

TOP-STABLE DEGENERATIONS OF FINITE DIMENSIONAL REPRESENTATIONS I

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Dedicated to Claus Michael Ringel on the occasion of his sixtieth birthday

ABSTRACT. Given a finite dimensional representation M of a finite dimensional algebra, two hierarchies of degenerations of M are analyzed in the context of their natural orders: the poset of those degenerations of M which share the top M/JM with M – here J denotes the radical of the algebra – and the sub-poset of those which share the full radical layering $(J^l M/J^{l+1} M)_{l \geq 0}$ with M . In particular, the article addresses existence of proper top-stable or layer-stable degenerations – more generally, it addresses the sizes of the corresponding posets including bounds on the lengths of saturated chains – as well as structure and classification.

1. INTRODUCTION

Let Λ be a basic finite dimensional algebra over an algebraically closed field K . Our objective is to understand major portions of the degeneration theory of the finite dimensional representations of Λ . This line of inquiry was triggered by Gabriel’s and Kac’s initial exploitation of the affine variety \mathbf{Mod}_d^Λ that parametrizes the d -dimensional representations of Λ ([8], [9], [16], [17]) towards a better understanding of hereditary algebras – equivalently, a better understanding of quiver representations. It was further pursued by Kraft, Riedtmann, Bongartz, Schofield, Skowroński, Zwara, among many others (a selection of basic references is [20], [21], [22], [23], [1], [3], [25], [26] [27]). In intuitive terms, degenerations of a Λ -module M result from a geometry-guided process of breaking its ‘structural bonds’, so as to incrementally unravel its structure and relate it to less complex ones for the purpose of analysis; for the definition, see Section 3. Yet, in general, the problem of classifying all degenerations of a given finite dimensional representation M is wild. By contrast, there are strong indications that the *top-stable* degenerations of M (i.e., those degenerations M' which satisfy $M/JM \cong M'/JM'$, where J denotes the Jacobson radical of Λ) should be classifiable, at least when $T = M/JM$ is squarefree (that is, when T has no simple summands of multiplicity ≥ 2). In tandem, we address the smaller collection of all *layer-stable* degenerations of M , namely those which share the entire radical layering $\mathbb{S}(M) = (J^l M/J^{l+1} M)_{l \geq 0}$ with M . Our primary goal in this article is to give a detailed general description of the hierarchy of top-stable degenerations of M in case T is squarefree (see Theorems 4.4, 4.5, and 5.1, 5.6) and to set up a framework for classification (see Conjectures 4.6).

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Here is an excerpt of the basic theory: Without loss of generality, we may assume Λ to be a path algebra modulo relations, $\Lambda = KQ/I$, where Q is a quiver and I an admissible ideal in the path algebra KQ . It turns out that, for any finite dimensional left Λ -module M with squarefree top, the following two numerical invariants govern the size of the poset of top-stable degenerations of M :

- The difference $\mathfrak{t} - \mathfrak{s}$, where \mathfrak{t} is the number of simple summands in the top of M and \mathfrak{s} the number of indecomposable summands of M , and
- the difference $\mathfrak{m} = \dim_K \text{Hom}_\Lambda(P, JM) - \dim_K \text{Hom}_\Lambda(M, JM)$, where P is a projective cover of M .

We call a module *local* if it has a simple top. The integer $\mathfrak{t} - \mathfrak{s}$, on one hand, measures the discrepancy from the case where M is a direct sum of local modules. On the other hand, if $M \cong P/C$, the integer \mathfrak{m} measures the discrepancy from the situation where all homomorphisms $P \rightarrow JP$ leave C invariant. The sum $\mathfrak{m} + \mathfrak{t} - \mathfrak{s}$ is a measure of how far C is from being stable under arbitrary endomorphisms of P , yet the two individual invariants, \mathfrak{m} and $\mathfrak{t} - \mathfrak{s}$, play fundamentally different roles relative to the degeneration theory of M . We note that \mathfrak{m} is always bounded from above by the sum of the multiplicities of the simple summands of T as composition factors of $JM/\text{soc}(JM)$.

Theorem A. Top-stable degenerations. *Suppose $T = M/JM$ is a direct sum of \mathfrak{t} pairwise non-isomorphic simple Λ -modules and P a projective cover of T . Write M in the form $M \cong P/C$ with $C \subseteq JP$.*

- (1) *The lengths of chains of proper top-stable degenerations of M are bounded above by $\mathfrak{m} + \mathfrak{t} - \mathfrak{s}$.*
- (2) *Existence: The following conditions are equivalent.*
 - (i) *M has a proper top-stable degeneration.*
 - (ii) *$\mathfrak{m} + \mathfrak{t} - \mathfrak{s} > 0$.*
 - (iii) *Either M is not a direct sum of local modules, or else C fails to be invariant under all homomorphisms $P \rightarrow JP$.*
 - (iv) *C fails to be invariant under all endomorphisms of P .*
- (3) *Unique existence: M has a unique proper top-stable degeneration if and only if M is a direct sum of local modules and $\mathfrak{m} = 1$.*
If $\mathfrak{m} = 0$ and $\mathfrak{t} - \mathfrak{s} = 1$, M has precisely two distinct proper top-stable degenerations. (For all values $\mathfrak{m} + \mathfrak{t} - \mathfrak{s} \geq 2$, infinitely many top-stable degenerations can be realized.)
- (4) *Bases: If M' is a top-stable degeneration of M , then M and M' share a basis consisting of paths in the quiver Q .*
To make this statement more precise, we assume (without loss of generality) that P is a left ideal of Λ . Then there exist submodules D and D' of P with $M \cong P/D$ and $M' \cong P/D'$, together with a set \mathcal{B} of paths in KQ (which is closed under right subpaths), such that $\{p + D \mid p \in \mathcal{B}\}$ is a basis for P/D and $\{p + D' \mid p \in \mathcal{B}\}$ is a basis for P/D' .
- (5) *The maximal top-stable degenerations of M always possess a fine moduli space, $\text{max-topdeg}(M)$, classifying them up to isomorphism. This moduli space is a projective variety of dimension at most $\max\{0, \mathfrak{m} + (\mathfrak{t} - \mathfrak{s}) - 1\}$.*
- (6) *The case $\mathfrak{m} = 0$: Let $M = \bigoplus_{1 \leq k \leq \mathfrak{s}} M_k$ be a decomposition of M into indecomposable summands. Then M has only finitely many top-stable degenerations, and each of them is*

a direct sum of top-stable degenerations of the M_k . Moreover, M degenerates top-stably to M' precisely when $M \leq M'$ in the Ext-order (see beginning of Section 3).

In particular: If $\text{Hom}_\Lambda(P, JP) = \text{Hom}_\Lambda(P, \text{soc}(JP))$, the degeneration order on the class of left Λ -modules with top T coincides with the Ext-order.

The upper bounds in Theorem A(1),(5) are attained for arbitrary values of \mathfrak{m} and $\mathfrak{t} - \mathfrak{s}$.

By way of preview of work in progress, we mention that, in the situation of repeated simple summands in the top T , the pressure placed on the structure of M by non-existence of proper top-stable degenerations exceeds expectations based on part A(2) of the theorem: In fact, suppose that $T = M/JM \cong S_1^{t_1} \oplus \cdots \oplus S_n^{t_n}$, where the S_i are pairwise non-isomorphic simples $\Lambda e_i/J e_i$ for suitable primitive idempotents e_i , and let P again be a projective cover of T . Then M is devoid of proper top-stable degenerations if and only if the following three conditions are satisfied:

- M is a direct sum of local modules, say $M = \bigoplus_{1 \leq i \leq n} \bigoplus_{1 \leq j \leq t_i} M_{ij}$, where $M_{ij} = \Lambda e_i/C_{ij}$, i.e., $(\sum_{1 \leq i \leq n} t_i) - \mathfrak{s} = 0$,
- $\dim \text{Hom}_\Lambda(P, JM) = \dim \text{Hom}_\Lambda(M, JM)$, i.e., $\mathfrak{m} = 0$, and
- For each $i \leq n$, the C_{ij} are linearly ordered with respect to inclusion.

Theorem B. Layer-stable degenerations. *Let M , \mathfrak{s} , and \mathfrak{t} be as in Theorem A.*

(1) *If M is a direct sum of local modules, i.e., if $\mathfrak{t} - \mathfrak{s} = 0$, then M has no proper layer-stable degenerations.*

(2) *If $\mathfrak{m} = 0$, then every minimal layer-stable degeneration of M (see 3.7) is of the form $M = U \oplus M/U$, where $U \subseteq M$ is a layer-stably embedded submodule (meaning $J^l U = J^l M \cap U$ for all l).*

(3) *On the other hand, for any positive integer r , there exists an indecomposable finite dimensional module M with squarefree top, over a suitable finite dimensional algebra Λ , such that M has a \mathbb{P}^r -family of pairwise non-isomorphic indecomposable layer-stable degenerations.*

Theorems A and B are excerpted from Theorems 5.1 and 5.6. For the case of local modules (i.e., $\mathfrak{t} = 1$), see Theorem 4.4. The second major result of Section 4, Theorem 4.5, addresses the vertical and horizontal size of the poset of top-stable degenerations of a local module. In rough terms, it says that “all” phenomena not excluded by either Theorem A or the conjectures outlined below actually occur. In particular, this means that every finite direct product of projective spaces can be realized as the moduli space, $\text{max-topdeg}(M)$, classifying the maximal top-stable degenerations of a local module M . Moreover, severe deviation from catenarity is a common phenomenon. The conjectures are backed by strong experimental evidence, collected with the computational methods which we briefly sketch prior to the examples in Sections 4.B and 5.B. These methods will be further elaborated in joint work with R. Donagi [7].

Conjectures. (Excerpt from Conjectures 4.6) We keep the notation of Theorem A. If $M = P/C$ has squarefree top of length \mathfrak{t} , the following hold:

(1) There exist finitely many non-negative integers $n_1 \leq \cdots \leq n_s$ such that conditions (a) and (b) below are satisfied.

(a) The irreducible components of the moduli space $\max\text{-topdeg}(M)$ coincide with its connected components, and each component is isomorphic to a direct product of projective spaces, $\prod_{i \in I} \mathbb{P}^{n_i}$, for a suitable subset I of $\{1, \dots, s\}$.

(b) For each $h \leq m$, the top-stable degenerations of height h above M (in the usual degeneration order) can be partitioned into finitely many disjoint families, each of which is bijectively parametrized by an orbit of the standard torus action on a product $\prod_{i \in I} \mathbb{P}^{n_i}$, for some $I \subseteq \{1, \dots, s\}$. (The torus we refer to is $\prod_{i \in I} (K^*)^{n_i+1}$.)

If s and $\sum_{i \leq s} n_i$ are chosen minimal with respect to (a) and (b), then the numbers s and n_1, \dots, n_s are isomorphism invariants of M .

(2) Suppose, more restrictively, that M is local (in particular, this implies $\mathfrak{t} - \mathfrak{s} = 0$). Then $\max\text{-topdeg}(M)$ is irreducible. If, in addition, we require the poset of top-stable degenerations of M to be finite, its cardinality (counting M) is bounded from below by $\mathfrak{m} + 1$ and from above by $2^{\mathfrak{m}}$. Moreover, M has a unique maximal top-stable degeneration in that case, and all saturated chains linking M to the latter have length \mathfrak{m} .

Our examples will show that the catenarity condition under (2) fails without the finiteness hypothesis – with arbitrarily large gaps between the lengths of maximal and minimal saturated chains in general. Moreover, all of the conjectural bounds are attained.

An ulterior motivation for the study of degenerations stems from their impact on possibilities of more comprehensive classification, in particular, on the problems of when the full classes of modules with fixed top T or fixed radical layering \mathbb{S} admit fine or coarse moduli spaces (see [13]). In fact, as is well-known, degenerations of modules constitute the foremost obstruction in the way of helpful quotients of stable subvarieties of \mathbf{Mod}_d^Λ by their GL_d -actions, for the following reason: As the GL_d -orbits are in one-to-one correspondence with the isomorphism types of d -dimensional Λ -modules, one looks for quotients that separate orbits; such separation is obviously ruled out if the orbits fail to be closed, that is, if the modules represented by these orbits have proper degenerations. Unfortunately – in contrast to the G -spaces typically arising in classical geometric problems – there are hardly any closed orbits in the module-theoretic setting, and hence the standard methods of invariant theory are not effective in this context. This was already addressed by King (see [18]).

Concerning methodology, the class of top-stable degenerations of a given module admits a twofold approach: one via the quasi-affine subvariety \mathbf{Mod}_d^T of the classical variety \mathbf{Mod}_d^Λ that parametrizes the d -dimensional Λ -modules with fixed top T , the other – our main resource here – via a projective variety \mathfrak{Grass}_d^T which parametrizes this same class of representations. The variety \mathfrak{Grass}_d^T is in turn partitioned by a group action whose orbits correspond bijectively to the isomorphism classes of the d -dimensional representations with top T . But, in contrast to the reductive group acting on \mathbf{Mod}_d^Λ , the group acting on \mathfrak{Grass}_d^T has a large unipotent radical in all interesting cases. This fact, in combination with completeness of \mathfrak{Grass}_d^T , brings a different cache of geometric methods to bear, as witnessed for instance by Proposition 4.2 and Corollary 5.4, next to the proofs of the main results. The two modes of approach combine smoothly, due to a result of Bongartz and the author (see Section 2).

In Section 2 we summarize prerequisites; the more technical ones will only be required for proofs and examples. In Section 3 we assemble a number of general facts about degen-

erations, not restricted to modules with squarefree tops, of which 3.4–3.7 appear to hold the highest independent interest. Crucial definitions concerning degenerations, in particular the concept of *height*, can be found at the end of Section 3. Section 4 is focused on the local case, i.e., the case of a simple top T , a situation which stands out through numerous special features; since all saturated chains of top-stable degenerations pass through direct sums of local modules, the “final degeneration stages” of nonlocal modules lead back to the local scenario. Section 5 addresses arbitrary squarefree tops. In Sections 4.B and 5.B we supplement the theory with examples.

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2. PREREQUISITES

General conventions:

Throughout, Λ will be a basic finite dimensional algebra over an algebraically closed field K . Hence, we may assume without loss of generality that $\Lambda = KQ/I$, where Q is a quiver and I an admissible ideal in the path algebra KQ . Moreover, L will denote the largest integer for which the power J^L of the Jacobson radical J of Λ does not vanish; in other words, $L+1$ is the Loewy length of Λ . The quiver Q provides us with a distinguished set of primitive idempotents e_1, \dots, e_n of Λ , which are in bijective correspondence with the vertices of Q ; notationally, we will often not distinguish between the vertices and the e_i . As is well-known, the factors $S_i = \Lambda e_i / J e_i$ then form a set of representatives for the simple left Λ -modules. An element x of a (left) Λ -module M will be called a *top element* of M if $x \notin JM$ and x is normed by some e_i , that is, $x = e_i x$.

Moreover, we will observe the following conventions: The product pq of two paths p and q in KQ stands for “first q , then p ”; in particular, pq is zero unless the end point of q coincides with the starting point of p . In this spirit, we call a path p_1 a *right subpath* of p if $p = p_2 p_1$ for some path p_2 . A *path in Λ* will be an element of the form $p + I$, where p is a path in KQ ; we will suppress the residue notation, provided there is no risk of ambiguity. Further, we will gloss over the distinction between the left Λ -structure of a module $M \in \Lambda\text{-mod}$ and its induced KQ -structure. (On the other hand, when we refer to paths in KQ , we will always point this out explicitly.)

Let a_1, \dots, a_r be a set of algebra generators for Λ over K ; for $\Lambda = KQ/I$ as above, a typical set of generators consists of the primitive idempotents e_1, \dots, e_n together with the (residue classes in Λ of the) arrows in Q . Recall that, for $d \in \mathbb{N}$, the classical affine variety of d -dimensional representations of Λ can be described in the form

$$\mathbf{Mod}_d^\Lambda = \{(x_i)_{i \leq r} \in \prod_{1 \leq i \leq r} \text{End}_K(K^d) \mid \text{the } x_i \text{ satisfy all relations satisfied by the } a_i\}.$$

Given a semisimple left Λ -module T , we denote by \mathbf{Mod}_d^T (or $\mathbf{Mod}_d^{\Lambda, T}$) the locally closed subvariety of \mathbf{Mod}_d^Λ which contains precisely the points representing modules M with top

T (that is, with $M/JM \cong T$). Thus the orbits of the GL_d -conjugation action on \mathbf{Mod}_d^T are in one-to-one correspondence with the isomorphism types of d -dimensional left Λ -modules that have top T .

Background on Grassmannians of d -dimensional modules with fixed top T :

We give a brief review of those constructions and results from [13] which will be pivotal in the sequel (see also [4] and [5] for background). For that purpose, we fix a finite dimensional semisimple module T , together with a positive integer d , and let P be a projective cover of T .

Our principal tool will be the following closed subvariety of the classical Grassmannian $\mathcal{Gr}(d', JP)$ of d' -dimensional subspaces of the K -space JP , where $d' = \dim_K P - d$:

$$\mathfrak{Grass}_d^T = \{C \in \mathcal{Gr}(d', JP) \mid C \text{ is a } \Lambda\text{-submodule of } JP\}.$$

This variety comes with an obvious surjection

$$\phi : \mathfrak{Grass}_d^T \longrightarrow \{\text{isomorphism types of } d\text{-dimensional modules with top } T\},$$

sending C to the class of P/C . Clearly, the fibres of ϕ coincide with the orbits of the natural $\mathrm{Aut}_\Lambda(P)$ -action on \mathfrak{Grass}_d^T . While the global geometry of the projective variety \mathfrak{Grass}_d^T cannot be reasonably compared with that of the quasi-affine variety \mathbf{Mod}_d^T , the “relative geometry” of the $\mathrm{Aut}_\Lambda(P)$ -stable subsets of \mathfrak{Grass}_d^T is tightly linked to that of the GL_d -stable subsets of \mathbf{Mod}_d^T (see [5]). Namely, if the orbits $\mathrm{Aut}_\Lambda(P).C$ and $\mathrm{GL}_d.x$ in \mathfrak{Grass}_d^T and \mathbf{Mod}_d^T , respectively, represent the same Λ -module up to isomorphism, the assignment $\mathrm{Aut}_\Lambda(P).C \mapsto \mathrm{GL}_d.x$ induces a bijection

$$\{\mathrm{Aut}_\Lambda(P)\text{-stable subsets of } \mathfrak{Grass}_d^T\} \rightarrow \{\mathrm{GL}_d\text{-stable subsets of } \mathbf{Mod}_d^T\}$$

which preserves openness, closures, connectedness, irreducibility, and types of singularities. This correspondence permits transfer of information between the two settings.

The first of the two isomorphism invariants of modules we use to organize \mathfrak{Grass}_d^T from a representation-theoretic viewpoint is the *radical layering*. It leads to a partition of \mathfrak{Grass}_d^T into locally closed subvarieties (in general, it is not a stratification, however).

Recall that the *radical layering* of a module M is the sequence

$$\mathbb{S}(M) = (M/JM, JM/J^2M, \dots, J^{L-1}M/J^L M, J^L M),$$

where we identify semisimple modules with their isomorphism classes; in effect, we are dealing with a matrix of discrete invariants keeping count of the multiplicities of the simple modules in the individual radical layers of M . Correspondingly, we consider the following locally closed subvarieties of \mathfrak{Grass}_d^T :

$$\mathfrak{Grass}(\mathbb{S}) = \{C \in \mathfrak{Grass}_d^T \mid \mathbb{S}(P/C) = \mathbb{S}\}$$

for any d -dimensional semisimple sequence \mathbb{S} with top T ; by this we mean any sequence $(\mathbb{S}_0, \dots, \mathbb{S}_L)$ of semisimple modules such that $\mathbb{S}_0 = T$, each \mathbb{S}_l embeds into $J^l P/J^{l+1} P$, and

$\sum_{0 \leq l \leq L} \dim \mathbb{S}_l = d$. Clearly, the above one-to-one correspondence between the $\text{Aut}_\Lambda(P)$ -stable subsets of \mathbf{Grass}_d^T and the GL_d -stable subsets of \mathbf{Mod}_d^Λ restricts to a correspondence between the $\text{Aut}_\Lambda(P)$ -stable subsets of $\mathbf{Grass}(\mathbb{S})$ and the GL_d -stable subsets of a locally closed subvariety of $\mathbf{Mod}_d^T \subseteq \mathbf{Mod}_d^\Lambda$, which we denote by $\mathbf{Mod}(\mathbb{S})$: It consists of those points in the classical variety \mathbf{Mod}_d^T which correspond to the modules with radical layering \mathbb{S} .

Further prerequisites, required for proofs and examples only:

For the remainder of this section, we will assume T to be *squarefree*. Thus

$$T = \bigoplus_{1 \leq i \leq t} S_i,$$

after potential reordering of the primitive idempotents. Correspondingly, we may take P to be $\bigoplus_{1 \leq i \leq t} \Lambda e_i$. In other words, P is the left ideal of Λ generated by $e_1 + \dots + e_t$.

The restriction to squarefree T allows for a simplified description of the second invariant of a module, which we use to organize \mathbf{Grass}_d^T for our purposes, namely its set of *skeletons*. Since each such skeleton consists of d paths of length at most L in the quiver Q , the set of all skeletons of d -dimensional modules is finite.

Definitions. For $C \in \mathbf{Grass}_d^T$, we call a set σ of paths of lengths $\leq L$ in KQ a *skeleton* of $M = P/C$ if the following two conditions are satisfied:

- (a) σ is closed under right subpaths, that is ' $p_2 p_1 \in \sigma \implies p_1 \in \sigma$ ';
- (b) for each $l \in \{0, \dots, L\}$, the set

$$\{p + C \mid p \in \sigma, \text{length}(p) = l\}$$

induces a K -basis for $J^l M / J^{l+1} M$.

Existence of at least one skeleton of M is obvious. Note, moreover, that condition (b) is equivalent to postulating that, for *every* sequence x_1, \dots, x_t of top elements of M with $e_i x_i = x_i$ and any $l \in \{0, \dots, L\}$, the set

$$\bigcup_{1 \leq i \leq t} \{p x_i \mid p = p e_i \in \sigma, \text{length}(p) = l\}$$

induces a basis for $J^l M / J^{l+1} M$. This guarantees that isomorphic modules have the same skeletons.

We further observe that (a) and (b) entail the following condition (c): $\sigma \cap \{e_1, \dots, e_t\} = \{e_1, \dots, e_t\}$. To obtain a suitable partition of \mathbf{Grass}_d^T , we refer to a set $\sigma \subseteq KQ$ as an *abstract d -dimensional skeleton* with top T if σ consists of d distinct paths in KQ satisfying conditions (a) and (c). An abstract skeleton need not be realizable as a skeleton of a module.

Every abstract d -dimensional skeleton σ with top T gives rise to an $\text{Aut}_\Lambda(P)$ -stable locally closed subset of \mathbf{Grass}_d^T , namely

$$\mathbf{Grass}(\sigma) := \{C \in \mathbf{Grass}_d^T \mid \sigma \text{ is a skeleton of } P/C\}.$$

A priori, we do not exclude those skeletons σ for which $\mathbf{Grass}(\sigma)$ is empty.

Each $\mathbf{Grass}(\mathbb{S})$ is the union of certain $\mathbf{Grass}(\sigma)$. Given \mathbb{S} , the appropriate choice of skeletons σ is as follows: An abstract skeleton σ is said to be *compatible with \mathbb{S}* in case, for each $l \in \{0, \dots, L\}$ and $i \in \{1, \dots, n\}$, the cardinality of the set of paths in σ that have length l and end in e_i equals the multiplicity of the simple module S_i in S_l . If σ is a skeleton of M , then clearly σ is compatible with $\mathbb{S}(M)$, and hence $\mathbf{Grass}(\mathbb{S})$ is the union of those $\mathbf{Grass}(\sigma)$ which correspond to skeletons compatible with \mathbb{S} .

Theorem 2.1. [13, Theorem 3.5] *Let \mathbb{S} be a d -dimensional semisimple sequence with squarefree top T . Then the sets $\mathbf{Grass}(\sigma)$, where σ runs through the skeletons compatible with \mathbb{S} , are $\text{Aut}_\Lambda(P)$ -stable and form an affine open cover of the variety $\mathbf{Grass}(\mathbb{S})$ representing the modules with radical layering \mathbb{S} . \square*

The $\text{Aut}_\Lambda(P)$ -stability of the resulting cover $(\mathbf{Grass}(\sigma))_\sigma$ of \mathbf{Grass}_d^T will be pivotal in the sequel. Caveat: While they are open in the pertinent subvarieties $\mathbf{Grass}(\mathbb{S})$, the affine charts $\mathbf{Grass}(\sigma)$ fail to be open in \mathbf{Grass}_d^T in general.

Connection with Schubert cells. We briefly comment on how this affine cover of \mathbf{Grass}_d^T relates to the Schubert cells of the classical Grassmannian $\mathcal{G}r(d', JP)$ in which \mathbf{Grass}_d^T is embedded as a closed subvariety (recall that $d' = \dim_K P - d$). Each $\mathbf{Grass}(\sigma)$ embeds into the big Schubert cell, $\text{Schu}(\sigma)$, which consists of the points in $\mathcal{G}r(d', JP)$ that complement $\bigoplus_{p \in \sigma} Kp$ in P . In fact, $\mathbf{Grass}(\sigma)$ is the intersection of $\text{Schu}(\sigma)$ with $\mathbf{Grass}(\mathbb{S})$, where \mathbb{S} is the unique semisimple sequence with which σ is compatible. (We point out that, by contrast, the intersection of $\text{Schu}(\sigma)$ with \mathbf{Grass}_d^T is usually not closed under the $\text{Aut}_\Lambda(P)$ -action.) In essence, the affine coordinates for $\mathbf{Grass}(\sigma)$ presented below are the Plücker coordinates relative to a conveniently chosen basis for JP that contains σ . Yet, we only need to retain a small portion of the numerical data recorded by these latter coordinates, owing to the fact that we are exclusively interested in submodules of JP . This restriction to submodules is also responsible for the fact that our coordinates are determined by polynomials that can readily be derived from the relations which we factor out of KQ to obtain Λ .

We conclude the section by describing the mentioned coordinate system for $\mathbf{Grass}(\sigma)$, relevant only for our analysis of examples. Starting with the quiver Q and a finite left ideal generating set \mathcal{R} of I , we can obtain polynomials describing $\mathbf{Grass}(\sigma)$ as outlined below; proofs can be found in [13, 3.10–3.13]. A σ -critical pair is a pair (α, p) , where p is a path in σ and α is an arrow such that αp is a path which fails to belong to σ . Given $C \in \mathbf{Grass}(\sigma)$, there are unique scalars $c_{\alpha p, q}$ with

$$\alpha p + C = \sum_{q \in \sigma(\alpha, p)} c_{\alpha p, q} (q + C),$$

where $\sigma(\alpha, p)$ is the set of all those paths $q \in \sigma$ which are at least as long as αp and have the same end point as αp . Clearly, the sets $\sigma(\alpha, p)$ for distinct critical pairs (α, p) will intersect in general.

We let N be the *disjoint* union of the $\sigma(\alpha, p)$, where (α, p) traces the σ -critical pairs. Whenever this promotes clarity, we will use the indexing $q = q(\alpha, p)$ to achieve formal disjointness.

Given $C \in \mathfrak{Grass}(\sigma)$, the point $c = (c_{\alpha p, q}) \in \mathbb{A}^N$ clearly determines $C \subseteq P = \bigoplus_{1 \leq i \leq t} \Lambda e_i$. Indeed,

$$C = \sum_{(\alpha, p) \text{ } \sigma\text{-critical}} \Lambda(\alpha p - \sum_{q \in \sigma(\alpha, p)} c_{\alpha p, q} q).$$

Due to [13, Theorem 3.14], the assignment $\mathfrak{Grass}(\sigma) \rightarrow \mathbb{A}^N$ that sends any point $C \in \mathfrak{Grass}(\sigma)$ to the corresponding point $c \in \mathbb{A}^N$ induces an isomorphism from $\mathfrak{Grass}(\sigma)$ onto the closed subvariety of \mathbb{A}^N determined by the following polynomials $\tau_q^\rho(X)$ in

$$K[X_{\alpha p, q} \mid (\alpha, p) \text{ } \sigma\text{-critical}, q \in \sigma(\alpha, p)].$$

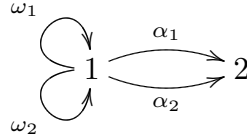
Let \mathfrak{A} be the polynomial ring with coefficients in KQ (noncommutative in general) in the variables $X_{\alpha p, q}$, where (α, p) ranges through the σ -critical pairs, and q through $\sigma(\alpha, p)$. Moreover, let \mathfrak{C} be the left ideal of \mathfrak{A} generated by the idempotents $e_{t+1}, \dots, e_n \in KQ$ annihilating T together with the differences

$$\alpha p - \sum_{q \in \sigma(\alpha, p)} X_{\alpha p, q} q.$$

Then $\mathfrak{A}/\mathfrak{C}$ is a free left module over the commutative polynomial ring $K[X_{\alpha p, q}]$ with basis $(q + \mathfrak{C})_{q \in \sigma}$. In particular, each element ρ in KQ is congruent, modulo \mathfrak{C} , to a sum $\sum_{q \in \sigma} \tau_q^\rho(X) q$ for unique polynomials $\tau_q^\rho(X)$ in $K[X_{\alpha p, q}]$. The affine vanishing set in \mathbb{A}^N of the $\tau_q^\rho(X)$, where ρ traces any left ideal generating set \mathcal{R} of I , depends only on σ , not on the choice of \mathcal{R} . As announced, this vanishing set is isomorphic to $\mathfrak{Grass}(\sigma)$ and will henceforth be identified with the latter variety (see [13, Section 6] for a proof).

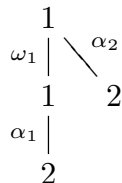
We illustrate the procedure for finding the described affine charts of \mathfrak{Grass}_d^T with an example. Throughout, $\hat{=}$ denotes congruence modulo \mathfrak{C} in \mathfrak{A} .

Example 2.2. Let $\Lambda = KQ/I$, where Q is the quiver



and $I \subseteq KQ$ the ideal generated by $\omega_i \omega_j$ for $i, j \in \{1, 2\}$ and $\alpha_1 \omega_1 - \alpha_2 \omega_2$. Clearly, the listed relations generate I as a left ideal of KQ .

For $T = S_1$, we consider the 4-dimensional abstract skeleton $\sigma = \{e_1, \omega_1, \alpha_1 \omega_1, \alpha_2\}$ with top T , compatible with the semisimple sequence $\mathbb{S} = (S_1, S_1 \oplus S_2, S_2)$. This skeleton can be visualized as the set of paths that start at the top of the following diagram:



In order to compute $\mathfrak{Grass}(\sigma)$, we list the σ -critical pairs, together with their sets $\sigma(\alpha, p)$. Obviously, there is no harm in ignoring pairs (α, p) with $\alpha p \in I$; the pairs that are left are (ω_2, e_1) with $\sigma(\omega_2, e_1) = \{\omega_1\}$, (α_1, e_1) with $\sigma(\alpha_1, e_1) = \{\alpha_2, \alpha_1\omega_1\}$, and (α_2, ω_1) with $\sigma(\alpha_2, \omega_1) = \{\alpha_1\omega_1\}$. This leads to the following basic equivalences in $KQ[X_1, \dots, X_4]$, where X_1, \dots, X_4 stand for $X_{\omega_2, \omega_1}, X_{\alpha_1, \alpha_2}, X_{\alpha_1, \alpha_1\omega_1}, X_{\alpha_2\omega_1, \alpha_1\omega_1}$; namely $\omega_2 \hat{=} X_1\omega_1$, $\alpha_1 \hat{=} X_2\alpha_2 + X_3\alpha_1\omega_1$, and $\alpha_2\omega_1 \hat{=} X_4\alpha_1\omega_1$. As one finds (by an easy algorithm described in [13, proof of Proposition 3.12]), all relations $\omega_i\omega_j$ are congruent to zero modulo \mathfrak{C} in \mathfrak{A} , while $\alpha_1\omega_1 - \alpha_2\omega_2$ is congruent to $(1 - X_4X_1)\alpha_1\omega_1$. Thus,

$$\mathfrak{Grass}(\sigma) = \{(c_1, c_2, c_3, c_4) \in \mathbb{A}^4 \mid 1 - c_1c_4 = 0\} \cong V(XY - 1) \times \mathbb{A}^2. \quad \square$$

3. GENERAL FACTS ABOUT TOP-STABLE AND LAYER-STABLE DEGENERATIONS

If M and M' are d -dimensional Λ -modules, represented by points x and x' in the classical affine variety \mathbf{Mod}_d^Λ , respectively, then M' is said to be a *degeneration* of M if x' belongs to the closure of the orbit $\mathrm{GL}_d \cdot x$ in \mathbf{Mod}_d^Λ . Following [1], we write $M \leq_{\mathrm{deg}} M'$ in that case. This notation reflects the fact that \leq_{deg} is a partial order on the isomorphism classes of d -dimensional modules. Moreover, we say that M' is a *top-stable degeneration* of M if $M \leq_{\mathrm{deg}} M'$ and $M/JM = M'/JM'$, and that M' is a *layer-stable degeneration* of M if $M \leq_{\mathrm{deg}} M'$ and $\mathbb{S}(M) = \mathbb{S}(M')$. The latter means that $J^l M/J^{l+1}M = J^l M'/J^{l+1}M'$ for all l . (Recall that we identify isomorphic semisimple modules.)

It has been the object of extensive efforts to characterize the partial order \leq_{deg} in purely algebraic terms. This order is known to be trapped between two algebraically defined partial orders which are easier to track, namely the ‘‘Hom-order’’, weaker than \leq_{deg} in general, given by ‘‘ $M \leq_{\mathrm{hom}} N \iff \dim_K \mathrm{Hom}_\Lambda(M, X) \leq \dim_K \mathrm{Hom}_\Lambda(N, X)$ for all X ’’ and the ‘‘extension-order’’, stronger than \leq_{deg} , arising as the transitive closure of the relation ‘‘ $M \leq_{\mathrm{ext}} N$ if $N \cong U \oplus V$ for some exact sequence $0 \rightarrow U \rightarrow M \rightarrow V \rightarrow 0$ ’’ (see, e.g., [22], or [3] for an overview). In case Λ has finite representation type, the degeneration order has been proved to coincide with the Hom-order; the final step in establishing this is due to Zwara [26] and is the culmination of a line of research going back to the 1980’s; see e.g. [1, 22, 23]. In general, however, the two algebraically defined orders \leq_{hom} and \leq_{ext} yield only a loose bracket framing \leq_{deg} . The following general description of \leq_{deg} , again due to Zwara [27] with roots in a result of Riedtmann [22], is in algebraic terms, but hard to track in general: Namely, $M \leq_{\mathrm{deg}} M'$ precisely when there exists an exact sequence of one of the following types: $0 \rightarrow M' \rightarrow M \oplus X \rightarrow X \rightarrow 0$ or $0 \rightarrow X \rightarrow M \oplus X \rightarrow M' \rightarrow 0$ for some module X . Note that the implication $(M \leq_{\mathrm{ext}} M' \implies M \leq_{\mathrm{deg}} M')$ specializes to $M \leq_{\mathrm{deg}} U \oplus M/U$ for every submodule U of M , and thus gives rise to the most accessible degenerations. This places a spotlight on indecomposable degenerations.

Fix $d \in \mathbb{N}$, and a semisimple Λ -module T with projective cover P . *None of the basic observations on layer- or top-stable degenerations assembled in this section requires squarefreeness of T .*

Suppose M and M' are represented by points x and x' in \mathbf{Mod}_d^T . Clearly, M' is a top-stable degeneration of M precisely when x' belongs to the relative closure of the orbit $\mathrm{GL}_d \cdot x$ in \mathbf{Mod}_d^T , and M' is a layer-stable degeneration of M precisely when x' belongs to the closure of $\mathrm{GL}_d \cdot x$ in the subvariety $\mathbf{Mod}(\mathbb{S})$. By the two-way transfer of geometric

properties between the GL_d -stable subsets of \mathbf{Mod}_d^T on one hand and the $\mathrm{Aut}_\Lambda(P)$ -stable subsets of \mathfrak{Grass}_d^T on the other (cf. Section 2), this description of the top-, resp. layer-stable, degenerations can be reformulated as follows:

Observation 3.1. *If $M \cong P/C$ with $C \in \mathfrak{Grass}_d^T$, then a module M' is a top-stable (resp., layer-stable) degeneration of M if and only if $M' \cong P/C'$, where C' belongs to the relative closure of the orbit $\mathrm{Aut}_\Lambda(P).C$ in \mathfrak{Grass}_d^T (resp., in $\mathfrak{Grass}(\mathbb{S})$). \square*

The next observation records some general facts concerning the acting group $\mathrm{Aut}_\Lambda(P)$ and its orbits. They were established in [13, Observation 2.3, Proposition 2.9, Lemma 2.10, and Proposition 4.1]; the structure of the $(\mathrm{Aut}_\Lambda(P))_u$ -orbits arises as a special case of a result of Rosenlicht [24] for arbitrary unipotent group actions. We start by noting that the unipotent radical of $\mathrm{Aut}_\Lambda(P)$ is

$$(\mathrm{Aut}_\Lambda(P))_u = \{\mathrm{id} + g \mid g \in \mathrm{Hom}_\Lambda(P, JP)\}.$$

Observation 3.2. *Abbreviating $(\mathrm{Aut}_\Lambda(P))_u$ to \mathcal{U} , we obtain:*

- $\mathrm{Aut}_\Lambda(P) \cong \mathcal{U} \rtimes \mathrm{Aut}_\Lambda(T)$ as algebraic groups. In particular, $\mathrm{Aut}_\Lambda(P)$ is a rational variety.
- For any point $C \in \mathfrak{Grass}_d^T$, say $C \in \mathfrak{Grass}(\mathbb{S})$, the \mathcal{U} -orbit $\mathcal{U}.C$ is closed in $\mathfrak{Grass}(\mathbb{S})$ and isomorphic to the affine space \mathbb{A}^m , where

$$m = \dim_K \mathrm{Hom}_\Lambda(P, JP/C) - \dim_K \mathrm{Hom}_\Lambda(P/C, JP/C).$$

- For any $C \in \mathfrak{Grass}_d^T$, the full orbit $\mathrm{Aut}_\Lambda(P).C$ has dimension

$$\dim_K \mathrm{Hom}_\Lambda(P, P/C) - \dim_K \mathrm{End}_\Lambda(P/C).$$

- For any $C \in \mathfrak{Grass}_d^T$, the orbit map $\mathrm{Aut}_\Lambda(P) \rightarrow \mathrm{Aut}_\Lambda(P).C$ is separable, and hence is a geometric quotient of $\mathrm{Aut}_\Lambda(P)$ by the stabilizer subgroup $\mathrm{Stab}_{\mathrm{Aut}_\Lambda(P)} C$. Restriction to $\mathcal{U}.C$ shows that the orbit map $\mathcal{U} \rightarrow \mathcal{U}.C$ is a geometric quotient of \mathcal{U} by $\mathrm{Stab}_{\mathcal{U}} C$. \square

The following straightforward observation supplements the remark about degenerations arising from short exact sequences. Given $M \in \Lambda\text{-mod}$, we call a submodule U of M *top-stably embedded* in M if $JM \cap U = JU$, *layer-stably embedded* in M in case $(J^l M) \cap U = J^l U$ for all $l \geq 1$. Clearly, the latter amounts to the same as to say that the embedding $U \hookrightarrow M$ induces monomorphisms $J^l U / J^{l+1} U \hookrightarrow J^l M / J^{l+1} M$ for all $l \geq 0$, whence our terminology.

Observation 3.3. *Suppose U is a submodule of M . Then U is top-stably (resp., layer-stably) embedded in M if and only if $U \oplus (M/U)$ is a top-stable (resp., layer-stable) degeneration of M . \square*

In particular, the refined extension order, selecting short exact sequences that remain exact on multiplication by powers of J , yields a partial order on the isomorphism classes of layer-stable degenerations of M .

The corollary to the next observation provides an initial piece of structural information on the top-stable degenerations of M in comparison with M : Roughly speaking, the degeneration process involves only upward mobility of simple composition factors relative to

the radical layering of M . The following partial order on the d -dimensional semisimple sequences with top T allows us to make this precise. Namely, we define $\mathbb{S} \leq \mathbb{S}'$ if and only if $\bigoplus_{l \leq r} \mathbb{S}_l$ is a direct summand of $\bigoplus_{l \leq r} \mathbb{S}'_l$ for all r .

The observation tying this order to degenerations is inspired by the Schubert cells in the classical Grassmannian $\mathcal{G}r(d', JP)$ containing \mathfrak{Grass}_d^T , in that its proof is based on intersection dimensions with partial flags of subspaces of JP .

Observation 3.4. *Suppose that \mathbb{S} is a d -dimensional semisimple sequence with top T . Then the union $\bigcup_{\mathbb{S}' \geq \mathbb{S}} \mathfrak{Grass}(\mathbb{S}')$ is closed in \mathfrak{Grass}_d^T .*

Proof. We write $J^l P / J^{l+1} P = \bigoplus_{1 \leq i \leq n} S_i^{t_{li}}$ and $\mathbb{S}_l = \bigoplus_{1 \leq i \leq n} S_i^{s_{li}}$ for each $l \in \{1, \dots, L\}$; note that $s_{li} \leq t_{li}$ by definition of semisimple sequences with top T . Moreover, for each vertex e_i of Q , we consider the following partial flag of subspaces of the K -space JP :

$$e_i J^L P \subseteq e_i J^{L-1} P \subseteq \dots \subseteq e_i JP.$$

Then the union of the $\mathfrak{Grass}(\mathbb{S}')$, with \mathbb{S}' tracing the semisimple sequences larger than or equal to \mathbb{S} under \leq , coincides with $\mathfrak{Grass}_d^T \cap (\bigcap_{1 \leq i \leq n} V_i)$, where V_i is the following subset of the classical Grassmannian $\mathcal{G}r(d', JP)$:

$$V_i = \{C \in \mathcal{G}r(d', JP) \mid \dim(C \cap e_i J^l P) \geq \sum_{k \geq l} (t_{ki} - s_{ki}) \text{ for all } 1 \leq l \leq L\}.$$

The latter sets are well known to be closed in $\mathcal{G}r(d', JP)$, and our claim follows. \square

Corollary 3.5. *Suppose M' is a top-stable degeneration of M . Then $\mathbb{S}(M') \geq \mathbb{S}(M)$. If M' fails to be a layer-stable degeneration of M , this inequality is strict. \square*

Finally, we note that we can always obtain the top-stable degenerations P/C' of a module P/C along projective curves, due to completeness of the orbit closure of C . Upgrades of this observation for simple, resp. squarefree, top T will be given in Proposition 4.2 and Corollary 5.4, respectively. The usefulness of this approach, both towards proofs and concrete examples, lies in the fact that the geometry of the $\text{Aut}_\Lambda(P)$ -orbits of \mathfrak{Grass}_d^T is particularly transparent in these cases. Moving subspace flags of the point $C \in \mathfrak{Grass}_d^T$ along in tandem, will moreover enable us to link up the structures of P/C and P/C' .

Let $\mathfrak{Flag}_d^T \subseteq \mathcal{G}r(1, JP) \times \mathcal{G}r(2, JP) \times \dots \times \mathcal{G}r(d', JP)$ denote the variety of subspace flags of the d' -dimensional Λ -submodules of JP , where $d' = \dim P - d$. In other words, if \mathcal{F} is the full flag variety of the K -space JP , and π denotes the projection from $\mathcal{G}r(1, JP) \times \dots \times \mathcal{G}r(\dim JP, JP)$ onto $\mathcal{G}r(1, JP) \times \dots \times \mathcal{G}r(d', JP)$, then \mathfrak{Flag}_d^T equals the intersection

$$\pi(\mathcal{F}) \cap \left(\mathcal{G}r(1, JP) \times \dots \times \mathcal{G}r(d' - 1, JP) \times \mathfrak{Grass}_d^T \right).$$

We observe that this intersection is closed in $\mathcal{G}r(1, JP) \times \dots \times \mathcal{G}r(d', JP)$, since \mathfrak{Grass}_d^T is closed in $\mathcal{G}r(d', JP)$ and the variety \mathcal{F} – whence also $\pi(\mathcal{F})$ – is complete. Thus \mathfrak{Flag}_d^T , in turn, is projective. Moreover, we note that the action of $\text{Aut}_\Lambda(P)$ on \mathfrak{Grass}_d^T naturally extends to an $\text{Aut}_\Lambda(P)$ -action on \mathfrak{Flag}_d^T .

The following result is due to János Kollár (personal communication). It extends to higher generality than required for our application; here we are only concerned with classical projective varieties, namely vanishing sets of families of homogeneous polynomials in projective space. Recall that an irreducible variety V is called unirational if its function field embeds into a finitely generated purely transcendental extension of the base field K , the latter amounting to the existence of a dominant rational map from some affine space \mathbb{A}^r to V (see, e.g., [6, Chapter AG, 13.7]). The analogous result for characteristic zero (based on Hironaka's resolution of singularities) was already obtained and used by Hanspeter Kraft in the context of degenerations of finite dimensional representations; see [20, p. 227].

Proposition 3.6. [J. Kollár] *Let V be a unirational projective variety of positive dimension (over an algebraically closed field K of arbitrary characteristic) and $c_1, \dots, c_s \in V$ any finite collection of points. Then there exists a curve $\psi : \mathbb{P}^1 \rightarrow V$, the image of which contains all c_i .*

Proof. We fix an embedding $V \subseteq \mathbb{P}^R$ of V into a projective space. By unirationality of V , there exists a dominant rational map from some \mathbb{P}^r to V , represented by (F, U) say, where U is a dense open subset of \mathbb{P}^r and

$$F(x) = (f_0(x), \dots, f_R(x))$$

with homogeneous polynomials $f_j \in K[X_0, \dots, X_r]$. First we cut the variety V with general hypersurfaces of \mathbb{P}^R containing c_1, \dots, c_s , to obtain an irreducible curve \mathcal{C} in V with $\{c_1, \dots, c_s\} \subset \mathcal{C}$ such that F induces a birational equivalence between \mathcal{C} and a suitable projective curve \mathcal{C}' in \mathbb{P}^r . (Feasibility is a consequence of a Bertini-type theorem, which is known to hold in arbitrary characteristic: Namely, if X is an irreducible subvariety of dimension at least 2 of a projective space \mathbb{P}^R , then, generically, hypersurface sections $X \cap H$ in \mathbb{P}^R are again irreducible; see [15]. In fact, the result has been generalized to not necessarily complete linear systems of divisors without fixed components on X , so as to yield irreducibility of their generic members. In our specific situation, one applies this version of Bertini's Theorem to the linear system of hyperplanes of a fixed degree $\geq s + 1$ containing the points c_1, \dots, c_s of V .) Let $h : \mathcal{D} \rightarrow \mathcal{C}'$ be the normalization of \mathcal{C}' . This provides us with a smooth projective curve \mathcal{D} with the property that

$$\{c_1, \dots, c_s\} \subset \overline{(F \circ h)(h^{-1}(U \cap \mathcal{C}'))} \subseteq V.$$

Since $\mathcal{D} \setminus h^{-1}(U \cap \mathcal{C}')$ is finite, smoothness allows us to extend $F \circ h$ to a morphism $H : \mathcal{D} \rightarrow V$ (see, e.g., [10, Chapter I, Proposition 6.8]), which yields $\{c_1, \dots, c_s\} \subset H(\mathcal{D})$.

Let $d_i \in \mathcal{D}$ with $H(d_i) = c_i$. Our claim will follow from the following two facts:

(1) For every $m \geq 1$, there exists a morphism $g : \mathbb{P}^1 \rightarrow \mathbb{P}^r$, together with points $b_1, \dots, b_s \in \mathbb{P}^1$, such that the Taylor expansion of g at b_i agrees, to order m , with the Taylor expansion of h at d_i , for all i .

(2) There is an integer $m \geq 1$ with the following property: If $g : B \rightarrow \mathbb{P}^r$ is any morphism from a smooth projective curve B and if $b_1, \dots, b_s \in B$ are points such that the Taylor expansion of g at b_i agrees, to order m , with the Taylor expansion of h at d_i , for all i , then $\{c_1, \dots, c_s\} \subset \overline{(F \circ g)(B \cap g^{-1}(U))}$.

For the first of these facts, we refer to [19, (4.1.2.4)], where (1) is stated for the base field $K = \mathbb{C}$. The proof given in [19, (5.2)] works in arbitrary characteristic, however.

The second can be seen as follows: Suppose that $g : B \rightarrow \mathbb{P}^r$ is a morphism from a smooth curve B , and let $b_1, \dots, b_s \in B$. If $G : B \rightarrow V$ is the extension of $F \circ (g|_{g^{-1}(U)})$ to B , then evidently $G(B) = \overline{(F \circ g)(B \cap g^{-1}(U))}$. We focus on a point d_i , pick a local coordinate t_i for \mathcal{D} at d_i , and consider h as given by power series h_j in t_i near d_i , that is, as given in the form $(h_0(t_i), \dots, h_r(t_i))$. Then

$$H(t_i) = (f_0(h_0(t_i), \dots, h_r(t_i)), \dots, f_R(h_0(t_i), \dots, h_r(t_i)))$$

in a neighborhood of d_i , punctured at d_i (the composition $F \circ h$ need not be defined at d_i), alias, for t_i near 0 but different from 0. Let $t_i^{m_i}$ be the minimal order of vanishing of the $f_j(h_0(t_i), \dots, h_r(t_i))$. Since $H(t_i) = (1/t_i^{m_i})H(t_i)$ in \mathbb{P}^r whenever $t_i \neq 0$ in said neighborhood, $H(0)$ is obtained by substituting $t_i = 0$ in the right-hand side of the previous equality. Thus, if $g_j(t_i)$ are power series expansions of the coordinate functions of g near b_i (where t_i also stands for a uniformizing parameter at b_i of the smooth curve B) such that

$$f_j(h_0(t_i), \dots, h_r(t_i)) \equiv f_j(g_0(t_i), \dots, g_r(t_i)) \pmod{t_i^{m_i+1}},$$

for $1 \leq j \leq R$, then H coincides with G at $t_i = 0$, and returning to the original variables, we obtain $c_i = H(d_i) = G(b_i)$. Hence $m \geq \max\{m_1, \dots, m_s\} + 1$ satisfies the requirement under (2). \square

The first of the observations below parallels Kraft's use of curves, over the base field \mathbb{C} , to reach a degeneration M' of a representation M of a complex algebra, by way of a rational curve connecting a point in the orbit of M in \mathbf{Mod}_d^Λ to a point representing M' ; he based his result on the special case of Proposition 3.6 for characteristic zero mentioned above. In the framework of \mathfrak{Grass}_d^T , we have the added advantage of a manageable format for such curves, improving the theoretical and computational accessibility of degenerations.

Observation 3.7. (1) For $C \in \mathfrak{Grass}_d^T$ and any top-stable degeneration P/C' of P/C , there exists a curve $\psi : \mathbb{P}^1 \rightarrow \overline{\mathrm{Aut}_\Lambda(P).C}$ such that $\psi^{-1}(\mathrm{Aut}_\Lambda(P).C)$ is dense in \mathbb{P}^1 and $C' \in \mathrm{Im}(\psi)$.

(2) If \mathcal{C} is a subspace flag of C and C' a point in the closure of $\mathrm{Aut}_\Lambda(P).C$ in \mathfrak{Flag}_d^T , there exists a curve $\psi : \mathbb{P}^1 \rightarrow \overline{\mathrm{Aut}_\Lambda(P).C}$ such that $\psi^{-1}(\mathrm{Aut}_\Lambda(P).C)$ is dense in \mathbb{P}^1 and $C' \in \mathrm{Im}(\psi)$.

(3) Suppose that $\mathcal{C} = (C_1, \dots, C_{d'}) \in \mathfrak{Flag}_d^T$ is a flag with $C_{d'} = C$. Moreover, let $\rho : U \rightarrow \mathrm{Aut}_\Lambda(P)$ be a morphism defined on a dense subvariety U of \mathbb{P}^1 . Then the morphism $\psi : U \rightarrow \overline{\mathrm{Aut}_\Lambda(P).C} \subseteq \mathfrak{Flag}_d^T$, defined by $\tau \mapsto (\rho(\tau).C_1, \dots, \rho(\tau).C_{d'})$, extends to a unique morphism $\bar{\psi} : \mathbb{P}^1 \rightarrow \overline{\mathrm{Aut}_\Lambda(P).C}$. The latter maps any point $\tau \in \mathbb{P}^1$ to a flag $\mathcal{D} = (D_1, \dots, D_{d'})$ of the point $D = D_{d'}$ in the orbit closure $\overline{\mathrm{Aut}_\Lambda(P).C}$. In case \mathcal{C} is a flag consisting of Λ -submodules of C , the flag \mathcal{D} consists of Λ -submodules of D .

Proof. (1) $\mathrm{Aut}_\Lambda(P)$ being a rational variety by Observation 3.2, each orbit closure is unirational. Thus Proposition 3.6 guarantees that any two points of an orbit can be connected by a rational curve. In other words, there exists a morphism $\psi : \mathbb{P}^1 \rightarrow \overline{\mathrm{Aut}_\Lambda(P).C}$, such

that $C, C' \in \text{Im}(\psi)$. Because $\text{Aut}_\Lambda(P).C$ is locally closed in \mathfrak{Grass}_d^T , our first claim follows. The proof of (2) is analogous.

(3) In light of the fact that \mathfrak{Flag}_d^T is projective, the extension statement for curves of flags is immediate. The final assertion under (3) follows from the fact that

$$\mathfrak{Flag}_d^T \cap (\mathfrak{Grass}_{\dim P-1}^T \times \cdots \times \mathfrak{Grass}_{\dim P-d'}^T)$$

is a closed subvariety of \mathfrak{Flag}_d^T . \square

The first part of Observation 3.7 has an analogue for the classical module variety: Namely, given any point $x \in \mathbf{Mod}_d^\Lambda$, each element in the closure of $\text{GL}_d.x$ can be realized in the form $\psi(a)$ for some morphism $\psi : U \rightarrow \overline{\text{GL}_d}.x$, where U is a nonempty open subset of \mathbb{A}^1 containing a , and $\psi(U \setminus \{a\}) \subseteq \text{GL}_d.x$; see [20] and [27]. The classical variety only sporadically lends itself to explicit computations of all degenerations of a given module, however. This is otherwise in the projective setting, as evidenced in Sections 4.B and 5.B.

Definitions and Remarks 3.8. Following standard terminology, we refer to a *proper* top-stable (resp., layer-stable) degeneration M' of M as *minimal top-stable* (resp., *minimal layer-stable*) in case there is no top-stable (resp., layer-stable) degeneration lying properly between M and M' in the degeneration order. Then M' is minimal also in the poset of *all* proper degenerations of M , since any degeneration of M which is trapped between M and M' is in turn top-stable (layer-stable).

Maximal top- or layer-stable degenerations are defined analogously, on waiving the condition that they be proper. Clearly a module M has no proper top-stable degenerations precisely when the set of (isomorphism classes of) maximal top-stable degenerations consists of M alone. Whereas the unique maximal degeneration of M , namely the direct sum of the simple composition factors of M , does not hold much interest, the maximal *top-stable* and the maximal *layer-stable* degenerations of M do; in particular, we will see that, in general, even modules with simple tops may have infinite families of maximal top-stable degenerations requiring parameter spaces of arbitrarily high dimension.

Finally, we say that M' is a top-stable (layer-stable) degeneration of M of *height* h , if h is the maximal length of a chain of top-stable (layer-stable) degenerations connecting M to M' . (As in the theory of prime ideals, we count as the Romans did, calling $M = M_0 <_{\text{deg}} M_1 <_{\text{deg}} \cdots <_{\text{deg}} M_l$ a chain of length l ; thus the minimal top-stable degenerations of M are those of height 1.) Clearly, the heights of the top-stable (layer-stable) degenerations of a d -dimensional module $M = P/C$ are bounded from above by the dimension of the orbit $\text{Aut}_\Lambda(P).C$ in \mathfrak{Grass}_d^T .

4. THE LOCAL CASE: T SIMPLE

Throughout this section, we assume T to be simple, i.e., $T = \Lambda e / J e$ for some primitive idempotent e , and $P = \Lambda e$. This means that every module with top T is local, i.e., has a unique maximal submodule. The local case affords several bonuses, sufficiently incisive to warrant separate treatment, all the more since all maximal top-stable degenerations of an arbitrary finite dimensional module are direct sums of local ones.

Responsible for these bonuses is the fact that, in this situation, $\text{Aut}_\Lambda(P)$ is the direct product $K^* \times (\text{Aut}_\Lambda(P))_u$, where the second factor is the unipotent radical of $\text{Aut}_\Lambda(P)$

identified in Section 3, and the first acts trivially on \mathfrak{Grass}_d^T . In particular, the $\text{Aut}_\Lambda(P)$ -orbits in \mathfrak{Grass}_d^T coincide with the $(\text{Aut}_\Lambda(P))_u$ -orbits, and hence are affine spaces, by Observation 3.2.

We start with a few elementary comments, the first giving an explicit incarnation of an isomorphism between the varieties \mathbb{A}^m and $\text{Aut}_\Lambda(P).C$, for $C \in \mathfrak{Grass}_d^T$. Since $T = \Lambda e/Je$, the group $(\text{Aut}_\Lambda(P))_u$ is anti-isomorphic to the multiplicative subgroup $e+eJe$ of the group of units of $e\Lambda e$, via the assignment which sends any $e+a$ to right multiplication of P by this element.

Now suppose that $\omega_1, \dots, \omega_\mu$ form a basis for eJe . We choose the ω_i to be oriented cycles from e to e , which is clearly always possible. Then the above isomorphism of groups induces an isomorphism $\bigoplus_{1 \leq i \leq \mu} K\omega_i \cong (\text{Aut}_\Lambda(P))_u$ of algebraic varieties, which maps any element $a = \sum_{1 \leq i \leq \mu} k_i \omega_i$ in eJe to right multiplication on P by $e+a$. To describe the orbits $\text{Aut}_\Lambda(P).C$ in these terms, let $\text{Stab}_{eJe} C$ be the K -space consisting of those elements $a \in eJe$ for which $Ca \subseteq C$, and denote the unipotent radical $(\text{Aut}_\Lambda(P))_u$ by \mathcal{U} . As a variety, $\text{Stab}_{eJe} C$ then becomes isomorphic to $\text{Stab}_{\mathcal{U}} C$ under the described assignment, whence $\dim \text{Aut}_\Lambda(P).C$ equals $\dim_K(eJe/\text{Stab}_{eJe} C)$; this dimension also equals the invariant \mathfrak{m} in the local case (again see Observation 3.2). Provided that $\omega_1, \dots, \omega_m$ form a K -basis of eJe modulo $\text{Stab}_{eJe} C$, this latter map in turn gives rise to an isomorphism $\bigoplus_{1 \leq i \leq m} K\omega_i \cong \mathcal{U}/\text{Stab}_{\mathcal{U}} C$. We thus obtain an explicit isomorphism between $\text{Aut}_\Lambda(P).C$ and \mathbb{A}^m as follows:

Lemma 4.1. *Let C and $\omega_1, \dots, \omega_m$ be as above. Then the map*

$$\bigoplus_{1 \leq i \leq m} K\omega_i \rightarrow \text{Aut}_\Lambda(P).C,$$

sending any element a in $\bigoplus_{1 \leq i \leq m} K\omega_i$ to the submodule $C(e+a)$ of P , is an isomorphism of varieties. \square

Based on the previous lemma, we can strengthen Observation 3.7. In light of projectivity of \mathfrak{Grass}_d^T , every morphism $\psi : U \rightarrow \mathfrak{Grass}_d^T$, where U is a dense subset of \mathbb{A}^1 , can be uniquely extended to $U \cup \{\infty\} \subseteq \mathbb{P}^1$. Therefore the notation $\lim_{\tau \rightarrow \infty} \psi(\tau)$ is unambiguous. Moreover, closedness of the flag variety \mathfrak{Flag}_d^T in $\mathcal{G}r(1, JP) \times \mathcal{G}r(2, JP) \times \dots \times \mathcal{G}r(d', JP)$ allows for an analogous limit notation for curves in \mathfrak{Flag}_d^T ; as before, $d' = \dim P - d$.

For the first assertion of Proposition 4.2, we only need to note that any curve $U \rightarrow \text{Aut}_\Lambda(P).C$ with $U \subseteq \mathbb{A}^1$ dense coincides, on a dense subset of U , with one of the form $\tau \mapsto C(e + \sum_{1 \leq i \leq m} r_i(\tau)\omega_i)$ for suitable rational functions $r_i(\tau)$ (Lemma 4.1). The final assertion rests on the fact that all $\text{Aut}_\Lambda(P)$ -orbits of \mathfrak{Grass}_d^T are affine.

Proposition 4.2. (1) *Let $C, C' \in \mathfrak{Grass}_d^T$ be such that P/C degenerates to P/C' , and let $\omega_1, \dots, \omega_m$ denote oriented cycles $e \rightarrow e$ in Q generating the K -space eJe modulo $\text{Stab}_{eJe} C$. Then there exist polynomials $q(\tau)$ and $p_i(\tau)$ in $K[\tau]$ for $i \leq m$ with $q \neq 0$ such that*

$$\lim_{\tau \rightarrow \infty} Ce(\tau) = C',$$

where, for each $\tau \in K$,

$$e(\tau) = q(\tau)e + \sum_{1 \leq i \leq m} p_i(\tau)\omega_i$$

is identified with an element of $e\Lambda e$, a unit in $e\Lambda e$ whenever $q(\tau) \neq 0$.

Moreover, given any flag $\mathcal{C} = (C_j)_{1 \leq j \leq d'}$ of subspaces C_j of C , the “limit”

$$\mathcal{C}' = \lim_{\tau \rightarrow \infty} \mathcal{C}e(\tau) = \left(\lim_{\tau \rightarrow \infty} C_j e(\tau) \right)_{j \leq d'}$$

is a flag of \mathcal{C}' . In particular, any subspace D of C gives rise to a $(\dim D)$ -dimensional subspace of $D' = \lim_{\tau \rightarrow \infty} De(\tau)$ of \mathcal{C}' . If D is a Λ -submodule of P , then so is D' .

(2) Conversely, given arbitrary polynomials $q(\tau)$ and $p_i(\tau)$ with $q \neq 0$, we set

$$e(\tau) = q(\tau)e + \sum_{1 \leq i \leq m} p_i(\tau)\omega_i \quad \text{for all } \tau \in \mathbb{A}^1.$$

If the curve $\tau \mapsto Ce(\tau)$ in \mathbf{Grass}_d^T , defined on the complement of the zero set of q in \mathbb{A}^1 , is nonconstant, its unique extension to \mathbb{P}^1 takes a value \mathcal{C}' such that P/\mathcal{C}' is a proper degeneration of P/C . \square

A proof of the following lemma has been communicated to the author by Ron Donagi; we include it with his permission.

Lemma 4.3. [R. Donagi] *Let V be an m -dimensional irreducible projective variety and W a nontrivial open subvariety permitting a non-constant regular function $W \rightarrow \mathbb{A}^1$. Then $V \setminus W$ has dimension $m - 1$.*

In particular, $\dim V \setminus W = m - 1$ whenever V is a projective variety containing a dense open subset W isomorphic to \mathbb{A}^m .

Proof. That $\dim(V \setminus W) \leq m - 1$ is clear.

For the converse, let $\phi : W \rightarrow \mathbb{A}^1$ be a nonconstant regular function as postulated, and let $\Gamma \subseteq V \times \mathbb{P}^1$ be the closure of its graph in $V \times \mathbb{P}^1$. Moreover, let \bar{p} and \bar{q} be the projections from $V \times \mathbb{P}^1$ onto V and \mathbb{P}^1 , respectively, and p, q their restrictions to Γ . Due to projectivity of Γ and our assumption on ϕ , the image of q is \mathbb{P}^1 , and therefore the fiber $q^{-1}(\infty) \subseteq \Gamma$ has dimension at least $\dim \Gamma - 1 = m - 1$ by [6, Chapter AG, Theorem 10.1]). Clearly, the restriction of p to $q^{-1}(\infty)$ is injective; indeed this restriction is just the embedding of $q^{-1}(\infty)$ into $\Gamma \cap (V \times \{\infty\})$ followed by the isomorphism $\bar{p} : V \times \{\infty\} \rightarrow V$. Consequently, the (closed) subvariety $p(q^{-1}(\infty))$ of V has dimension at least $m - 1$ as well. Clearly, it is contained in $V \setminus W$, and the missing inequality follows. \square

4.A. The basic theorems for the local case.

We have seen that all $\text{Aut}_\Lambda(P)$ -orbits of \mathbf{Grass}_d^T are full affine spaces in the local case. For the sake of symmetry – compare with the more general statement addressing the geometry of the $\text{Aut}_\Lambda(P)$ -orbits in the twin Theorem 5.1 – we include this fact as part (1) of the theorem below. For a proof of our first theorem we only need to assemble the information accumulated so far. In the subsequent theorem, we provide more detail on the possible sizes and shapes of the posets of top-stable degenerations. This second result, proved in Section 4.B, demonstrates that our conjectures concerning the general form of the poset of top-stable degenerations of a local module cannot be simplified.

Theorem 4.4. *Suppose $T = \Lambda e/Je$ is simple and $P = \Lambda e$.*

(1) **Structure of the $\text{Aut}_\Lambda(P)$ -orbits and chain lengths.** *For each $C \in \mathfrak{Grass}_d^T$,*

$$\text{Aut}_\Lambda(P).C \cong \mathbb{A}^{\mathfrak{m}} \quad \text{where } \mathfrak{m} = \dim_K \text{Hom}_\Lambda(P, P/C) - \dim_K \text{End}_\Lambda(P/C).$$

In particular, the lengths of chains of top-stable degenerations of P/C are bounded above by \mathfrak{m} .

(2) **Layer-stable degenerations.** *Local modules do not have any proper layer-stable degenerations.*

(3) **Top-stable degenerations.** *Let M be a local module with top T , say $M \cong P/C$ with $C \in \mathfrak{Grass}_d^T$.*

(a) **Existence.** *The following conditions are equivalent:*

- (i) *M does not have any proper top-stable degenerations.*
- (ii) *The orbit $\text{Aut}_\Lambda(P).C$ is a singleton.*
- (iii) *$\dim_K \text{End}_\Lambda(M)$ equals the multiplicity of T as a composition factor of M .*
- (iv) *C is invariant under all endomorphisms of P , that is, $C\omega \subseteq C$ for all oriented cycles ω from e to e in the quiver Q .*

In particular, these equivalent conditions are satisfied if all appearances of T as a composition factor of JM occur in the socle of JM .

(b) **Unique existence.** *M has a unique proper top-stable degeneration if and only if $\mathfrak{m} = 1$, that is, if and only if the multiplicity of the simple module T in M exceeds $\dim_K \text{End}_\Lambda(M)$ by 1.*

(c) **Bases.** *If M' is a top-stable degeneration of M , then M and M' share a basis consisting of paths. More precisely, there exist submodules D and D' of P with $M \cong P/D$ and $M' \cong P/D'$, together with a set \mathcal{B} of paths in KQ (which is closed under right subpaths), such that $\{p + D \mid p \in \mathcal{B}\}$ is a basis for P/D and $\{p + D' \mid p \in \mathcal{B}\}$ a basis for P/D' .*

(d) **The maximal top-stable degenerations of M always possess a fine moduli space classifying them up to isomorphism, namely**

$$\text{max-topdeg}(M) = \{C' \in \overline{\text{Aut}_\Lambda(P).C} \mid \text{Aut}_\Lambda(P).C' \text{ is a singleton}\},$$

and the latter is a projective variety of dimension at most $\max\{0, \mathfrak{m} - 1\}$. More specifically, the assignment $C' \mapsto P/C'$ yields a universal (bijective) parametrization of the maximal top-stable degenerations of M , up to isomorphism.

As a special case of part (2), we rediscover the fact that uniserial modules fail to have proper uniserial degenerations (see [2]). Moreover, we note that the conditions in Theorem 4.4 characterizing existence and unique existence of proper top-stable degenerations can be readily checked in concrete situations, given C and a quiver presentation of Λ ; see Section 4.B.

Proof. For (1), see Lemma 4.1, and for (2), see [13, Corollary 4.3].

(3a) The equivalences can be readily derived from [13, Theorem 4.2]; we include the short argument for the convenience of the reader. In view of part (1), the equivalence of statements (ii)–(iv) is straightforward, as is the implication ‘(ii) \implies (i)’. For the converse, we only need to combine Observation 3.1 with the fact that the closure of $\text{Aut}_\Lambda(P).C$ in \mathfrak{Grass}_d^T is projective. The supplementary statement under (3a) is an obvious consequence of the equivalences.

(3b) First suppose that $\text{Aut}_\Lambda(P).C$ is one-dimensional. That this ensures existence of a proper top-stable degeneration of P/C follows from part (a). Moreover, since $\text{Aut}_\Lambda(P).C \cong \mathbb{A}^1$ by part (1), the closure of $\text{Aut}_\Lambda(P).C$ in the projective variety \mathfrak{Grass}_d^T contains only a single point outside $\text{Aut}_\Lambda(P).C$: Indeed, let $\mathbb{A}^1 \cong \text{Aut}_\Lambda(P).C \rightarrow \overline{\text{Aut}_\Lambda(P).C}$ be the embedding. By [10, Chapter I, Proposition 6.8], this embedding can be extended to a morphism $\mathbb{P}^1 \rightarrow \overline{\text{Aut}_\Lambda(P).C}$, and by [6, Chapter AG, Section 7.4], the image of \mathbb{P}^1 under this extension is closed, whence $\overline{\text{Aut}_\Lambda(P).C} \setminus \text{Aut}_\Lambda(P).C$ is a singleton. This guarantees uniqueness. The converse is due to the fact that, for $\dim \text{Aut}_\Lambda(P).C \geq 2$, the variety $\overline{\text{Aut}_\Lambda(P).C} \setminus \text{Aut}_\Lambda(P).C$ has dimension at least 1; combine Lemma 4.3 with part (1) for verification. Thus, if this set difference does not contain an orbit of positive dimension, it contains at least two distinct orbits. If, on the other hand, it does contain an orbit of positive dimension, $\text{Aut}_\Lambda(P).C'$ say, then P/C' in turn has a proper top-stable degeneration by part (a), which again yields two distinct orbits in the boundary of $\text{Aut}_\Lambda(P).C$.

(3c) Let $M' \cong P/D'$, and pick any skeleton σ of M' . Then D' belongs to $\mathfrak{Grass}(\sigma)$ and hence to the open subset

$$\text{Schu}(\sigma) = \{E \in \mathfrak{Grass}_d^T \mid P/E \text{ has basis } \{p + E \mid p \in \sigma\}\}$$

of \mathfrak{Grass}_d^T . Since $M \cong P/C$, this means that the intersection $\text{Schu}(\sigma) \cap \overline{\text{Aut}_\Lambda(P).C}$ is nonempty and open in $\overline{\text{Aut}_\Lambda(P).C}$. The orbit $\text{Aut}_\Lambda(P).C$ being open in its closure and irreducible by part (1), we conclude that $\text{Schu}(\sigma) \cap \text{Aut}_\Lambda(P).C \neq \emptyset$, which yields a point $D \in \text{Aut}_\Lambda(P).C$ such that P/D has basis $\{p + D \mid p \in \sigma\}$ as required; see Section 2.

(3d) That $\text{max-topdeg}(M)$ is a fine moduli space for the maximal top-stable degenerations of M follows from (3a) and [13, Theorem 4.4]. Projectivity of $\text{max-topdeg}(M)$ follows from closedness in $\overline{\text{Aut}_\Lambda(P).C}$, which, in turn, is due to the lower semicontinuity of the orbit dimension. Since $\dim \text{Aut}_\Lambda(P).C = \mathfrak{m}$ and $\text{max-topdeg}(M)$ is either a singleton or contained in $\overline{\text{Aut}_\Lambda(P).C} \setminus \text{Aut}_\Lambda(P).C$, the bound on the dimension is obvious. \square

Next, we provide evidence for the fact that the size of the poset of top-stable degenerations of a local module $M = P/C$ may be very large, both “horizontally” and “vertically”, within the limits imposed by the equality $\dim \overline{\text{Aut}_\Lambda(P).C} \setminus \text{Aut}_\Lambda(P).C = \dim \text{Aut}_\Lambda(P).C - 1$ (Lemma 4.3).

Theorem 4.5. *Let n_1, \dots, n_s be arbitrary positive integers, and $l \geq s$. Then there is a finite dimensional algebra Λ , together with a local Λ -module $M = P/C$, such that the poset of top-stable degenerations of M has the following properties:*

$$\text{max-topdeg}(M) \cong \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_s},$$

the integer l is the minimal length of a saturated chain of top-stable degenerations of M , and $\dim \text{Aut}_\Lambda(P).C = l + \sum_{1 \leq i \leq s} n_i$.

In addition, we may require that either

- (i) the lengths of the saturated chains of top-stable degenerations of P/C trace all values between l and $(l - s) + \sum_{1 \leq i \leq s} n_i$, or else
- (ii) the saturated chains of top-stable degenerations of P/C have constant length l .

Theorem 4.5 will be established in Section 4.B, Examples 4.10. Beyond the listed properties, our families of examples exhibit the following feature, which we believe to be representative of the general situation, in a sense made more precise below: For every value $h \leq \dim \text{Aut}_\Lambda(P).C$ occurring as the height of a top-stable degeneration of P/C , the set of (isomorphism classes of) top-stable degenerations of that height is the disjoint union of finitely many families, each of which is bijectively parametrized by some torus orbit in a direct product of projective spaces. By a *torus orbit* in a product $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$ we will mean any orbit of the canonical $(K^*)^{n_1+1} \times \cdots \times (K^*)^{n_s+1}$ -action.

We conclude the section with a number of conjectures, setting up as tight a framework for classification of the poset of top-stable degenerations of local modules as can be expected in this generality. The first conjecture pertains to modules with arbitrary squarefree tops, the second to local modules.

Conjectures 4.6. For any module $M = P/C$ with squarefree top, the following hold:

Conjecture (1) There exist finitely many non-negative integers $n_1 \leq \cdots \leq n_s$, repetitions allowed, such that the following conditions are satisfied:

(a) The irreducible components of the fine moduli space $\text{max-topdeg}(M)$ coincide with the connected components, and each component is isomorphic to a direct product of projective spaces, $\prod_{i \in I} \mathbb{P}^{n_i}$, for a suitable subset I of $\{1, \dots, s\}$.

(b) For each $h \leq \dim \text{Aut}_\Lambda(P).C$, the top-stable degenerations of height h above M can be partitioned into finitely many disjoint families, each of which is bijectively parametrized by a torus orbit in a product $\prod_{i \in I} \mathbb{P}^{n_i}$ for some $I \subseteq \{1, \dots, s\}$.

If s and $\sum_{1 \leq i \leq s} n_i$ are minimal with respect to (a) and (b), then n_1, \dots, n_s are isomorphism invariants of M , and

(c) the difference $\dim \text{Aut}_\Lambda(P).C - \sum_{1 \leq i \leq s} n_i$ coincides with the minimum of the lengths of saturated chains of top-stable degenerations of M .

Conjecture (2) Now suppose that M has simple top, and let $\mathfrak{m} = \dim \text{Hom}_\Lambda(P, JM) - \dim \text{Hom}_\Lambda(M, JM)$. Then $\text{max-topdeg}(M)$ is irreducible, that is, the universal family of isomorphism classes of maximal top-stable degenerations of M is a $(\prod_{i \in I} \mathbb{P}^{n_i})$ -family for a suitable subset $I \subseteq \{1, \dots, s\}$. If, in addition, the poset of (isomorphism types of) top-stable degenerations of M is finite, then its cardinality (counting M) is bounded from below by $\mathfrak{m} + 1$ and from above by $2^{\mathfrak{m}}$. More precisely, for $0 \leq j \leq \mathfrak{m}$, the number of top-stable degenerations of M of height j lies between 1 and $\binom{\mathfrak{m}}{j}$. In particular, this means that M has a unique maximal top-stable degeneration, and that the latter has height \mathfrak{m} . Moreover, all saturated chains linking M to its unique maximal top-stable degeneration have the same length. (This catenarity condition fails when either of the hypotheses is removed.)

Suppose M is a module with squarefree top and s, n_1, \dots, n_s are as postulated in Conjecture (1) above, chosen to be minimal. The requirement that M have only finitely many top-stable degenerations then becomes equivalent to $s = 1$ and $n_1 = 0$; for an infinite poset

of top-stable degenerations, the integers n_i are all positive, due to minimality. Provided that the conjectures can be confirmed, all bounds are sharp. This will be seen in Section 4.B as an application of the theory developed so far. In all of the local examples given here, $\dim \max\text{-topdeg}(M) = n_1 + \cdots + n_s$ for the unique choice of the n_i as in Conjectures 4.6. This is not a universal phenomenon, however: $\dim \max\text{-topdeg}(M)$ may be strictly smaller than $\sum_{i \leq s} n_i$; in fact, the equality $\dim \max\text{-topdeg}(M) = 0$ (i.e., a unique maximal top-stable degeneration) is compatible with an infinite poset of top-stable degenerations of M ; see [7]. Essentially all of the “slack” left in Conjectures 4.6 can be filled with examples.

4.B. Prototypical examples and proof of Theorem 4.5.

We begin with a number of theoretical remarks which are pivotal for all our computations. Suppose that P/C degenerates to P/C' , where, as above, $P = \Lambda e$ is a local module; that is, e is a vertex of the quiver Q , and $C, C' \in \mathfrak{Grass}_d^T$. As in 4.A, we let $\omega_1, \dots, \omega_m$ be oriented cycles from e to e which generate eJe modulo $\text{Stab}_{eJe} C = \{a \in eJe \mid Ca \subseteq C\}$. From Proposition 4.2 we know that there exist rational functions $p_i(\tau)$ such that

$$\lim_{\tau \rightarrow \infty} Ce(\tau) = C',$$

where $e(\tau) = e + \sum_{1 \leq i \leq m} p_i(\tau)\omega_i$ is identified with an element of $e + eJe \cong (\text{Aut}_\Lambda(P))_u$ whenever $\tau \in \mathbb{A}^1$ lies outside the union of the pole sets of the $p_i(\tau)$. We will write τ for either a variable over K or a scalar as convenience dictates. Adopting the convention $\deg p = \deg f - \deg g$ if f, g are nonzero polynomials in $K[\tau]$ with $p = f/g$, and $\deg p = -\infty$ when $p = 0$, we note that $P/C \cong P/C'$ in case none of the $p_i(\tau)$ has positive degree.

Proposition 4.2 provides elementwise access to C' by way of the following observation: Fix a K -basis (y_j) for JP . For any element $x \in JP$, there then exist unique rational functions $\rho_j(\tau)$ with the property that $xe(\tau) = \sum_j \rho_j(\tau)y_j$ whenever $\tau \in \mathbb{A}^1$ lies outside the pole sets of the $p_i(\tau)$. Note that, for $x \neq 0$, at least one of the ρ_j is nonzero since, for the pertinent values of τ , right multiplication by $e(\tau)$ is an automorphism of P . We define the $e(\tau)$ -degree of a nonzero element x in JP as $\max\{\deg \rho_j(\tau)\}$. The $e(\tau)$ -degree does not depend on our choice of a basis for P , but only depends on x . Moreover, if $x \neq 0$, we let $\tilde{\rho}_j(1/\tau)$ be the rational function in $1/\tau$ for which

$$\tilde{\rho}_j(1/\tau) = (1/\tau)^{e(\tau) - \deg(x)} \rho_j(\tau).$$

By construction, all of the $\tilde{\rho}_j(1/\tau)$ are defined at $1/\tau = 0$, and at least one of the values $\tilde{\rho}_j(0)$ is nonzero. Thus

$$x' = \sum_j \tilde{\rho}_j(0)y_j$$

is a well-defined element of JP , nonzero whenever x is nonzero. As we will see next, x' is determined by x up to a factor in K^* , irrespective of our choice of reference basis.

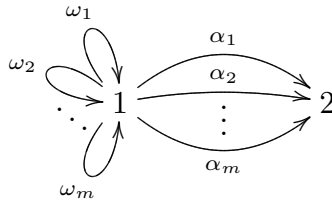
Lemma 4.7. *Keep the above notation. Then $\lim_{\tau \rightarrow \infty} (Kx)e(\tau) = Kx'$ for every nonzero element $x \in JP$. In particular, Kx' is a one-dimensional subspace of C' whenever $x \in C \setminus \{0\}$.*

Proof. Consider the curve $\tau \mapsto K(\sum_j \tilde{\rho}_j(1/\tau)y_j)$ in $\mathcal{G}r(1, JP)$. It is defined at the point ∞ of \mathbb{P}^1 and coincides with $\tau \mapsto Kxe(\tau)$ on a dense subvariety of \mathbb{P}^1 ; indeed,

$Kxe(\tau) = K(1/\tau)^{e(\tau) - \deg(x)}xe(\tau) = K(\sum_j \tilde{\rho}_j(1/\tau)y_j)$ for all $\tau \in K^*$ for which the $\rho_j(\tau)$ are defined. \square

We will – mostly tacitly – make use of this fact in the following. In each of the examples, completeness of the given list of top-stable degenerations of M can be verified via Proposition 4.2(1). We will carry out this argument in Example 4.8 and leave it to the reader in the other instances. In the first two of the three families of examples of this section, the posets of top-stable degenerations are finite. They show that, for any value of $\mathfrak{m} = \dim \text{Aut}_\Lambda(P).C$, there may be as many as $2^{\mathfrak{m}}$ top-stable degenerations of P/C or as few as $\mathfrak{m} + 1$. These are the upper and lower bounds conjectured in 4.6(2).

Example 4.8. Let $m \geq 1$ and $\Lambda = KQ/I$, where Q is the following quiver



and $I \subseteq KQ$ the ideal generated by $\omega_i\omega_j$ for all $i, j \in \{1, \dots, m\}$ and $\alpha_j\omega_i$ for all i, j with $i \neq j$. For $T = S_1$, we consider the point $C = \sum_{1 \leq i \leq m} \Lambda\alpha_i$ in \mathbf{Grass}_{2m+1}^T . Then $\text{Aut}_\Lambda(P).C \cong \mathbb{A}^m$, that is, $m = \mathfrak{m}$ in this example.

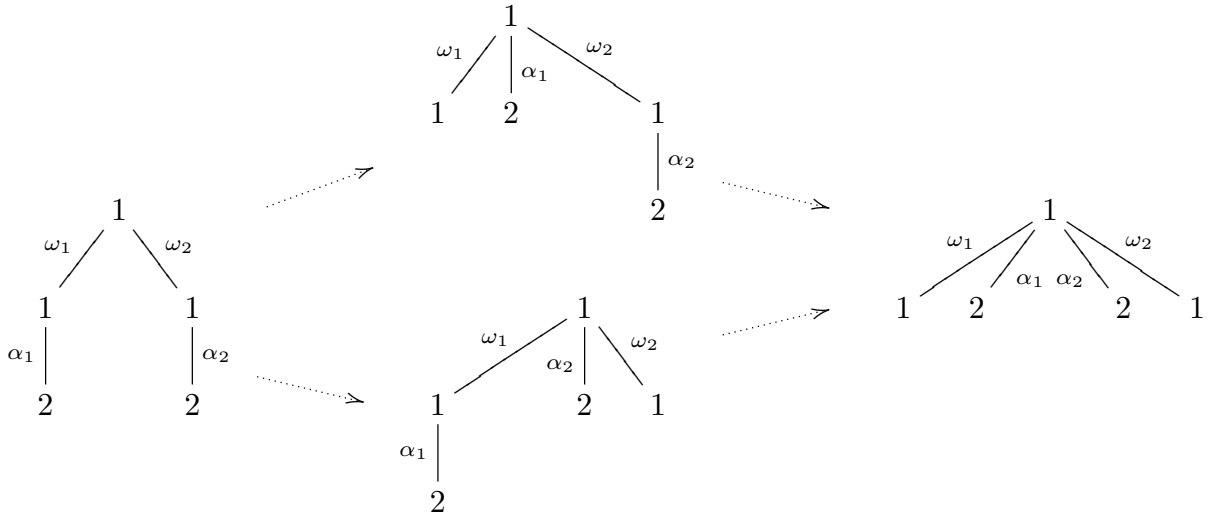
For each subset $\mathcal{I} \subseteq \{1, \dots, m\}$, the preceding comments yield the following top-stable degeneration $P/C(\mathcal{I})$ of $M = P/C$. Namely, $C(\mathcal{I})$ equals

$$\lim_{\tau \rightarrow \infty} C(e + \sum_{i \in \mathcal{I}} \tau\omega_i) = \lim_{\tau \rightarrow \infty} \left(\sum_{i \notin \mathcal{I}} \Lambda\alpha_i + \sum_{i \in \mathcal{I}} \Lambda((1/\tau)\alpha_i + \alpha_i\omega_i) \right) = \sum_{i \notin \mathcal{I}} \Lambda\alpha_i + \sum_{i \in \mathcal{I}} \Lambda\alpha_i\omega_i.$$

Similarly, one shows that $P/C(\mathcal{I}_1) \leq_{\deg} P/C(\mathcal{I}_2)$ is equivalent to $\mathcal{I}_1 \subseteq \mathcal{I}_2$.

To see that all proper top-stable degenerations of M are among the $P/C(\mathcal{I})$ with $\mathcal{I} \neq \emptyset$, let P/C' be any such degeneration, and let $e(\tau) = e + \sum_i p_i(\tau)\omega_i$ be such that $C' = \lim_{\tau \rightarrow \infty} Ce(\tau)$. Then the maximum of the degrees of the $p_i(\tau)$ is positive, and for $\mathcal{I} = \{i \leq m \mid \deg p_i(\tau) > 0\}$, one verifies $C' \cong C(\mathcal{I})$.

Therefore, there are precisely 2^m distinct top-stable degenerations of M in total, including $M = P/C(\emptyset)$. In fact, for each $h \leq m$, the number of isomorphism types of top-stable degenerations of height h equals $\binom{m}{h}$ as postulated in 4.6(2); in particular, there is a unique maximal top-stable degeneration of M . For $\mathfrak{m} = 2$, the full poset can be visualized as follows. Here a dotted arrow from $P/C(\mathcal{I}_1)$ to $P/C(\mathcal{I}_2)$ indicates that the latter is a minimal degeneration of the former. For our graphing conventions, we refer to [11] and [12].

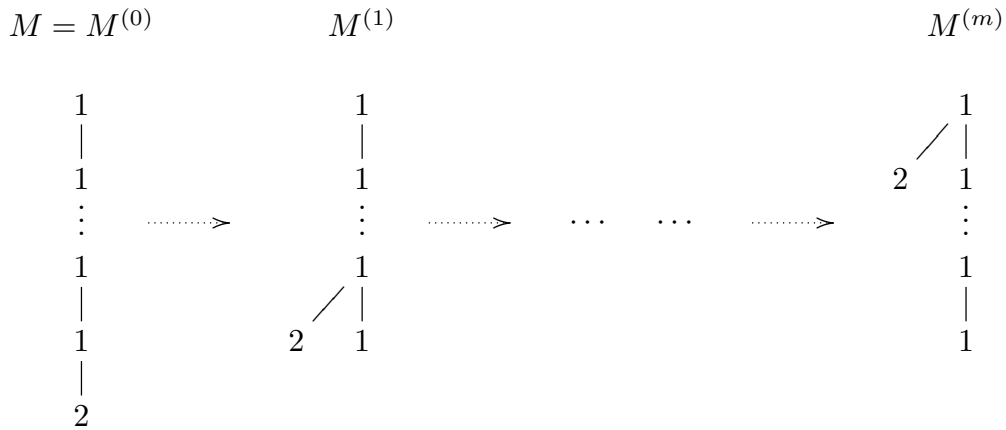


Next, we realize the lower bound on the number of top-stable degenerations of P/C given in Conjecture 4.6(2) for any value of $\mathfrak{m} = \dim \text{Aut}_\Lambda(P).C$. As is predicted by the conjecture, every number h between 1 and \mathfrak{m} arises as the height of precisely one top-stable degeneration in this situation. The case $\mathfrak{m} = 1$ is covered by the previous example.

Example 4.9. Let $m \geq 2$ and $\Lambda = KQ/I$, where Q is the quiver

$$\omega \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} 1 \xrightarrow{\alpha} 2$$

and $I = \langle \omega^{m+1} \rangle$. Once more, we set $T = S_1$. We focus on the module $M = P/C$, where C is the point $\sum_{0 \leq i \leq m-1} \Lambda \alpha \omega^i$ in \mathbf{Grass}_{m+2}^T and observe that again $\text{Aut}_\Lambda(P).C \cong \mathbb{A}^m$. Using the above remarks (see [7] for detail), one checks that the top-stable degenerations of $M = M^{(0)}$ are precisely the local modules $M^{(j)} = P/C^{(j)}$, where $C^{(j)} = \sum_{0 \leq i \leq m, i \neq m-j} \Lambda \alpha \omega^i$ for $1 \leq j \leq m$, as depicted below.



The third class of examples illustrates the infinite scenario and proves Theorem 4.5.

Examples 4.10 / Proof of Theorem 4.5. As in Theorem 4.5, let l, n_1, \dots, n_s be positive integers with $l \geq s$. In part (a), we specialize to the case $s = l = 1$ to construct the basic building blocks for the general case. In part (b), we subsequently duplicate and assemble them to meet the initial requirements of the theorem, combined with variable saturated chain lengths of top-stable degenerations as postulated in (i) for $l = s$. In part (c) we first address the case $s = l$, combined with constant saturated chain length as postulated in (ii), and in (d) finally, we indicate how to realize (i) and (ii) for $l > s$.

(a) We return to the algebra Λ of Example 4.8 with $m = n_1 + 1$ and again let $T = S_1$. But this time, we consider the point $C = \sum_{1 \leq i \leq m} \Lambda \alpha_i + \Lambda(\sum_{1 \leq i \leq m} \alpha_i \omega_i)$ in \mathbf{Grass}_{2m}^T .

The situation is akin to that of 4.8 in that $\text{Aut}_\Lambda(P).C \cong \mathbb{A}^m$, so again $\mathfrak{m} = m$, and each subset $\mathcal{I} \subseteq \{1, \dots, m\}$ leads to a family of degenerations of $M = P/C$. Once more, all top-stable degenerations of M are among the ones we display in the following. Given \mathcal{I} and a vector $k = (k_1, \dots, k_m) \in K^m$ with $k_i \neq 0$ precisely when $i \in \mathcal{I}$, we find:

$$\begin{aligned} C(\mathcal{I}, k) &:= \lim_{\tau \rightarrow \infty} C(e + \sum_{1 \leq i \leq m} k_i \tau \omega_i) \\ &= \lim_{\tau \rightarrow \infty} \left(\sum_{i \notin \mathcal{I}} \Lambda \alpha_i + \sum_{i \in \mathcal{I}} \Lambda((1/\tau)\alpha_i + k_i \alpha_i \omega_i) + \Lambda\left(\sum_{1 \leq i \leq m} \alpha_i \omega_i\right) \right) \\ &= \lim_{\tau \rightarrow \infty} \left(\sum_{i \notin \mathcal{I}} \Lambda \alpha_i + \sum_{i \in \mathcal{I}} \Lambda((1/\tau)\alpha_i + k_i \alpha_i \omega_i) + \Lambda\left(\sum_{i \notin \mathcal{I}} \alpha_i \omega_i - \sum_{i \in \mathcal{I}} (1/\tau k_i) \alpha_i\right) \right). \end{aligned}$$

If \mathcal{I} is a proper subset of $\{1, \dots, m\}$, we derive

$$C(\mathcal{I}) := C(\mathcal{I}, k) = \sum_{i \notin \mathcal{I}} \Lambda \alpha_i + \sum_{i \in \mathcal{I}} \Lambda \alpha_i \omega_i + \Lambda\left(\sum_{i \notin \mathcal{I}} \alpha_i \omega_i\right),$$

regardless of the choice of $k_i \in K$. In particular, the case where $\{1, \dots, m\} \setminus \mathcal{I}$ is a singleton $\{j\}$ yields a maximal top-stable degeneration $P/C(\mathcal{I})$, since $P/C(\mathcal{I})$ has Loewy length 2 in this situation; indeed, $C(\mathcal{I}) = \Lambda \alpha_j + \sum_{1 \leq i \leq m} \Lambda \alpha_i \omega_i$.

If, on the other hand, $\mathcal{I} = \{1, \dots, m\}$, the last summand in the final line of the above limit computation equals $\Lambda(\sum_{i \in \mathcal{I}} (1/k_i) \alpha_i)$ for $\tau \neq 0$, and consequently $C(\mathcal{I}, k)$ takes on the form

$$C(\mathcal{I}, k) = \sum_{1 \leq i \leq m} \Lambda \alpha_i \omega_i + \Lambda\left(\sum_{1 \leq i \leq m} (1/k_i) \alpha_i\right).$$

Clearly, each such $P/C(\mathcal{I}, k)$ is again a maximal top-stable degeneration of P/C , its isomorphism type depending only on the point $(1/k_1, \dots, 1/k_m) \in \mathbb{P}^{m-1}$. These latter degenerations thus form a family parametrized by the torus orbit of dimension $m - 1$ in \mathbb{P}^{m-1} .

As for the degenerations $P/C(\mathcal{I})$, where the cardinality of $\{1, \dots, m\} \setminus \mathcal{I}$ exceeds 1: They are non-maximal, and their top-stable degenerations can in turn be determined along the above lines, with the set $\mathcal{I}' = \{1, \dots, m\} \setminus \mathcal{I}$ taking over the role played by \mathcal{I} in the previous paragraphs. In particular: Whenever $|\{1, \dots, m\} \setminus \mathcal{I}_1| \geq 2$, the relation “ $P/C(\mathcal{I}_1) \leq_{\text{deg}} P/C(\mathcal{I}_2)$ ” is equivalent to “ $\mathcal{I}_1 \subseteq \mathcal{I}_2$ ”. Thus, each set \mathcal{I} with $|\mathcal{I}'| \geq 2$ leads to another infinite family of maximal top-stable degenerations $P/C(\mathcal{I}, k)$ with k_i nonzero precisely

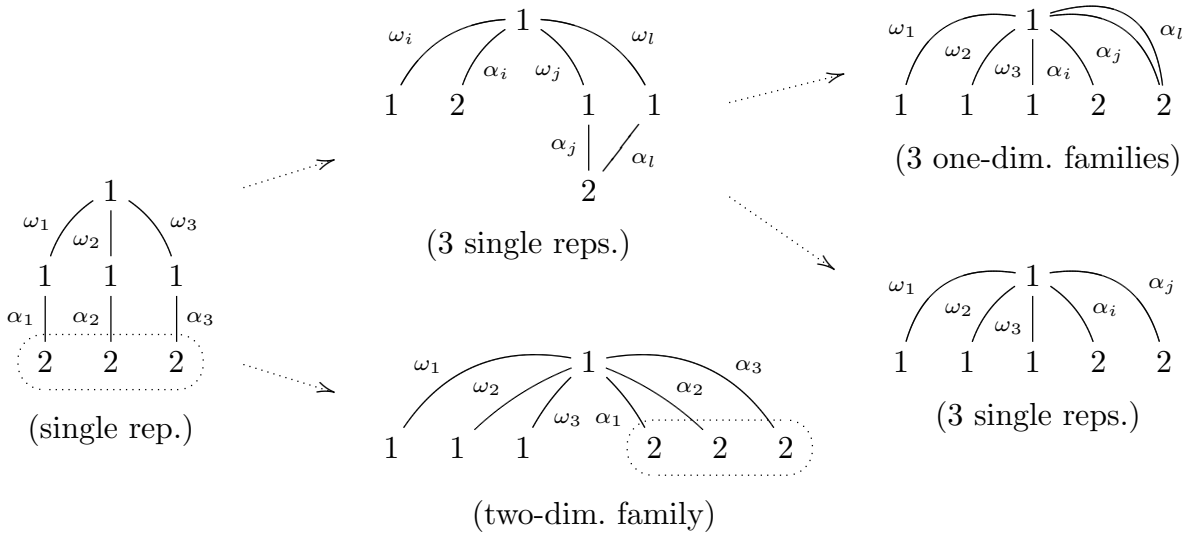
when $i \in \mathcal{I}'$. It is accordingly parametrized by the points of a torus orbit of dimension $|\mathcal{I}'| - 1$ in \mathbb{P}^{m-1} ; namely, $C(\mathcal{I}, k) = \sum_{1 \leq i \leq m} \Lambda \alpha_i \omega_i + \Lambda(\sum_{i \in \mathcal{I}'} (1/k_i) \alpha_i)$.

We conclude that the non-maximal top-stable degenerations of M are precisely the $P/C(\mathcal{I})$ for which the cardinality of $\{1, \dots, m\} \setminus \mathcal{I}$ is at least 2, while the maximal top-stable degenerations of M form a \mathbb{P}^{m-1} -family via

$$\overline{(a_1, \dots, a_m)} \mapsto \sum_{1 \leq i \leq m} \Lambda \alpha_i \omega_i + \Lambda \left(\sum_{1 \leq i \leq m} a_i \alpha_i \right).$$

This latter family breaks up into subfamilies indexed by torus orbits of dimension $m - h$, consisting of degenerations of fixed height h , respectively. Moreover, we glean: The minimum of the heights of the maximal top-stable degenerations of M is 1, and the maximum is $m - 1$.

For $m = 3$, the hierachy of top-stable degenerations of M thus takes on the following form:



Here the dotted enclosure in the graph of M indicates that the C -residues of the paths $\alpha_1 \omega_1, \alpha_2 \omega_2, \alpha_3 \omega_3$ are linearly dependent, whereas any two of them are linearly independent. The graph in the lower-most row is to be interpreted analogously; but, while for M the linear dependence relationship is prescribed, namely $\sum_{1 \leq i \leq 3} \alpha_i \omega_i = 0$, in the lower-most graph, it takes on the form $\sum_{1 \leq i \leq 3} a_i \alpha_i = 0$ for arbitrary $(a_1, a_2, a_3) \in (K^*)^3$, to the effect that this latter graph represents a family indexed by the dense torus orbit in \mathbb{P}^2 (a formal description was given above). Similarly, the top graph in the last column stands for a family of modules with $a_j \alpha_j + a_l \alpha_l = 0$, where the pair (a_j, a_l) traces the homogeneous coordinates of a 1-dimensional torus orbit in \mathbb{P}^2 , namely the orbit of the element with homogeneous coordinates 1 in positions j, l and 0 in position i . Throughout, the indices i, j, l trace all choices yielding $\{i, j, l\} = \{1, 2, 3\}$.

(b) Let n_1, \dots, n_s be given, and let Q be the quiver with $s + 1$ vertices, labeled 1 and a_1, \dots, a_s for $1 \leq i \leq s$. Moreover, suppose Q has $\sum_{1 \leq i \leq s} (n_i + 1)$ loops ω_{ij} at 1, where

$1 \leq i \leq s$, $1 \leq j \leq n_i + 1$, next to another set of $\sum_{1 \leq i \leq s} (n_i + 1)$ arrows, $\alpha_{ij} : 1 \rightarrow a_i$, one for each eligible choice of i and j . The ideal $I \subseteq KQ$ is generated by $\omega_{ij}\omega_{kl}$ for all legitimate choices of i, j, k, ℓ and by the paths $\alpha_{ij}\omega_{kl}$ for $(i, j) \neq (k, \ell)$. Again $P = \Lambda e_1$, but this time,

$$C = \sum_{\text{all } i, j} \Lambda \alpha_{ij} + \sum_{1 \leq i \leq s} \Lambda \left(\sum_{1 \leq j \leq n_i + 1} \alpha_{ij} \omega_{ij} \right) \subseteq JP.$$

Mimicking the argumentation under (a), one checks that the maximal top-stable degenerations of $M = P/C$ form a $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$ -family via

$$\left(\overline{(k_{11}, \dots, k_{1, n_1 + 1})}, \dots, \overline{(k_{s1}, \dots, k_{s, n_s + 1})} \right) \mapsto \left(\sum_{i, j} \Lambda \alpha_{ij} \omega_{ij} + \sum_{1 \leq i \leq s} \Lambda \left(\sum_{1 \leq j \leq n_i + 1} k_{ij} \alpha_{ij} \right) \right).$$

Moreover, the minimum of the heights of the maximal top-stable degenerations of M is s , and the heights of the maximal top-stable degenerations trace all values between $l = s$ and $\sum_{1 \leq i \leq s} n_i$.

(c) To realize examples analogous to those under (b), but satisfying the additional condition that all maximal top-stable degenerations of M have the same height $l = s$, we once more modify the quiver Q . The vertices are as under (b), and again we suppose that there are $\sum_{1 \leq i \leq s} (n_i + 1)$ distinct loops ω_{ij} at 1, where $1 \leq i \leq s$ and $1 \leq j \leq n_i + 1$; but now we place precisely one arrow from 1 to each of the other vertices, say $\alpha_i : 1 \rightarrow a_i$. We factor the following ideal I out of the path algebra KQ to obtain Λ : namely I is generated by the products $\omega_{ij}\omega_{kl}$ for arbitrary choices of i, j and k, l , and by the paths $\alpha_l\omega_{ij}$ for $l \neq i$. Moreover, we let $M = P/C$, where $C = \sum_{1 \leq i \leq s} \Lambda \alpha_i$. By means of Proposition 4.7, it is straightforward to check that the dimension of $\text{Aut}_\Lambda(P).C$ is $\mathfrak{m} = (\sum_{1 \leq i \leq s} n_i) + s$. Along the previous pattern, one moreover shows that the following family (P/C_k) , parametrized by the points $k \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$, gives the maximal top-stable degenerations of M , each isomorphism type arising precisely once: Let $k = (\overline{k^{(1)}}, \dots, \overline{k^{(s)}})$ be a point in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$, and let $(k_{i1}, \dots, k_{i, n_i + 1})$ be homogeneous coordinates for $\overline{k^{(i)}}$; then $C_k \subseteq P = \Lambda e_1$ is generated by the sums $\sum_{1 \leq j \leq n_i + 1} k_{ij} \alpha_i \omega_{ij}$ for $1 \leq i \leq s$. It is obvious that all of the degenerations P/C_k of M have height s .

(d) For $l > s$, we use the basic setup of (b) or (c) and, in each case, tack on $l - s$ replicas of Example 4.8 for $m = 1$. This means adding on $l - s$ new vertices, $l - s$ new loops at 1, and one arrow from 1 to each of the new vertices, next to additional relations modeled on 4.8 and relations ensuring that the new building blocks interfere neither with each other nor with the old picture. The modification brings the minimal saturated chain length up to l and the orbit dimension $\dim \text{Aut}_\Lambda(P).C$ to $l + \sum_{1 \leq i \leq s} n_i$, yet adds no new invariants n_k if $s \geq 1$. Instead, it effects a parallel-shift of the top-stable degeneration posets under (b) and (c). \square

5. DEGENERATIONS OF MODULES WITH SQUAREFREE TOPS

From now on, we let $T \in \Lambda\text{-mod}$ be a squarefree semisimple module of dimension \mathfrak{t} , say

$$T = \Lambda e_1 / J e_1 \oplus \cdots \oplus \Lambda e_{\mathfrak{t}} / J e_{\mathfrak{t}} \quad \text{and} \quad P = \Lambda e_1 \oplus \cdots \oplus \Lambda e_{\mathfrak{t}}$$

after renumbering of the vertices of Q . Note that P is a left ideal of Λ . Clearly, the group $\text{Aut}_\Lambda(T)$, which arises as a semidirect factor of $\text{Aut}_\Lambda(P)$ (see Observation 3.2), is a \mathfrak{t} -dimensional torus in the present situation. We fix the following incarnation \mathcal{T} of this torus inside $\text{Aut}_\Lambda(P)$ which, a priori, is only unique up to conjugation in the full automorphism group: namely, we let \mathcal{T} consist of the automorphisms of P sending z to $z(\sum_{1 \leq i \leq \mathfrak{t}} a_i e_i)$ where $(a_1, \dots, a_{\mathfrak{t}}) \in (K^*)^{\mathfrak{t}}$; any such automorphism will be identified with the tuple $(a_1, \dots, a_{\mathfrak{t}})$. Then $\text{Aut}_\Lambda(P) = (\text{Aut}_\Lambda(P))_u \rtimes \mathcal{T}$.

5.A. The basic theorems for squarefree T .

In contrast to the local case, there may be a plethora of layer-stable degenerations of modules with non-simple squarefree top. In fact, for any $r \geq 0$, one can find a module $M = P/C$ having a \mathbb{P}^r -family of such degenerations (see Example 5.9), and there is no uniform upper bound on the lengths of chains of indecomposable layer-stable degenerations.

Theorem 5.1. Top-stable degenerations. *Let T be a direct sum of \mathfrak{t} pairwise non-isomorphic simple modules, $P = \bigoplus_{1 \leq i \leq \mathfrak{t}} \Lambda e_i$ its projective cover, d a positive integer, and $M = P/C$ with $C \in \mathfrak{Grass}_d^T$.*

(1) **Structure of the $\text{Aut}_\Lambda(P)$ -orbits and chain lengths.** *The $\text{Aut}_\Lambda(P)$ -orbits of \mathfrak{Grass}_d^T are isomorphic to direct products of full affine spaces and tori. More precisely,*

$$\text{Aut}_\Lambda(P).C \cong \mathbb{A}^{\mathfrak{m}} \times (K^*)^{\mathfrak{t}-\mathfrak{s}},$$

where $\mathfrak{s} = \mathfrak{s}(M)$ is the number of indecomposable summands of M , and

$$\mathfrak{m} = \mathfrak{m}(M) = \dim_K \text{Hom}_\Lambda(P, JM) - \dim_K \text{Hom}_\Lambda(M, JM).$$

In particular, the lengths of chains of top-stable degenerations of M are bounded above by $\mathfrak{t} - \mathfrak{s}$ plus the sum of the multiplicities of the simple summands of T as composition factors of $JM/\text{soc}(JM)$.

(2) **Existence.** *The following conditions are equivalent:*

- (i) M has no proper top-stable degenerations.
- (ii) $\text{Aut}_\Lambda(P).C$ is a singleton, i.e., $\mathfrak{m} = \mathfrak{t} - \mathfrak{s} = 0$.
- (iii) M is a direct sum of local modules, and every homomorphism in $\text{Hom}_\Lambda(P, JP)$ leaves C invariant. (The latter condition is equivalent to requiring that $Cu \subseteq C$ for every nontrivial path u starting in one of the vertices $e_1, \dots, e_{\mathfrak{t}}$ and ending in one of these vertices.)
- (iv) C is invariant under all endomorphisms of P .

(3) **Unique existence.** *M has a unique proper top-stable degeneration if and only if $\text{Aut}_\Lambda(P).C \cong \mathbb{A}^1$, that is, precisely when $\mathfrak{m} = 1$ and $\mathfrak{t} - \mathfrak{s} = 0$. In particular, this condition forces M to be a direct sum of local modules.*

If $\mathfrak{m} = 0$ and $\mathfrak{t} - \mathfrak{s} = 1$, then M has precisely two distinct proper top-stable degenerations.

(4) **Bases.** *If M' is a top-stable degeneration of M , then M and M' share a basis consisting of paths. More precisely, there exist submodules D and D' of P with $M \cong P/D$ and $M' \cong P/D'$, together with a set \mathcal{B} of paths in KQ (which is closed under right subpaths), such that $\{p + D \mid p \in \mathcal{B}\}$ is a basis for P/D and $\{p + D' \mid p \in \mathcal{B}\}$ is a basis for P/D' .*

(5) **The maximal top-stable degenerations of M always possess a fine moduli space classifying them up to isomorphism, namely**

$$\max\text{-topdeg}(M) = \{C' \in \overline{\text{Aut}_\Lambda(P).C} \mid \text{Aut}_\Lambda(P).C' \text{ is a singleton}\},$$

and the latter is a projective variety of dimension at most $\max\{0, \mathfrak{m} + (\mathfrak{t} - \mathfrak{s}) - 1\}$.

(6) **In case $\mathfrak{m} = 0$, the degeneration order equals \leq_{ext} .** Suppose $\mathfrak{m} = 0$, and let $M = \bigoplus_{1 \leq k \leq \mathfrak{s}} M_k$ be a decomposition of M into indecomposable summands. Then every top-stable degeneration of M is a direct sum of top-stable degenerations of the M_k . Moreover, the poset of top-stable degenerations of M is finite, and $M \leq_{\text{deg}} M'$ if and only if $M \leq_{\text{ext}} M'$, for all d -dimensional modules M' with top T .

In particular: If $\text{Hom}_\Lambda(P, JP) = \text{Hom}_\Lambda(P, \text{soc}(P))$, the degeneration order on the class of left Λ -modules with top T coincides with the Ext-order.

On the side, we point out that the situation where M has precisely two proper top-stable degenerations is not restricted to $\mathfrak{m} = 0$ and $\mathfrak{t} - \mathfrak{s} = 1$. It may also occur when $\mathfrak{m} = 2$ and $\mathfrak{t} - \mathfrak{s} = 0$ (see Example 4.9 for $\mathfrak{m} = 2$). Further we note that, in the nonlocal case, the fine moduli space for the maximal top-stable degenerations, $\max\text{-topdeg}(P/C)$, is usually reducible. Both of our examples in Section 5.B attest to this.

We smooth the road towards a proof of the first part of Theorem 5.1. A proof of the first lemma can be found in [13, Proposition 2.9(3)]

Lemma 5.2. For any $C \in \mathfrak{Grass}_d^T$, there exists a point $C' \in \text{Aut}_\Lambda(P).C$ such that the T -orbit $T.C'$ is isomorphic to the torus $(K^*)^{\mathfrak{t}-\mathfrak{s}}$, where $\mathfrak{s} = \mathfrak{s}(P/C)$. In fact, this is true whenever $P = \bigoplus_{1 \leq i \leq \mathfrak{s}} P_i$ and $C = \bigoplus_{1 \leq i \leq \mathfrak{s}} C_i$ with $C_i \subseteq P_i$ such that $\bigoplus_{1 \leq i \leq \mathfrak{s}} P_i/C_i$ is a decomposition of P/C into indecomposable summands. \square

Lemma 5.3. If $M \in \Lambda\text{-mod}$ has squarefree top and $\mathfrak{s} = \mathfrak{s}(M)$, then

$$\dim_K \text{End}_\Lambda(M) = \mathfrak{s} + \dim_K \text{Hom}_\Lambda(M, JM).$$

Proof. Suppose $M = \bigoplus_{1 \leq i \leq \mathfrak{s}} M_i$ is a decomposition of M into indecomposable summands. Since the top of M_i does not share any simples with that of M_j for $i \neq j$ by hypothesis, $\text{End}_\Lambda(M)/\text{Hom}_\Lambda(M, JM) \cong \prod_i (\text{End}_\Lambda(M_i)/\text{Hom}_\Lambda(M_i, JM_i))$. We may thus focus on the situation where M is indecomposable.

Set $T = M/JM$. Moreover, let \mathcal{E} denote the Λ -endomorphism ring of M and \mathcal{I} the ideal $\text{Hom}_\Lambda(M, JM)$. Then \mathcal{I} is nilpotent and the factor \mathcal{E}/\mathcal{I} embeds canonically into $\text{End}_\Lambda(T)$ as a K -algebra. Since T is squarefree, the algebra $\text{End}_\Lambda(T)$ is a direct product of copies of K , and so is \mathcal{E}/\mathcal{I} ; say \mathcal{E}/\mathcal{I} is a direct product of s copies of K . If s were larger than 1, there would be a nontrivial idempotent in \mathcal{E}/\mathcal{I} , which would lift to a nontrivial idempotent of \mathcal{E} . But this is ruled out by our indecomposability assumption. \square

Proof of Theorem 5.1, part (1). (1) In light of Lemma 5.2, we may assume without loss of generality that $T.C \cong (K^*)^{\mathfrak{t}-\mathfrak{s}}$. Consequently, $\dim \text{Stab}_T C = \mathfrak{s}$.

Setting $(\text{Aut}_\Lambda(P))_u = \mathcal{U}$, we start by showing $\text{Stab}_{\text{Aut}_\Lambda(P)} C = \text{Stab}_{\mathcal{U}} C \rtimes \text{Stab}_T C$. For that purpose, we abbreviate $\text{Aut}_\Lambda(P)$ to \mathcal{A} , $\text{Stab}_{\text{Aut}_\Lambda(P)} C$ to \mathcal{A}_0 , $\text{Stab}_{\mathcal{U}} C$ to \mathcal{U}_0 , and $\text{Stab}_T C$ to \mathcal{T}_0 . Clearly, $\mathcal{U}_0 \rtimes \mathcal{T}_0$ is a subgroup of \mathcal{A}_0 . Moreover, the algebraic

group \mathcal{A}_0 is connected – we will ascertain this in the next paragraph – and $\mathcal{U}_0 \times \mathcal{T}_0$ is closed in \mathcal{A}_0 , since \mathcal{U}_0 and \mathcal{T}_0 are closed in \mathcal{U} and \mathcal{T} , respectively (see, e.g., [6, Chapter I, 1.7]). Therefore the desired equality $\mathcal{A}_0 = \mathcal{U}_0 \times \mathcal{T}_0$ is equivalent to equality of the corresponding dimensions $\dim \mathcal{A}_0$ and $\dim \mathcal{U}_0 \times \mathcal{T}_0$. But this dimension equality is an immediate consequence of Observation 3.2: Indeed, in view of $\mathcal{A} \cdot C \cong \mathcal{A}/\mathcal{A}_0$ and $\mathcal{U} \cdot C \cong \mathcal{U}/\mathcal{U}_0$, we obtain $\dim \mathcal{A}_0 = \dim_K \text{End}_\Lambda(M) + \dim_K \text{Hom}_\Lambda(P, C)$ and $\dim \mathcal{U}_0 \times \mathcal{T}_0 = \dim_K \text{Hom}_\Lambda(P, C) + \dim_K \text{Hom}_\Lambda(M, JM) + \mathfrak{s}$, whence Lemma 5.3 fills in what we need.

Thus we only need to back our connectedness claim to secure the desired stabilizer equality. For that purpose, it suffices to observe that the open immersion of \mathcal{A} into the affine space $\text{Hom}_\Lambda(P, P)$ restricts to an open immersion of \mathcal{A}_0 into that subspace of $\text{Hom}_\Lambda(P, P)$ which consists of the maps leaving C invariant. In other words, as a variety, \mathcal{A}_0 is isomorphic to an open subvariety of a linear K -space and consequently connected.

The group \mathcal{T}_0 is connected as well: Indeed, the obvious open immersion of \mathcal{T} into \mathbb{A}^t restricts to an open immersion of \mathcal{T}_0 into the K -subspace $\{(a_1, \dots, a_t) \mid C(\sum_{1 \leq i \leq t} a_i e_i) \subseteq C\}$. Since connected subgroups of tori are direct factors (see, e.g., [14, Ex.6 on p.108]), we infer that $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_0$ for some torus \mathcal{T}_1 .

Next, we deduce that the variety $\mathcal{A} \cdot C \cong \mathcal{A}/\mathcal{A}_0$ is isomorphic to the direct product $(\mathcal{U}/\mathcal{U}_0) \times (\mathcal{T}/\mathcal{T}_0)$. In light of the splitting of \mathcal{T} , we obtain a projection map

$$\pi_1 : \mathcal{A} = (\mathcal{U} \cdot \mathcal{T}_1) \times \mathcal{T}_0 \rightarrow \mathcal{U} \cdot \mathcal{T}_1$$

of varieties; it is the geometric quotient of \mathcal{A} by the canonical right action of \mathcal{T}_0 (by [6, Chapter II, Proposition 6.6], for instance). Let $\pi_2 : \mathcal{U} \cdot \mathcal{T}_1 \rightarrow (\mathcal{U} \cdot \mathcal{T}_1)/\mathcal{U}_0$ be the geometric quotient of the subgroup $\mathcal{U} \cdot \mathcal{T}_1$ of \mathcal{A} by its right \mathcal{U}_0 -multiplication, and observe that the map $\pi_2 \circ \pi_1$, passing to consecutive quotients, has the same fibres as the quotient of \mathcal{A} by \mathcal{A}_0 – here again, we let \mathcal{A}_0 act by right multiplication on \mathcal{A} . Consequently, $\mathcal{A}/\mathcal{A}_0 \cong (\mathcal{U} \cdot \mathcal{T}_1)/\mathcal{U}_0$. In light of the obvious variety isomorphisms $(\mathcal{U} \cdot \mathcal{T}_1)/\mathcal{U}_0 = (\mathcal{U} \times \mathcal{T}_1)/\mathcal{U}_0 \cong \mathcal{U}/\mathcal{U}_0 \times \mathcal{T}_1 \cong \mathcal{U}/\mathcal{U}_0 \times \mathcal{T}/\mathcal{T}_0$, we thus obtain the desired isomorphism $\mathcal{A} \cdot C \cong \mathcal{U}/\mathcal{U}_0 \times \mathcal{T}/\mathcal{T}_0$.

The second factor of this direct product is $(K^*)^{t-\mathfrak{s}}$. As for the first: From Observation 3.2 we obtain $\mathcal{U}/\mathcal{U}_0 \cong \mathbb{A}^m$ with \mathfrak{m} as postulated.

The final statement of part (1) is clear. \square

The following consequence of the proof of Theorem 5.1(1) generalizes Proposition 4.2; it provides another specific instance of the general approach to finding top-stable degenerations suggested by Observation 3.7. We set $e = e_1 + \dots + e_t$, whence P takes on the form Λe and $\text{Aut}_\Lambda(P)$ is identified with the group of units of $e\Lambda e$. We start with the prerequisites for describing the relevant curves in the latter group. Given $i, j \in \{1, \dots, t\}$, not necessarily distinct, we let $(\omega_{ij}^{(r)})_{r \in R_{ij}}$ denote a collection of paths in Q , each from e_j to e_i , such that the $\omega_{ij}^{(r)}$, with $1 \leq i, j \leq t$ and $r \in R_{ij}$, generate eJe modulo $\text{Stab}_{eJe} C = \{a \in eJe \mid Ca \subseteq C\}$. In particular, this means that each $\omega_{ij}^{(r)}$ has positive length and satisfies $\omega_{ij}^{(r)} = e_i \omega_{ij}^{(r)} e_j$.

Corollary 5.4. *We keep the limit notation of Proposition 4.2.*

(1) *Suppose that P/C degenerates to P/C' for some $C' \in \mathfrak{Grass}_d^T$. Then there exist nonzero polynomials $q_1(\tau), \dots, q_t(\tau)$ in $K[\tau]$, next to polynomials $p_{ij}^{(r)}(\tau)$, such that*

$$\lim_{\tau \rightarrow \infty} C e(\tau) = C',$$

where

$$e(\tau) = \sum_{1 \leq i \leq t} q_i(\tau) e_i + \sum_{1 \leq i, j \leq t} \sum_{r \in R_{ij}} p_{ij}^{(r)}(\tau) \omega_{ij}^{(r)}$$

is identified with a unit in $e\Lambda e$ whenever $\tau \in \mathbb{A}^1$ lies outside the zero sets of the q_i .

Moreover, given any flag $\mathcal{C} = (C_\mu)_{1 \leq \mu \leq d'}$ of subspaces C_μ of C ,

$$\mathcal{C}' = \left(\lim_{\tau \rightarrow \infty} C_1 e(\tau), \dots, \lim_{\tau \rightarrow \infty} C_{d'} e(\tau) \right)$$

is a flag of subspaces of C' . In particular, for any subspace D of C , the limit $D' = \lim_{\tau \rightarrow \infty} D e(\tau)$ is a subspace of C' of the same dimension. If D is a Λ -submodule of P , then so is D' .

(2) Conversely, given polynomials as in (1), let $U \subseteq \mathbb{A}^1$ be a dense subset on which all of the q_i are nonzero, and denote by ψ the extension to \mathbb{P}^1 of the curve $U \rightarrow \overline{\text{Aut}_\Lambda(P).C}$ which sends $\tau \in U$ to $Ce(\tau)$. Then either ψ is constant, or else $\text{Im}(\psi)$ contains a point in $\overline{\text{Aut}_\Lambda(P).C} \setminus \text{Aut}_\Lambda(P).C$.

Proof. Without loss of generality, we may assume C to be chosen as in the proof of Theorem 5.1(1). Indeed, passage to a different point in the orbit $\text{Aut}_\Lambda(P).C$ is harmless, since each such point is of the form $C(\sum_{1 \leq i \leq t} b_i e_i + \sum_{1 \leq i, j \leq t} \sum_{r \in R_{ij}} a_{ij}^{(r)} \omega_{ij}^{(r)})$ for suitable $a_{ij}^{(r)} \in K$ and $b_i \in K^*$. We pick a sequence $\epsilon_1, \dots, \epsilon_{t-s}$ in \mathcal{T} such that $\mathcal{T}/\text{Stab}_{\mathcal{T}}(C) = \prod_k K^* \epsilon_k$. In view of $\text{Aut}_\Lambda(P).C \cong \mathcal{U}.C \times \mathcal{T}.C$, we then obtain an isomorphism of varieties, analogous to that of Lemma 4.1:

$$\left(\bigoplus_{i,j,r} K \omega_{ij}^{(r)} \right) \times \left(\prod_k K^* \epsilon_k \right) \cong \text{Aut}_\Lambda(P).C.$$

It maps $(\omega_{ij}^{(r)}, \epsilon_k)$ to $C \left(\left(\sum_{1 \leq l \leq t} \epsilon_{kl} e_l \right) + \omega_{ij}^{(r)} \right)$, if $\epsilon_k = (\epsilon_{k1}, \dots, \epsilon_{kt})$ with $\epsilon_{kl} \in K^*$. Consequently, any curve $U \rightarrow \text{Aut}_\Lambda(P).C$, where U is a dense subset of \mathbb{A}^1 , has the form $C \mapsto Ce(\tau)$ for suitable rational functions $q_i(\tau)$ and $p_{ij}^{(r)}(\tau)$. Clearing denominators yields polynomials $q_i(\tau)$ and $p_{ij}^{(r)}(\tau)$ as required.

For (2), it suffices to note that the variety $\text{Aut}_\Lambda(P).C$ is quasi-affine and hence has no complete subvarieties of positive dimension. \square

Suppose P/C degenerates to P/C' . In parallel to the approach presented in Lemma 4.7, we compute individual relations in C' from relations in C as follows: Let $e(\tau)$ be as in Corollary 5.4. Again, we fix a K -basis (y_k) for JP and define the $e(\tau)$ -degree of any element $x \in JP$ as in Section 4.B. As before, if $x e(\tau) = \sum_k \rho_k(\tau) y_k$, we let $\widetilde{\rho}_k(1/\tau)$ be the rational function in $1/\tau$ for which $\widetilde{\rho}_k(1/\tau) = (1/\tau)^{e(\tau) - \deg(x)} \rho_k(\tau)$ and set $x' = \sum_k \widetilde{\rho}_k(0) y_k$.

Lemma 5.5. *Let x be a nonzero element of JP . Keeping the above notation, we obtain:*

(1) $\lim_{\tau \rightarrow \infty} (Kx) e(\tau) = Kx'$.

This is a one-dimensional subspace of C' whenever $x \in C$.

Now suppose that all $p_{ij}^{(r)}(\tau)$ are zero, i.e., $e(\tau) = \sum_{1 \leq i \leq t} q_i(\tau) e_i$. Then:

(2) The $e(\tau)$ -degree of x equals the maximum of the degrees $\deg q_i(\tau)$ for which $xe_i \neq 0$. Moreover,

$$Kx' = K\left(\sum_{i \in \mathcal{L}} \tilde{q}_i(0)xe_i\right),$$

where $\mathcal{L} = \{i \leq \mathfrak{t} \mid \deg q_i(\tau) = e(\tau)\text{-deg}(x)\}$ and $\tilde{q}_i(1/\tau)$ is the rational function in $1/\tau$ for which $\tilde{q}_i(1/\tau) = (1/\tau)^{e(\tau) - \deg(x)} q_i(\tau)$. In particular, the $e(\tau)$ -degrees of x and x' coincide.

(3) For every subspace $D \subseteq JP$, there exists a basis (d_k) such that

$$\lim_{\tau \rightarrow \infty} De(\tau) = \bigoplus_k \lim_{\tau \rightarrow \infty} (Kd_k)e(\tau) = \bigoplus_k Kd'_k.$$

More strongly, if r belongs to $\mathbb{Z} \cup \{-\infty\}$ and $D_r \subseteq D$ is the subspace consisting of all elements in D of $e(\tau)$ -degree at most r , any basis for D_r of the above description can be supplemented to a basis for D with the same property.

Proof. Part (1) is proved as Lemma 4.7. For (2), we observe that the nonzero candidates among the elements xe_i are linearly independent, whence it is harmless to incorporate them in the reference basis (y_k) of JP ; indeed, in light of part (1), a basis change leaves x' invariant up to a factor in K^* . This makes the first assertion under (2) obvious. For τ in a dense subset of K , the product $(1/\tau)^{e(\tau) - \deg(x)} xe(\tau)$ consequently takes on the form $\sum_{i, xe_i \neq 0} \tilde{q}_i(1/\tau)xe_i$, whence passage to the limit “ $\tau \rightarrow \infty$ ” proves the second assertion under (2).

(3) By hypothesis, $e(\tau) = \sum_{1 \leq i \leq \mathfrak{t}} q_i(\tau)e_i$ with polynomials $q_i \in K[\tau]$. Let $r_1 < \dots < r_s$ denote the distinct $e(\tau)$ -degrees attained on the nonzero elements of D , and consider the partial flag $D_{-\infty} = 0 \subsetneq D_{r_1} \subsetneq \dots \subsetneq D_{r_s} = D$, where each $D_j \subseteq D$ is the subspace consisting of the elements of D that have $e(\tau)$ -degree at most j . For every $j \in \{r_1, \dots, r_s\}$, we moreover define $\mathcal{L}_j = \{1 \leq i \leq \mathfrak{t} \mid \deg q_i(\tau) = j\}$ and $f_j = \sum_{i \in \mathcal{L}_j} e_i$. Part (2) of the lemma then implies that $D_j = D(\sum_{i \leq j} f_i)$. Refine the partial flag $(D_j)_j$ of D to a full subspace flag $0 = B_0 \subsetneq B_1 \subsetneq \dots \subsetneq B_t = D$, and suppose that, for some $m < t$, the family (b_k) is a basis for B_m with the following properties: it contains bases for those subspaces D_j which are contained in B_m , and

$$B'_m = \lim_{\tau \rightarrow \infty} B_m e(\tau) = \bigoplus_k \lim_{\tau \rightarrow \infty} (Kb_k)e(\tau).$$

Pick $x \in B_{m+1} \setminus B_m$. Assuming that $\lim_{\tau \rightarrow \infty} (Kx)e(\tau)$ is contained in B'_m , our choice of the b_k provides us with scalars $a_k \in K$ such that $x' = \sum_k a_k b'_k$; here x' and b'_k result from x and b_k as described above. Set $y = \sum_k a_k b_k \in B_m$, and let $l = e(\tau)\text{-deg}(x)$, $l_k = e(\tau)\text{-deg}(b_k)$. The construction of our flag guarantees that $l_k \leq l$ and that all elements in D of $e(\tau)$ -degree $< l$ belong to B_m . In view of part (2) of the lemma, $x' = x'f_l$ and $b'_k = b'_k f_{l_k}$, whence linear independence of the b'_k yields $a_k = 0$ whenever $l_k < l$. Since, by construction, $y \neq 0$ and $D_{l-1} \subset B_m$ has basis $\{b_k \mid e(\tau) - \deg(b_k) < l\}$, we glean that the $e(\tau)$ -degree of y also equals l and that $y' = \sum_k a_k b'_k = x'$. In other words, $\sum_{i \in \mathcal{L}_l} \tilde{q}_i(0)ye_i = \sum_{i \in \mathcal{L}_l} \tilde{q}_i(0)xe_i$. Since $\tilde{q}_i(0) \neq 0$ for all $i \in \mathcal{L}_l$, we infer $ye_i = xe_i$ for $i \in \mathcal{L}_l$. In light of $xe_i = ye_i = 0$ whenever $i \in \bigcup_{j > l} \mathcal{L}_j$, we conclude that the $e(\tau)$ -degree of $x - y$ is strictly

smaller than l ; again we invoke part (2) of the lemma. But, as we noted above, this places the difference $x - y$ into B_m , which is incompatible with our choice of x outside B_m . An obvious induction now completes the proof of (3). \square

From part (3) of Lemma 5.5 we glean in particular that the “torus degenerations” of P/C can be obtained along simplicistic curves in $\text{Aut}_\Lambda(P).C$, because only certain coefficients of the $q_i(\tau)$ matter, next to their degree differential. Indeed, $\lim_{\tau \rightarrow \infty} C(\sum_{1 \leq i \leq t} q_i(\tau)e_i)$ remains unchanged if we replace the $q_i(\tau)$ by monomials $a_i(1/\tau)^{m_i}$ with suitable coefficients $a_i \in K^*$ such that $m_i < m_j$ if and only if $\deg q_i > \deg q_j$.

Lemma 5.6. *Let \mathcal{T} be the torus in $\text{Aut}_\Lambda(P)$, as introduced at the beginning of Section 5, and let $C \in \mathfrak{Grass}_d^T$ be a point with $\dim \mathcal{T}.C > 0$. Then $\overline{\mathcal{T}.C} \setminus \mathcal{T}.C$ contains at least two points belonging to different $\text{Aut}_\Lambda(P)$ -orbits. In particular, P/C has at least two nonisomorphic proper top-stable degenerations.*

Proof. By part (1) of Theorem 5.1 – already established – we can write $M = P/C$ in the form $U \oplus V$, where U is indecomposable and nonlocal. Without loss of generality, $C = A \oplus B$ with $A \subseteq \bigoplus_{1 \leq r \leq u} \Lambda e_r$ and $B \subseteq \bigoplus_{u+1 \leq r \leq t} \Lambda e_r$ such that $U \cong (\bigoplus_{1 \leq r \leq u} \Lambda e_r)/A$ and $V = (\bigoplus_{u+1 \leq r \leq t} \Lambda e_r)/B$ for a suitable index $u \in \{2, \dots, t\}$.

First we consider, for any $\tau \in K^*$, the automorphism $f_\tau \in \mathcal{T}$ defined by $f_\tau(e_1) = \tau e_1$ and $f_\tau(e_r) = e_r$ for $r \geq 2$. The curve $K^* \rightarrow \mathcal{T}.C$, given by $\tau \mapsto f_\tau(C)$, has a unique extension $\mathbb{P}^1 \rightarrow \overline{\mathcal{T}.C}$, due to completeness of $\overline{\mathcal{T}.C}$. We denote the value of this extension at infinity by $C' = \lim_{\tau \rightarrow \infty} f_\tau(C)$. First we note that C' is again a d' -dimensional subspace of JP . To see that C' does not belong to $\mathcal{T}.C$, we let $\pi_1 : P \rightarrow \Lambda e_1$ be the projection along $\bigoplus_{r \geq 2} \Lambda e_r$. Setting $\mu = \dim \pi_1(C)$, we pick elements $a_1, \dots, a_\mu \in A$ such that $\pi_1(a_1), \dots, \pi_1(a_\mu)$ form a basis for $\pi_1(A) = \pi_1(C)$. We supplement these elements with a basis $a_{\mu+1}, \dots, a_\nu$ for $A \cap \text{Ker}(\pi_1)$ to obtain a basis a_1, \dots, a_ν for A . Finally, we add on a basis $b_{\nu+1}, \dots, b_{d'}$ for B , which results in a basis $a_1, \dots, a_\nu, b_{\nu+1}, \dots, b_{d'}$ for C . Clearly, $a_{\mu+1}, \dots, a_\nu, b_{\nu+1}, \dots, b_{d'}$ are fixed by all f_τ , and hence belong to C' . Moreover, the following spaces are contained in C' (cf. Observation 3.7): Namely, the one-dimensional subspaces $\lim_{\tau \rightarrow \infty} f_\tau(Ka_r)$ for r between 1 and μ . If $a_r = \sum_{1 \leq i \leq t} \lambda_{ri} e_i$ with $\lambda_{ri} \in \Lambda$, then the latter space equals

$$\lim_{\tau \rightarrow \infty} K(\lambda_{r1} e_1 + \sum_{i \geq 2} (1/\tau) \lambda_{ri} e_i) = K \lambda_{r1} e_1 = K \pi_1(a_r).$$

Since the elements $\pi_1(a_1), \dots, \pi_1(a_\mu), a_{\mu+1}, \dots, a_\nu, b_{\nu+1}, \dots, b_{d'}$ of C' are linearly independent by construction, they form a basis for C' . Consequently, $C' = A_1 \oplus A_2 \oplus B$, where $A_1 = \pi_1(C) = \pi_1(A)$ and $A_2 = A \cap (\bigoplus_{2 \leq r \leq u} \Lambda e_r) = \sum_{\mu+1 \leq r \leq \nu} Ka_r$. We thus obtain the following decomposition of the Λ -module P/C' :

$$P/C' = (\Lambda e_1/A_1) \oplus \left(\left(\bigoplus_{2 \leq r \leq u} \Lambda e_r \right) / A_2 \right) \oplus \left(\left(\bigoplus_{u+1 \leq r \leq d'} \Lambda e_r \right) / B \right).$$

This module is not isomorphic to P/C , since it has more indecomposable summands than P/C .

Analogously, one obtains a point $C'' = A'_1 \oplus A'_2 \oplus B \in \overline{\mathcal{T}.C} \setminus \mathcal{T}.C$ with $A'_1 = A \cap \Lambda e_1$ and $A'_2 = \pi'(A)$, where π' is the projection of P onto $\bigoplus_{2 \leq r \leq u} \Lambda e_r$ along $\Lambda e_1 \oplus \bigoplus_{u+1 \leq r \leq t} \Lambda e_r$.

Since U is indecomposable, $\dim A'_1 < \dim A_1$, which shows P/C' and P/C'' to be nonisomorphic. \square

Proof of Theorem 5.1, parts (2)–(6).

(2). That (i) implies (ii) follows from part (1), in view of the fact that closedness of the quasi-affine variety $\text{Aut}_\Lambda(P).C$ in the complete variety \mathfrak{Grass}_d^T forces the dimension of the former to be zero. The converse is trivial. To see that (ii) is equivalent to (iii), note that the equality $\mathfrak{t} = \mathfrak{s}$ just means that M is a direct sum of local modules. Moreover, $\mathfrak{m} = 0$ is tantamount to the equality $\mathcal{U}.C = \{C\}$ (cf. Observation 3.2); the latter in turn says $(\text{id} + f)(C) = C$ for all $f \in \text{Hom}_\Lambda(P, JP)$, which is the same as $f(C) \subseteq C$ for all $f \in \text{Hom}_\Lambda(P, JP)$. The implication (iv) \implies (ii) is clear, since (iv) means $\text{Aut}_\Lambda(P).C$ is a singleton; for (ii) \implies (iv), let $f \in \text{End}_\Lambda(P)$, and consider the automorphism $g = a \text{id} - f$ of P , where $a \in K$ fails to be an eigenvalue of f . Invariance of C under g then amounts to invariance of C under f .

(3). By the proof of part (1), we may, without loss of generality, assume that the variety $\text{Aut}_\Lambda(P).C$ is isomorphic to the product $\mathcal{U}.C \times \mathcal{T}.C$, with the second factor isomorphic to $(K^*)^{\mathfrak{t}-\mathfrak{s}}$.

Whenever $\dim \mathcal{T}.C > 0$, Lemma 5.6 guarantees the existence of two distinct proper top-stable degenerations of M . So, on the assumption that M has a unique proper top-stable degeneration, $\text{Aut}_\Lambda(P).C$ equals $\mathcal{U}.C \cong \mathbb{A}^{\mathfrak{m}}$, and as in the proof of Theorem 4.4, part (3b), we derive $\mathfrak{m} = 1$. That, conversely, $\text{Aut}_\Lambda(P).C \cong \mathbb{A}^1$ forces M to have a unique top-stable degeneration, can also be shown as in the local case.

To justify the final assertion under (3), suppose $\mathfrak{m} = 0$ and $\mathfrak{t} - \mathfrak{s} = 1$. The existence of at least two nonisomorphic proper top-stable degenerations of M follows from Lemma 5.6. On the other hand, $|\overline{\mathcal{T}.C} \setminus \mathcal{T}.C| \leq 2$, since the embedding $\mathcal{T}.C \cong K^* \rightarrow \overline{\mathcal{T}.C}$ extends to a morphism $\mathbb{P}^1 \rightarrow \overline{\mathcal{T}.C}$ and the image of the latter is closed (see, e.g., [6, Chapter AG, Section 7.4]). This completes the proof of (3).

Proofs for parts (4) and (5) carry over verbatim from the local situation.

(6). Now suppose that $\mathfrak{m} = 0$, meaning that the unipotent radical of $\text{Aut}_\Lambda(P)$ acts trivially on C . Thus $\text{Aut}_\Lambda(P).C = \mathcal{T}.C$. Once again, we assume that C is chosen as in the proof of Theorem 5.1(1). In other words, there exists a partition $\mathcal{I}_1 \cup \dots \cup \mathcal{I}_\mathfrak{s}$ of $\mathcal{I} = \{1, \dots, \mathfrak{t}\}$ with the following property: For $P_k = \bigoplus_{i \in \mathcal{I}_k} \Lambda e_i$, we obtain a decomposition $C = \bigoplus_{1 \leq k \leq \mathfrak{s}} C_k$ with $C_k \subseteq P_k$ which, in turn, gives rise to a decomposition $\bigoplus_{1 \leq k \leq \mathfrak{s}} P_k/C_k$ of M into indecomposable direct summands.

We first prove that every top-stable degeneration of M is a direct sum of top-stable degenerations of the P_k/C_k . Let $C' \in \mathfrak{Grass}_d^T$ with $P/C \leq_{\text{deg}} P/C'$. According to Corollary 5.4, we choose $e(\tau)$ such that $C' = \lim_{\tau \rightarrow \infty} C e(\tau)$. In our present scenario, we may take $e(\tau)$ to be of the form $\sum_{1 \leq i \leq \mathfrak{t}} q_i(\tau) e_i$ for suitable polynomials $q_i(\tau) \in K[\tau]$, and, in view of the remark following Lemma 5.5, we may simplify $e(\tau)$ further as follows: Namely, if the degrees of the $q_i(\tau)$ take $r + 1$ different values, $m_0 > \dots > m_r$, we may replace $q_i(\tau)$ by $a_i(1/\tau)^\nu$ if $\deg q_i = m_\nu$, where the a_i are nonzero scalars determined by the q_i .

Clearly, $P_k e(\tau) \subseteq P_k$, whenever $\tau \in K^*$. Consequently, $\tau \mapsto C_k e(\tau)$ defines a curve from K^* to $\mathcal{G}r(\dim C_k, JP_k)$. Due to closedness of $\mathcal{G}r(\dim C_k, JP_k)$ in $\mathcal{G}r(\dim C_k, JP)$, we infer that $\lim_{\tau \rightarrow \infty} C_k e(\tau)$ is a $(\dim C_k)$ -dimensional subspace of JP_k . A comparison of dimensions shows that C' is the direct sum of these limits. Call them C'_k , respectively.

Therefore, $M' = P/C'$ decomposes in the form $M' = \bigoplus_{1 \leq k \leq \mathfrak{s}} P_k/C'_k$ with $P_k/C_k \leq_{\text{deg}} P_k/C'_k$ as required.

Next we show that $M \leq_{\text{ext}} M'$. (Our argument will actually prove that the following holds, irrespective of the value of \mathfrak{m} : In case $C' \in \overline{\mathcal{T}}\mathcal{C}$, the degeneration P/C' satisfies $P/C \leq_{\text{ext}} P/C'$.) The preceding paragraph legitimizes restriction to the case where M is indecomposable. Moreover, it is harmless to assume $\mathfrak{t} > 1$, for otherwise $\mathcal{T}\mathcal{C}$ is a singleton by part (1).

The crucial step consists of showing that either $P/C \cong P/C'$, or else there is a nonempty proper subset \mathcal{L} of $\mathcal{I} = \{1, \dots, \mathfrak{t}\}$ such that the top-stably embedded submodule $U = \sum_{i \in \mathcal{L}} \Lambda(e_i + C)$ of P/C is isomorphic to a direct summand U' of P/C' , with M/U degenerating to a direct sum complement of U' in P/C' .

The $e(\tau)$ -degrees of the nonzero elements of C vary between $-r$ and 0 in our present setting. For each integer k in this range, we consider the submodule D_k of C consisting of all elements having $e(\tau)$ -degree at most k . Evidently, $D_0 = C$. According to Lemma 5.5(3), we can find a basis B of C which includes a basis B_k for each D_k and has the property that

$$D'_k := \lim_{\tau \rightarrow \infty} D_k e(\tau) = \bigoplus_{b \in B_k} \lim_{\tau \rightarrow \infty} (Kb) e(\tau) = \bigoplus_{b \in B_k} Kb'$$

for all k , where the b' are as in Lemma 5.5. Note that D'_k is a submodule of C' by Corollary 5.4. We introduce another partition of the set \mathcal{I} : Namely \mathcal{I} is the disjoint union of the subsets $\mathcal{L}_k = \{1 \leq i \leq \mathfrak{t} \mid \deg q_i(\tau) = k\}$ for $-r \leq k \leq 0$.

Suppose $b \in B_k \setminus B_{k-1}$ for some $k > -r$. This means $be_i \neq 0$ for at least one index $i \in \mathcal{L}_k$ and $be_i = 0$ for all $i \in \bigcup_{j > k} \mathcal{L}_j$. To adjust the notation to that of Lemma 5.5(2) for an inspection of b' , we set $\tilde{q}_i(1/\tau) = q_i(\tau)(1/\tau)^k$. Whenever $i \in \mathcal{L}_k$, this gives $\tilde{q}_i(0) = a_i$. Lemma 5.5 thus yields $\lim_{\tau \rightarrow \infty} (Kb) e(\tau) = K(\sum_{i \in \mathcal{L}_k} a_i be_i)$, and hence this latter space is contained in $C' \cap \bigoplus_{i \in \mathcal{L}_k} \Lambda e_i$. For $k = -r$, our choice of B thus implies $D'_{-r} = \lim_{\tau \rightarrow \infty} D_{-r} e(\tau) \subseteq C' \cap Q_1$, where $Q_1 = \bigoplus_{i \in \mathcal{L}_{-r}} \Lambda e_i$. Moreover, we see that D'_{-r} is isomorphic to D_{-r} via a torus automorphism of Q_1 , which guarantees that Q_1/D_{-r} is isomorphic to Q_1/D'_{-r} . If we denote by D the subspace of C spanned by the $b \in B \setminus B_{-r}$, we similarly see that $D' := \lim_{\tau \rightarrow \infty} D e(\tau)$ is contained in $C' \cap Q_2$, where $Q_2 = \bigoplus_{i \in \bigcup_{-r < k \leq 0} \mathcal{L}_k} \Lambda e_i$. Let f be the sum of the idempotents generating Q_2 ; i.e., right multiplication by f projects P onto Q_2 along Q_1 . Then $Df = Cf$, and we find that this submodule of JP is shifted to D' as we move it to ∞ along our given curve: Indeed, by construction,

$$D' = \bigoplus_{b \in B \setminus B_{-r}} \lim_{\tau \rightarrow \infty} (Kb) e(\tau) = \bigoplus_{b \in B \setminus B_{-r}} \lim_{\tau \rightarrow \infty} (Kb) f e(\tau),$$

where the last limit equals $\lim_{\tau \rightarrow \infty} (Cf) e(\tau)$, due to the dimension equality $\dim D = \dim Cf$. In particular, D' is a submodule of C' , since Cf is a submodule of C . This confirms that Q_2/Cf degenerates to Q_2/D' .

In case $r = 0$, i.e., in case all $q_i(\tau)$ have the same degree, C' equals $C(\sum_{1 \leq i \leq \mathfrak{t}} a_i e_i)$ and consequently belongs to $\text{Aut}_\Lambda(P)\mathcal{C}$, that is, $P/C \cong P/C'$. So suppose that $r > 0$, and set $\mathcal{L} = \mathcal{L}_{-r}$. Then the submodule $U = \sum_{i \in \mathcal{L}} \Lambda(e_i + C)$ of P/C satisfies the requirements

spelled out above: Indeed, $C' = D'_{-r} \oplus D'$ by our choice of B , and therefore $M' = (Q_1/D'_{-r}) \oplus (Q_2/D')$. The first summand is isomorphic to $Q_1/D_{-r} = Q_1/(Q_1 \cap C) \cong U$. For the second, we obtain $M/U \cong (Q_2/Cf) \leq_{\text{deg}} (Q_2/D')$ as explained above.

To deduce that $P/C \leq_{\text{ext}} P/C'$, it now suffices to keep in mind that D' equals the limit $\lim_{\tau \rightarrow \infty} (Cf)(\sum_{i \in \mathcal{I} \setminus \mathcal{L}_{-r}} q_i(\tau)e_i)$. This puts D' into the closure of the torus orbit of Cf in $\mathcal{G}r(\dim Cf, JQ_2)$, whence an obvious induction completes the argument.

Finally, we observe that there are only 2^t possibilities for the submodule $U \subseteq M$ described above, whence we conclude that M has only finitely many top-stable degenerations, up to isomorphism. \square

Theorem 5.7. Layer-stable degenerations. *Let T be a direct sum of \mathfrak{t} pairwise non-isomorphic simple modules, $P = \bigoplus_{1 \leq i \leq t} \Lambda e_i$ its projective cover, d a positive integer, and $M = P/C$ with $C \in \mathfrak{Grass}_d^T$. Let $\mathfrak{s} = \mathfrak{s}(M)$ and $\mathfrak{m} = \mathfrak{m}(M)$.*

(1) Existence.

(a) *If M is a direct sum of local modules, that is if $\mathfrak{t} - \mathfrak{s} = 0$, then M has no proper layer-stable degenerations.*

(b) *If $\mathfrak{m} = 0$, the following statements are equivalent:*

(i) *M has a proper layer-stable degeneration.*

(ii) *One of the indecomposable summands of M has a nonzero proper layer-stably embedded submodule.*

In this case, M has only finitely many proper layer-stable degenerations, the minimal ones being of the form $U \oplus M/U$, where U is layer-stably embedded in M .

(c) *For any positive integer r , there exists an indecomposable finite dimensional module M with squarefree top, over a suitable finite dimensional algebra Λ , such that M has a \mathbb{P}^r -family of pairwise non-isomorphic indecomposable layer-stable degenerations.*

(2) Bases. *If $M' = P/C'$ is a layer-stable degeneration of M , then M and M' share a skeleton (cf. Section 2).*

Proof. (1)(a) is due to the fact that \mathcal{T} acts trivially on C , whenever $M = P/C$ is a direct sum of local modules: This means $\text{Aut}_\Lambda(P).C = \mathcal{U}.C$, where \mathcal{U} is the unipotent radical of $\text{Aut}_\Lambda(P)$. But, letting $\mathbb{S} = \mathbb{S}(M)$ be the radical layering of M , we know from Observation 3.2 that $\mathcal{U}.C$ is closed in $\mathfrak{Grass}(\mathbb{S})$, whence Observation 3.1 allows us to conclude that M has no proper layer-stable degeneration.

In light of Observation 3.3, the assertion (1)(b) is a special case of part (6) of Theorem 5.1, and (1)(c) will be justified in Example 5.9.

(2). Let $M' = P/C'$ be a layer-stable degeneration of P/C and σ any skeleton of M' . Layer-stability means that C' belongs to the closure of $\text{Aut}_\Lambda(P).C$ in $\mathfrak{Grass}(\mathbb{S})$, where $\mathbb{S} = \mathbb{S}(M) = \mathbb{S}(M')$ (Observation 3.1). Since $\mathfrak{Grass}(\sigma)$ is open in $\mathfrak{Grass}(\mathbb{S})$ (Theorem 2.1), we infer that $\mathfrak{Grass}(\sigma)$ intersects $\text{Aut}_\Lambda(P).C$ nontrivially. This means that $\text{Aut}_\Lambda(P).C \subseteq \mathfrak{Grass}(\sigma)$, because $\mathfrak{Grass}(\sigma)$ is stable under the $\text{Aut}_\Lambda(P)$ -action, and thus proves our claim. \square

5.B Examples and proof of Theorem 5.7, part (1)(c).

Our approach to computing the layer-stable degenerations of a module $M = P/C$ (without computing all of its top-stable degenerations) is based on the fact that the closure

of $\text{Aut}_\Lambda(P).C$ in $\mathfrak{Grass}(\mathbb{S}(M))$ is the union of the closures of the orbit $\text{Aut}_\Lambda(P).C$ in the affine varieties $\mathfrak{Grass}(\sigma)$, where σ runs through the skeletons of M . In writing $e = e_1 + \cdots + e_t$, we obtain $\text{Aut}_\Lambda(P).C$ as the set of points Cu in \mathfrak{Grass}_d^T , where u runs through the units in $e\Lambda e$. One can pare down the computation required by noticing that, in fact, $\text{Aut}_\Lambda(P).C$ consists of the points Cu , where u is a unit in $e\Lambda e$ which is a K -linear combination of (I -residue classes of) paths in σ ; this is due to the fact that every module isomorphic to M again has skeleton σ and thus results from M via an isomorphism having a lift to $\text{Aut}_\Lambda(P)$ of the described ilk.

Example 5.8 illustrates the finite scenario. In contrast to the situation $\mathfrak{m} = 0$ described in Theorem 5.7, even an indecomposable module M with only finitely many top-stable degenerations may have indecomposable candidates among its proper layer-stable degenerations. Our first example exhibits this phenomenon. Diagram 5.7 shows the full poset of proper top-stable degenerations of the considered module.

Example 5.8. Let Q be the following quiver:

$$2 \xrightarrow{\beta} 1 \xleftarrow{\gamma} 3 \quad \text{with a curved arrow } \alpha \text{ from } 1 \text{ to } 2$$

Define $\Lambda = KQ/\langle \alpha^3 \rangle$. Moreover, let $T = S_1 \oplus S_2 \oplus S_3$, and $d = 5$. The module whose top-stable degenerations we want to find is $M = \Lambda/C$, where $C = \Lambda(\alpha - \beta) + \Lambda(\alpha - \gamma)$. Clearly, M has precisely three distinct skeletons, namely:

$$\sigma_1 = \{e_1, e_2, e_3, \alpha, \alpha^2\}, \quad \sigma_2 = \{e_1, e_2, e_3, \beta, \alpha\beta\}, \quad \sigma_3 = \{e_1, e_2, e_3, \gamma, \alpha\gamma\}.$$

Let us abbreviate σ_1 to σ and start by determining polynomials for $\mathfrak{Grass}(\sigma)$. There are precisely two relevant σ -critical pairs, namely (β, e_2) and (γ, e_3) (we may ignore the pair (α, α^2) since $\alpha^3 M = 0$), with $\sigma(\beta, e_2) = \{\alpha, \alpha^2\} = \sigma(\gamma, e_3)$, leading to the basic congruences

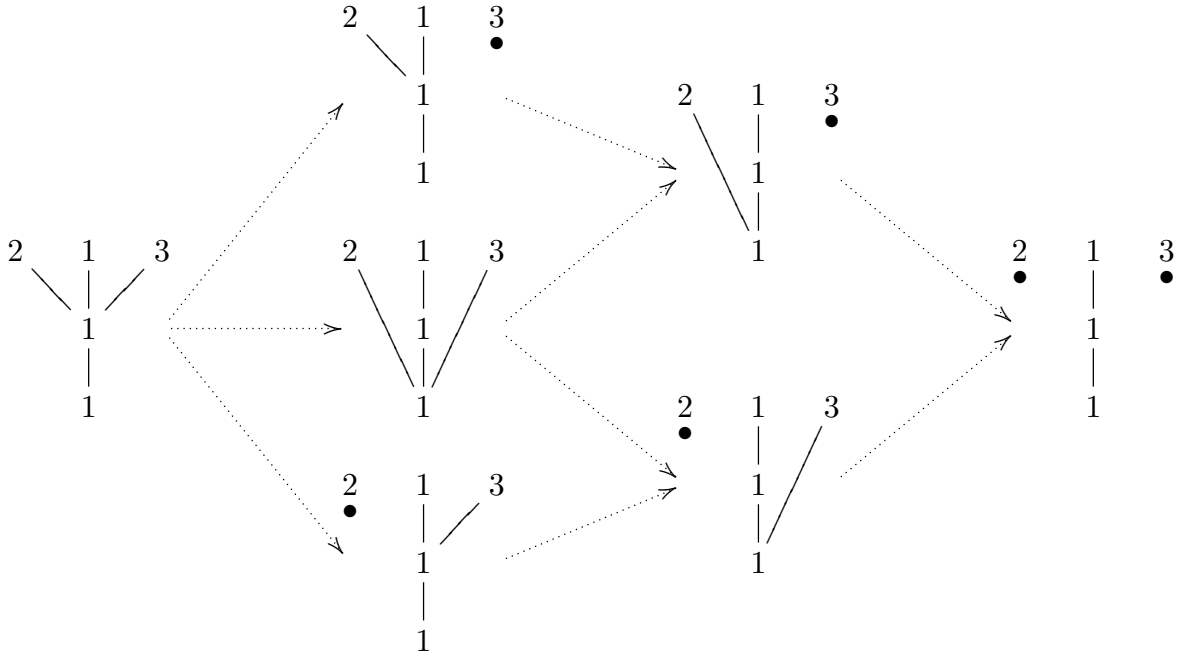
$$\beta \hat{=} X_1\alpha + X_2\alpha^2 \quad \text{and} \quad \gamma \hat{=} X_3\alpha + X_4\alpha^2.$$

The procedure employed in Example 2.2 yields $\mathfrak{Grass}(\sigma) = \mathbb{A}^4$. Note that M corresponds to the point $C = (1, 0, 1, 0)$ in the present coordinatization. Using the comments we made at the beginning of this section, we further obtain

$$\text{Aut}_\Lambda(P).C = \{C' = (c'_1, c'_2, c'_3, c'_4) \mid c'_1, c'_3 \neq 0, c'_1 c'_4 = c'_2 c'_3\}$$

in the coordinates of $\mathfrak{Grass}(\sigma)$. Thus, $\text{Aut}_\Lambda(P).C \cong \mathbb{A}^1 \times (K^*)^2$ in the format of Theorem 5.1(1).

In computing the closure $\overline{\text{Aut}_\Lambda(P).C}^\sigma$ of $\text{Aut}_\Lambda(P).C$ in $\mathfrak{Grass}(\sigma)$, one arrives at the following degenerations of M ; a graph of M is displayed on the left:



A justification of this picture is as follows: One finds the complement of $\text{Aut}_\Lambda(P).C$ in the $\mathbf{Grass}(\sigma)$ -closure $\overline{\text{Aut}_\Lambda(P).C}^\sigma$ to have the following connected components U, V, W :

$$\begin{aligned}
 U &= \{(k_1, k_2, 0, 0) \mid k_1, k_2 \in K\}, & V &= \{(0, 0, k_3, k_4) \mid k_3, k_4 \in K\}, \\
 W &= \{(0, k_2, 0, k_4) \mid k_2, k_4 \in K\}.
 \end{aligned}$$

To further decompose U into $\text{Aut}_\Lambda(P)$ -orbits, we compute the orbit of $(1, 0, 0, 0)$ to be $U_1 = \{k \in U \mid k_1 \in K^*\} \cong K^* \times \mathbb{A}^1$, leaving the difference $U \setminus U_1 = \{(0, k_2, 0, 0) \mid k_2 \in K\}$. The latter, finally, falls into a one-dimensional orbit, $U_2 = \{(0, k_2, 0, 0) \mid k_2 \in K^*\} \cong K^*$, and the point $U_3 = \{(0, 0, 0, 0)\}$, these being the orbits corresponding to the modules $((\Lambda e_1 \oplus \Lambda e_2)/\Lambda(\alpha^2 - \beta)) \oplus S_3$ and $\Lambda e_1 \oplus S_2 \oplus S_3$, respectively. The degenerations resulting from this decomposition of U into orbits are depicted in the uppermost sequence of dotted arrows in the above diagram. With V and W one deals analogously, and on eliminating overlaps, one obtains the above hierarchy of degenerations with skeleton $\sigma = \sigma_1$.

To obtain the full collection of layer-stable degenerations of M afforded by the closure of $\text{Aut}_\Lambda(P).C$ in $\mathbf{Grass}(\mathbb{S})$, one repeats these computations for the skeleta σ_2 and σ_3 of M . The complete hierarchy of proper top-stable degenerations of M is given in Diagram 5.8 below. The four minimal top-stable degenerations can be found in the (expanded) left-most column. The next column to the right contains the five top-stable degenerations of height 2 above M , the right-most column the three maximal top-stable degenerations. \square

Our final example is a variation on the theme of the previous one, showing that, for any positive integer r , there exists a module M with squarefree top over a suitable finite dimensional algebra such that M has a \mathbb{P}^{r-1} -family of distinct indecomposable layer-stable degenerations.

The poset of proper top-stable degenerations for Example 5.8

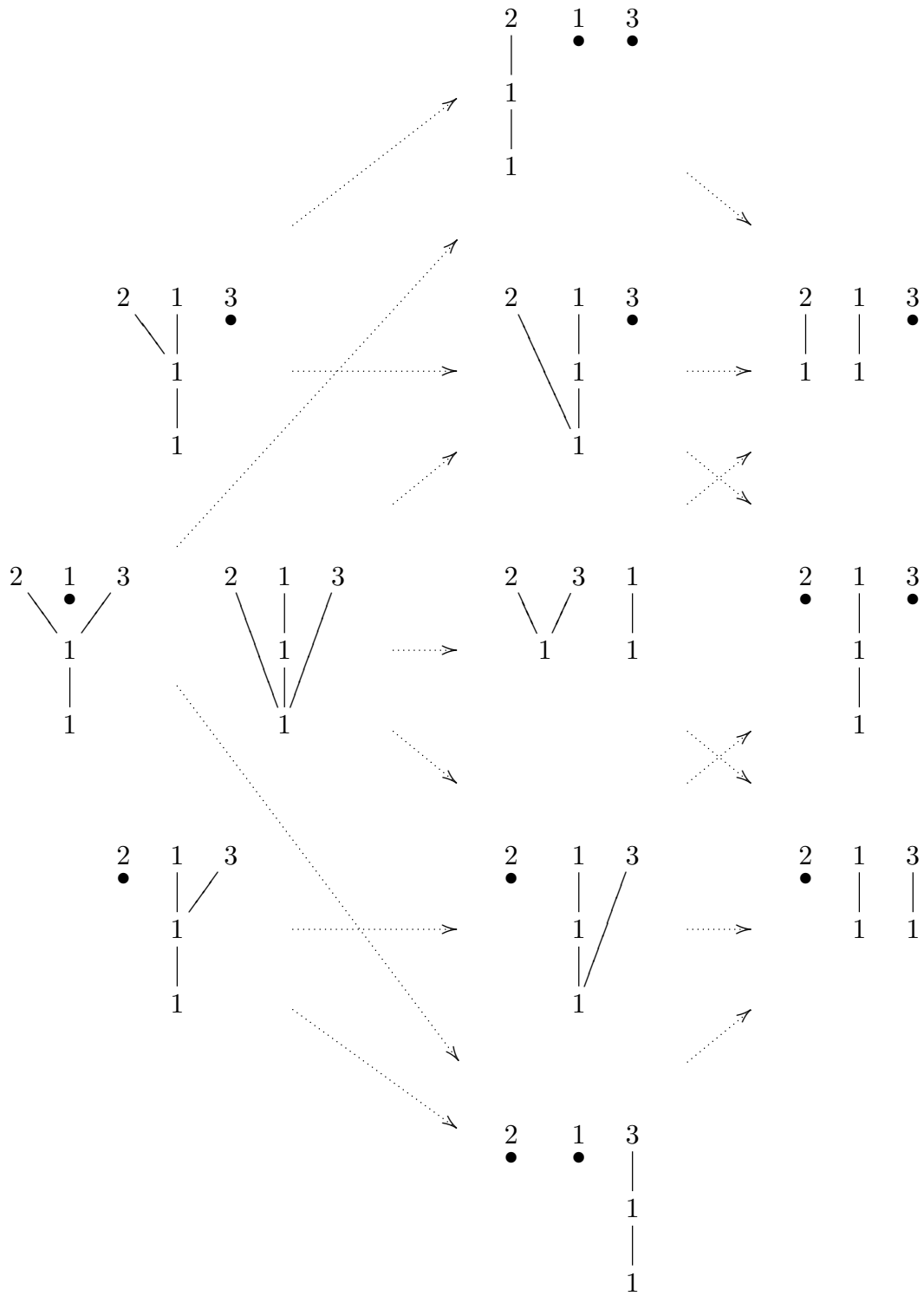
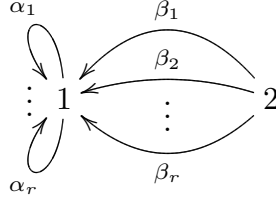


Diagram 5.8

Example 5.9. Fix $r \geq 1$, and let Q be the quiver



Moreover, let $\Lambda = KQ/I$, where I is the ideal of KQ generated by α_i^3 and $\alpha_i^2\beta_i$ for $1 \leq i \leq r$, and $\alpha_i\alpha_j$ and $\alpha_i\beta_j$ for $i \neq j$. Moreover, let $P = \Lambda = \Lambda e_1 \oplus \Lambda e_2$ and $T = P/JP$. We focus on the module $M = P/C$ with

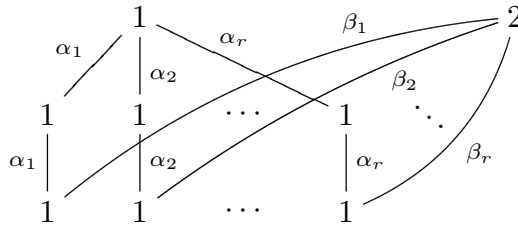
$$C = \sum_{1 \leq i \leq r} \Lambda(\alpha_i - \beta_i) \in \mathfrak{Grass}_d^T,$$

where $d = 2r + 2$. Here $\text{Aut}_\Lambda(P).C \cong \mathbb{A}^r \times K^*$.

The following is a \mathbb{P}^{r-1} -family of layer-stable degenerations of M : Namely, for any point $k \in \mathbb{P}^{r-1}$ with homogeneous coordinates (k_1, \dots, k_r) , we set $M_k = P/C_k$, where

$$C_k = \lim_{\tau \rightarrow \infty} C(e_1 + \tau e_2 + \sum_{1 \leq i \leq r} \tau k_i \alpha_i) = \sum_{1 \leq i \leq r} \Lambda(k_i \alpha_i^2 - \beta_i).$$

Clearly $M_k \cong M_{k'}$ if and only if $k = k'$ in \mathbb{P}^{r-1} . For $(k_1, \dots, k_r) \in (K^*)^r$, the degeneration M_k has the following graph:



Beyond the listed ones, M has precisely two further proper layer-stable degenerations, namely $S_1 \oplus \Lambda e_2$ and $\Lambda e_1 \oplus S_2$. \square

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