replace each variable X_{i_j} by the arrow $\alpha_{i_j}^j$. This process results in a homogeneous linear combination g_s of paths in Q .

As before, $\mathbb{S} = (S_0, \dots, S_d)$ and $P = \Lambda e_0$.

In light of Examples 5.4, $\text{Grass}(\mathbb{S})$ is a subvariety of \mathbb{P}^n ; indeed, since our present ideal I contains that of 5.4, each module M with radical layering \mathbb{S} is uniquely determined by M/J^2M with radical layering (S_0, S_1) . Moreover, in 5.4, we explicitly provide an assignment sending $\bar{k} \in \mathbb{P}^n$ to a module with radical layering (S_0, S_1) .

So we only need to show that the given variety V coincides with the set of those points $\bar{k} = (k_0 : \cdots : k_n) \in \mathbb{P}^n$ for which there exists $C \in \text{Grass}(\mathbb{S})$ – over the present incarnation of Λ – with the property that the factor M/J^2M of $M = P/C$ is determined by \bar{k} . Without loss of generality, we may assume that V is not contained in the hyperplane $k_0 = 0$, whence the affine variety V_* obtained from V by dehomogenizing at the variable X_0 completely determines V . Hence our task is reduced to showing that the set of those points $C \in \text{Gr-Grass}(\mathbb{S})$ for which $\alpha_0^0(e_0 + C) \neq 0$ in P/C coincides with V_* . The latter is exactly the affine patch $\text{Gr-Grass}(\sigma)$ where σ is the skeleton introduced in 5.4.

The basic congruences leading to polynomials for $\text{Gr-Grass}(\sigma)$ are as listed in the last paragraph of 5.4, and since the relations in 5.4 are among those we factored out of KQ in the present example, we again obtain $X_i^r = X_i^0$ for $0 \leq i \leq n$ and all $r \in \{0, \dots, d-1\}$. Moreover, as before, any path $q = \alpha_{i_r}^r \cdots \alpha_{i_0}^0$ in Λ is congruent to $X_{i_r}^r \cdots X_{i_0}^0 \alpha_0^r \cdots \alpha_0^0$. Under the legitimized identification of X_i^r with X_i for any r , substitution of the basic congruences into the relations g_r thus leads to the dehomogenizations of the polynomials f_r relative to the variable X_0 . \square

6. Appendix: Graded modules generated in mixed degrees

The purpose of this appendix is to show that, in Theorem 4.3, our restriction to graded modules generated in degree zero is superfluous.

Again, we let $\Lambda = KQ/I$ be graded by path lengths, and

$$T = \bigoplus_{1 \leq i \leq n} S_i^{t_i} = \bigoplus_{1 \leq r \leq t} \Lambda e(r)/Je(r),$$

where $t = \sum_i t_i$. As before, $P = \bigoplus_{1 \leq r \leq t} \Lambda z_r \rightarrow T$ is a projective cover of T sending the top elements $z_r = e(r)z_r$ to the residue classes $e(r) + Je(r)$, but now we assume the simple modules $\Lambda e(r)/Je(r)$ to be homogeneous of degree $h(r)$, respectively, and call $h = (h(1), \dots, h(t))$ the *degree vector* of T . Correspondingly, we choose the grading of P so as to make the above projective cover homogeneous; in other words, we assume z_r to be homogeneous of degree $h(r)$ for $r \leq t$. Any factor module M of P by a homogeneous submodule contained in JP is then said to have *top degree vector* h . A full sequence of top elements of M consists of a generating set m_1, \dots, m_t such that $m_r = e(r)m_r$ is homogeneous of degree $h(r)$; this provides the framework for carrying over the concept of a skeleton of M . Our goal is to verify that, in case the graded d -dimensional Λ -modules with top T and top degree vector h have a coarse moduli space, all of the considered modules are direct sums of graded local submodules generated in the degrees $h(1), \dots, h(t)$. We have already dealt with the special case $h = (0, \dots, 0)$ in Theorem 4.3. In the following, we will outline the considerations required to adjust the argument to an arbitrary top degree vector.

To that end, we modify the definitions of Sections 2 and 3 as follows: Let \mathbb{S} be a d -dimensional semisimple sequence with top T and, as in Section 2, denote by d_{li} the multiplicity of the simple module S_i in \mathbb{S}_l . Moreover, let P_{li} be the K -subspace of P generated by all elements of the form pz_r , where p is a path of length $l - h(r)$ ending in the vertex e_i . Again $\text{Gr-Grass}(\mathbb{S})_{li}$ denotes the classical Grassmannian of all $(\dim P_{li} - d_{li})$ -dimensional subspaces of P_{li} . But now, we define

$$\text{Gr-Grass}(\mathbb{S}) = \{C \in \text{Grass}(\mathbb{S}) \mid C = \bigoplus_{1 \leq l \leq L, 1 \leq i \leq n} C_{li} \text{ with } C_{li} \in \text{Gr-Grass}(\mathbb{S})_{li}\},$$

and let Gr-Grass_d^T be the union of the $\text{Gr-Grass}(\mathbb{S})$, where \mathbb{S} runs through the d -dimensional semisimple sequences with top T . The same arguments as used in Section 2 guarantee projectivity of these subvarieties of Grass_d^T . The acting group $\text{Gr-Aut}_\Lambda(P)$ is once more the group of homogeneous Λ -automorphisms of P ; note, however, that in mixed degrees, this group may have nontrivial unipotent radical.

Defining $\text{Gr-Grass}(\sigma)$ as the intersection $\text{Grass}(\sigma) \cap \text{Gr-Grass}_d^T$ – this conforms with Section 3 – one establishes analogues of Facts 3.3; the only adjustment required

in the proof concerns the last sentence: Namely, in the present situation, the image of $\text{Gr-Grass}(\sigma)$ under ψ consists of those points

$$c = (c_{\alpha p^{(r)}, q^{(s)}})_{(\alpha, p^{(r)}) \text{ } \sigma\text{-critical, } q^{(s)} \in \sigma(\alpha, p^{(r)})}$$

in $\psi(\text{Grass}(\sigma))$ for which $c_{\alpha p^{(r)}, q^{(s)}} = 0$ whenever

$$\text{length}(q^{(s)}) + h(s) \neq \text{length}(\alpha p^{(r)}) + h(r).$$

Observation 3.4 remains unchanged; keep in mind that, for $C \in \text{Gr-Grass}_d^T$, the grading of the factor module P/C inherited from P now has top degree vector $h = (h(1), \dots, h(\mathfrak{t}))$. Observation 3.5 should be replaced by the remark that the family of Proposition 3.4 has the local universal property relative to families of graded d -dimensional modules with top T and the specified top degree vector h . As in Criterion 4.1, this setup allows us to apply [8, Proposition 2.13] to conclude that existence of a coarse moduli space for the considered graded modules with top T (resp., with radical layering \mathbb{S}) implies closedness of the $\text{Gr-Aut}_\Lambda(P)$ -orbits of Gr-Grass_d^T (resp., of $\text{Gr-Grass}(\mathbb{S})$). A replica of the argument backing part (1) of Theorem 4.3 finally shows that the latter closedness conditions force all d -dimensional graded modules with top degree vector h and top T (resp., all d -dimensional graded modules with top degree vector h and radical layering \mathbb{S}) to be direct sums of graded local summands generated in the degrees $h(1), \dots, h(\mathfrak{t})$. But, as we already saw, this legitimizes waiving of the hypothesis that $h(r) = 0$ for all r in Theorem 4.3.

Conclusion: For any choice of h , the d -dimensional modules with top T and top degree vector h have a coarse moduli space if and only if the d -dimensional modules with top T generated in degree 0 do, and in case of existence, the two moduli spaces coincide.

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