DUALITIES FROM ITERATED TILTING

Birge Huisgen-Zimmermann

Abstract. First a duality theory is developed for arbitrary finite dimensional algebras Λ and Λ'. It provides a characterization of the contravariant equivalences which link resolving subcategories of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$, the category of finitely generated left Λ-modules of finite projective dimension, to resolving subcategories of $\mathcal{P}^{<\infty}(\text{mod-}\Lambda')$. The pertinent theorem expands Miyashita's work on tilting. As a consequence, we find: If there are resolving subcategories of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{P}^{<\infty}(\text{mod-}\Lambda')$, respectively, which are dual via functors satisfying a strict exactness condition, then Λ and Λ' are derived equivalent.

The core of the paper addresses the tilting theory of truncated path algebras, i.e., of path algebras modulo ideals generated by all paths of a given fixed length in the underlying quiver. (These algebras provide a natural environment for the study of finite dimensional representations of quivers with oriented cycles in that, for growing Loewy length, they reflect the combinatorics of the quiver in undiluted form.)

If Λ is a truncated path algebra, the category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is known to be contravariantly finite in Λ-mod, whence Λ has a strong tilting module. It is shown here that all algebras Δ obtained from Λ via iterated strong tilting retain these assets, their strong tilting modules being explicitly available from the quiver and Loewy length of Λ. The iteration process becomes periodic with period 2 after the initial tilting step. While structurally the algebras Δ that arise from an iteration of strong tilting have little in common with the original truncated algebra Λ, we decode their homological properties by combining the mentioned dualities with an algebraic-combinatorial approach to their $\mathcal{P}^{<\infty}$-categories. This analysis permits us to recognize the Δ-modules of finite projective dimension in terms of their intrinsic structure.

1. Introduction and Main Results

Among the tools for comparing the categories of representations of two finite dimensional algebras, tilting functors have become staples. They give rise to partial equivalences of the module categories they connect, as well as to triangle equivalences on the level of the derived module categories; see [**6**, **3**, **8**, **15**, **7**, **24**, **2**, **14**, **10**, **25**, **22**] for background.

We adopt Miyashita's definitive concept of a *tilting module*. (For the italicized terms, we refer to Section 2.A.) In particular, any tilting (left) module over a finite dimensional algebra Λ belongs to the full subcategory $\mathcal{P}^{<\infty}(\Lambda\text{-mod}) := \{X \in \Lambda\text{-mod} \mid \operatorname{p\,dim} X < \infty\}$. Among the tilting modules, distinguished specimens dubbed *strong* were first placed under a spotlight by Auslander and Reiten in [**4**]. They are the tilting modules T which are Ext-*injective relative to the objects in* $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$, meaning that $\operatorname{Ext}^i_\Lambda(\mathcal{P}^{<\infty}(\Lambda\text{-mod}), T) = 0$ for all $i \geq 1$; in light of Section 2.C, this description of strong tilting modules is equivalent to the original definition. Strong tilting modules are unique up to repeats of indecomposable summands; i.e., there is at most one *basic* strong tilting object in Λ-mod, existence being equivalent to *contravariant finiteness* of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$. Subsequently, the basic strong tilting modules were re-encountered from a different angle by Happel-Unger in [**16**] and further explored for Auslander-Gorenstein algebras by Iyama-Zhang in [**21**]. Existence provided, the basic strong tilting module in Λ-mod is the unique smallest object in the set of all basic tilting objects in Λ-mod under a natural partial order.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Our objectives here are twofold:

(1) The first is to investigate and apply *dualities* induced by tilting modules. Let $T \in \Lambda$-mod be any tilting module, and $\widetilde{\Lambda} = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$ the corresponding tilted algebra. Next to a solidly explored family of covariant equivalences among certain subcategories of Λ-mod and $\widetilde{\Lambda}$-mod, Miyashita exhibited weakened clones of Morita dualities relating suitable subcategories of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ to subcategories of $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ ([**24**, Theorem 3.5]); the latter have received little attention to date. These "partial" contravariant equivalences morph into a fully-fledged duality

$$(\dagger) \qquad \mathrm{Hom}_\Lambda(-,T): \ \mathcal{P}^{<\infty}(\Lambda\text{-mod}) \ \longleftrightarrow \ \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda}) \ : \mathrm{Hom}_{\widetilde{\Lambda}}(-,T)$$

precisely when ${}_\Lambda T_{\widetilde{\Lambda}}$ is a tilting bimodule which is strong on both sides. Our Theorem 1 (Section 2.B) provides a converse in the following broader context: Namely, given arbitrary finite dimensional algebras Λ, Λ' and *resolving subcategories* $\mathcal{C} \subseteq \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{C}' \subseteq \mathcal{P}^{<\infty}(\text{mod-}\Lambda')$, any pair of "strictly exact" inverse dualities $\mathcal{C} \longleftrightarrow \mathcal{C}'$ is induced by a tilting bimodule ${}_\Lambda T_{\Lambda'}$ via the restricted Hom-functors $\mathrm{Hom}_\Lambda(-,T)|_{\mathcal{C}}$ and $\mathrm{Hom}_{\Lambda'}(-,T)|_{\mathcal{C}'}$; that a contravariant functor F between full subcategories of module categories, say $F : \mathcal{D} \to \mathcal{D}'$ with $\mathcal{D} \subseteq \Lambda$-Mod and $\mathcal{D}' \subseteq$ Mod-Λ', be *strictly exact* means that F takes any short exact sequence in Λ-Mod with terms in $\mathcal{D}$ to a short exact sequence in Mod-Λ'. Observe that, for dualities $\mathcal{C} \longleftrightarrow \mathcal{C}'$ with $\mathcal{C}$ and $\mathcal{C}'$ as above, strict exactness is automatic provided $\mathcal{C}$ and $\mathcal{C}'$ are closed also under cokernels of injective morphisms in the ambient module categories, as is the case for $\mathcal{C} = \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{C}' = \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ in ($\dagger$).

We point to an immediate consequence of the result on dualities: If there are resolving subcategories of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{P}^{<\infty}(\text{mod-}\Lambda')$ which are dual by way of strictly exact functors, then Λ and Λ' are derived equivalent.

Subsequently, the dualities induced by strong tilting modules are explored in depth for truncated path algebras, i.e., for algebras of the form $\Lambda = KQ/I$, where K is a field, Q a quiver and $I \subseteq KQ$ the ideal generated by all paths of some fixed length ≥ 2. In this scenario, existence of a strong tilting module T is guaranteed by [**11**, Theorem 4.1], but in general the righthand side of ($\dagger$) needs to be replaced by a *proper* resolving subcategory of $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$; again $\widetilde{\Lambda} = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$. Still, we will find the process of strong tilting to always allow for iteration when it is anchored in a truncated algebra. Indeed, mod-$\widetilde{\Lambda}$ will be seen to, in turn, feature a strong tilting module $\widetilde{T}$, as does $\widetilde{\widetilde{\Lambda}}$-mod for $\widetilde{\widetilde{\Lambda}} = \mathrm{End}_{\widetilde{\Lambda}}(\widetilde{T})$, etc. Assuming T to be basic, we further prove that the sequence of algebras thus obtained becomes periodic, returning to $\widetilde{\Lambda}$ after two additional tilting steps. On the other hand, the strong tilts of Λ move "far away" from Λ; in particular, they exhibit a steep increase in Loewy lengths and complexity of quivers and relations, in exchange for increased homological mirror-symmetry. In tandem with exploiting the (partial) dualities among the $\mathcal{P}^{<\infty}$-categories of Λ and the successive strongly tilted algebras, we arrive at a thorough understanding of the objects of these $\mathcal{P}^{<\infty}$-categories, which allows us to single them out in terms of their intrinsic structure rather than relying on resolutions.

(2) The second objective is part of a more encompassing goal: Namely, to advance the representation theory of truncated path algebras to a level which is comparable to that attained for hereditary Λ. (Note that all finite dimensional hereditary algebras over an algebraically closed field are Morita equivalent to path algebras of acyclic quivers, the latter being truncated by way of the cutoff at maximal path length.) This program – well under way with regard to an exploration of the (geometrically) generic representations in the irreducible components of the parametrizing varieties $\mathbf{Rep}_{\mathbf{d}}(\Lambda)$ (see, e.g., [**13**]) – is motivated by the following straightforward fact: Given an arbitrary path algebra modulo relations, $\Lambda = KQ/I$, there is a unique truncated path algebra

Λ_{trunc} with $\Lambda = \Lambda_{\mathrm{trunc}}/(\text{suitable ideal})$ such that Λ_{trunc} has the same quiver and Loewy length as Λ. The resulting inclusion $\Lambda\text{-mod} \subseteq \Lambda_{\mathrm{trunc}}\text{-mod}$ affords a profitable back and forth between the Λ- and Λ_{trunc}-structures of the Λ-modules under scrutiny; see [**13**, Section 7] for illustration.

More concretely, our second objective here is to exhibit the exceptional behavior relative to tilting which is specific to truncated path algebras. In the process, we will detect a homological kinship between truncated and hereditary algebras, which surfaces only after the first round of strong tilting.

Main results of Sections 4-8. We provide capsule versions of these results, formulated in somewhat rough terms. Throughout these sections, Λ *stands for a truncated path algebra*. The starting point is provided by the following facts (briefly reviewed in Section 3.A). The category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is always contravariantly finite in Λ-mod [**11**]. The corresponding basic strong tilting module ${}_\Lambda T$ is well understood, as are the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the simple left Λ-modules, which form the basic building blocks of the Λ-modules of finite projective dimension. This yields a concrete grip on the strongly tilted algebra $\widetilde{\Lambda} = \mathrm{End}_\Lambda(T)^{\mathrm{op}} = K\widetilde{Q}/\widetilde{I}$. In particular, there is a natural one-to-one correspondence connecting the vertices $e_1, \dots, e_n$ of Q to the vertices $\widetilde{e}_1, \dots, \widetilde{e}_n$ of $\widetilde{Q}$ (see the prelude to 4.A); *our indexing of the $\widetilde{e}_i$ in the theorems below reflects this bijection*. We identify the vertices of Q and $\widetilde{Q}$ with full sequences of primitive idempotents of Λ and $\widetilde{\Lambda}$, respectively, and denote by J and $\widetilde{J}$ the respective Jacobson radicals. The following attributes "precyclic", "postcyclic" and "critical" pertaining to vertices of Q indicate their placement relative to oriented cycles: A vertex e_i is *precyclic* if there is a directed path in Q which starts in e_i and ends on an oriented cycle; *postcyclic* is dual, and *critical* means "both pre- and postcyclic".

As was further shown in [**11**], the tilting bimodule ${}_\Lambda T_{\widetilde{\Lambda}}$ is strong on both sides if and only if all precyclic vertices of Q are critical. But the followup question that imposes itself, namely whether $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ is still contravariantly finite in mod-$\widetilde{\Lambda}$ when Q fails to be "cycle-symmetric" in this sense (in other words, the problem of whether mod-$\widetilde{\Lambda}$ always has its own strong tilting module) was left open. In the present paper, we answer it in the positive, based on the announced structural analysis of the right $\widetilde{\Lambda}$-modules of finite projective dimension.

Theorem A. (See Proposition 9 and Corollary 15.) *Let $\widetilde{S}_i$ be the simple right $\widetilde{\Lambda}$-module $\widetilde{e}_i\widetilde{\Lambda}/\widetilde{e}_i\widetilde{J}$. Then* $\mathrm{p\,dim}\,\widetilde{S}_i < \infty$ *precisely when the corresponding vertex e_i of Q is non-critical.*

For comparison: Clearly, a simple left Λ-module $\Lambda e_i/Je_i$ has finite projective dimension in Λ-mod if and only if e_i is non-precyclic. Theorem A smooths the road to a description of arbitrary objects in $\mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$. Here is a preview of the initial step.

Theorem B. (See Theorem 13 for the announced intrinsic characterization of the $\widetilde{\Lambda}$-modules of finite projective dimension, in terms of their "critical cores".)
If $\widetilde{\mu} = \sum_{e_i \text{ critical}} \widetilde{e}_i$, *the following statements are equivalent for an arbitrary right $\widetilde{\Lambda}$-module $\widetilde{M}$ in* Mod-$\widetilde{\Lambda}$*:*

- $\mathrm{p\,dim}\,\widetilde{M} < \infty$.
- *The right $\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$-module $\widetilde{M}\widetilde{\mu}$ is projective over the corner algebra $\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$.*

The homological kinship between truncated and hereditary algebras becomes more apparent after the first round of strong tilting in that typically the radical of a projective right $\widetilde{\Lambda}$-module sports an abundance of projective submodules (see Corollary 16 and the example in 3.B).

The structural understanding of the $\widetilde{\Lambda}$-modules of finite projective dimension enables us to confirm passage of contravariant finiteness from $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ to $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$.

Theorem C. (See Theorem 19 ff.) *The category* $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ *is contravariantly finite in* mod-$\widetilde{\Lambda}$. *Moreover, the minimal* $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$*-approximations of the simple right* $\widetilde{\Lambda}$*-modules are both theoretically understood and computationally accessible from* $\widetilde{Q}$ *and* $\widetilde{I}$*, as is the corresponding basic strong tilting module* $\widetilde{T} \in \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$.

Now set $\widetilde{\widetilde{\Lambda}} = \text{End}_{\widetilde{\Lambda}}(\widetilde{T})$, where $\widetilde{T}$ is as in Theorem C, and consider the tilting bimodule ${}_{\widetilde{\widetilde{\Lambda}}}\widetilde{T}_{\widetilde{\Lambda}}$. The obvious next question, namely whether $\widetilde{T}$ is strong on both sides, has an affirmative answer as well; this confirms the symmetrizing effect of strong tilting.

Theorem D. Iterated strong tilting. (See Theorem 21.) *With the notation introduced above, the tilting bimodule* ${}_{\widetilde{\widetilde{\Lambda}}}\widetilde{T}_{\widetilde{\Lambda}}$ *is strong on both sides, whence the functors* $\text{Hom}_{\widetilde{\Lambda}}(-, \widetilde{T})$ *and* $\text{Hom}_{\widetilde{\widetilde{\Lambda}}}(-, \widetilde{T})$ *restrict to inverse dualities* $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda}) \longleftrightarrow \mathcal{P}^{<\infty}(\widetilde{\widetilde{\Lambda}}\text{-mod})$.

In particular, another round of strong tilting applied to $\widetilde{\widetilde{\Lambda}}$ *yields* $\text{End}_{\widetilde{\widetilde{\Lambda}}}(\widetilde{T})^{\text{op}} \cong \widetilde{\Lambda}$. *Thus, following the initial step* $\Lambda \rightsquigarrow \widetilde{\Lambda}$*, iterated strong tilting of* Λ *is periodic with period* 2.

Overview. Section 2. Tilting, contravariant finiteness, and dualities: **2.A**. Background; **2.B**. Dualities of subcategories of $\mathcal{P}^{<\infty}$-categories for arbitrary Λ; **2.C**. Characterizations of strong tilting modules and finitistic dimensions. **Section 3.** Setting the stage for truncated path algebras Λ: **3.A**. The intrinsic homology of Λ-Mod; **3.B**. Reference example. **Section 4.** The strongly tilted category Mod-$\widetilde{\Lambda}$: **4.A**. The endofunctors ∇ and Δ, and the critical core; **4.B**. The simples in $\mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$. **Section 5.** Auxiliaries. **Section 6.** Structure of the projective $\widetilde{\Lambda}$-modules. **Section 7.** Contravariant finiteness of $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$. **Section 8.** Iterated strong tilting.

Acknowledgement: The author cordially thanks Manuel Saorín for his extensive and very helpful comments on this work.

2. Tilting, contravariant finiteness, and dualities

Throughout this section, Λ *will denote an arbitrary finite dimensional algebra over a field* K. We fix a full sequence $e_1, \dots, e_n$ of primitive idempotents of Λ. By J we denote the Jacobson radical of Λ and by S_i the simple module $\Lambda e_i / J e_i$. Given an object M in the category Λ-mod of finitely generated left Λ-modules, an element $x \in M$ is called a *top element* in case $x \notin JM$ and there exists $i \le n$ such that $x = e_i x$; in this situation, we say that x is *normed by* e_i. A *full sequence of top elements of* M consists of top elements $x_1, \dots, x_t$ which are linearly independent modulo JM and generate M. The subcategory $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ of Λ-mod was introduced in Section 1; its "big" companion, $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$, consists of *all* left Λ-modules of finite projective dimension.

2.A. Background on strong tilting modules and contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$.

Following Miyashita [**24**], we call a left Λ-module T a *tilting module* in case (i) T belongs to $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$, (ii) $\text{Ext}^i(T, T) = 0$ for $i \ge 1$, and (iii) there exists an exact sequence $0 \to {}_\Lambda\Lambda \to T_0 \to \cdots \to T_m \to 0$ with $T_j \in \text{add}(T)$. (Recall that $\text{add}(T)$ consists of the direct summands of finite powers of T.) Any sequence as in (iii) is referred to as a *finite coresolution* of ${}_\Lambda\Lambda$ by objects in $\text{add}(T)$; we retain this terminology when ${}_\Lambda\Lambda$ is replaced by any left Λ-module. Call T *basic* if T has no indecomposable direct summands of multiplicity ≥ 2.

Given a tilting module ${}_\Lambda T$ with $\widetilde{\Lambda} = \text{End}_\Lambda(T)^{\text{op}}$, the properties of the bimodule ${}_\Lambda T_{\widetilde{\Lambda}}$ are known to be left-right symmetric in the following sense: $T_{\widetilde{\Lambda}}$ is in turn a tilting module, and Λ is canonically isomorphic to $\text{End}_{\widetilde{\Lambda}}(T)$, i.e., the K-algebra map $\Lambda \to \text{End}_{\widetilde{\Lambda}}(T)$, $\lambda \mapsto (x \mapsto \lambda x)$, is an isomorphism (see [**24**, Proposition 1.4]); in other words, ${}_\Lambda T_{\widetilde{\Lambda}}$ is a *balanced bimodule*.

According to Auslander-Reiten [**4**], a tilting module ${}_\Lambda T$ is *strong* if every object in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ has a finite coresolution by objects in $\mathrm{add}(T)$; due to Proposition 4 in Section 2.C, this definition is equivalent to the description of a strong tilting module given in the introduction (namely, relative Ext-injectivity of ${}_\Lambda T$ in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$). It was shown in [**4**] that Λ-mod never contains more than one basic strong tilting module and that existence is equivalent to contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ (abbreviated to $\mathcal{P}^{<\infty}$ for the moment) in Λ-mod. This means that every object M in Λ-mod has a (right) $\mathcal{P}^{<\infty}$-*approximation* in the following sense: There is a homomorphism $\phi : \mathcal{A} \to M$ such that $\mathcal{A} \in \mathcal{P}^{<\infty}$ and every map in $\mathrm{Hom}_\Lambda(\mathcal{P}^{<\infty}, M)$ factors through ϕ. Whenever such an approximation of M exists, there is a minimal one, $\mathcal{A}(M)$, which is uniquely determined by M up to isomorphism. As was established in extenso, contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ guarantees a particularly transparent homological behavior of Λ-Mod; see, e.g., [**5**, **4**, **20**, and **18**]; for classes of examples, see [**19**, **9**, **11**].

We state the existence/uniqueness result for a strong tilting module for easy reference.

Reference Theorem I. [**4**, Proposition 6.3; use Theorem 5.5, loc.cit., to correct the statement of 6.3] *The category* Λ-mod *has a strong tilting module if and only if* $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ *is contravariantly finite in* Λ-mod. *In the positive case, the basic strong tilting module is the direct sum of the distinct indecomposable objects* $C \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ *which satisfy* $\mathrm{Ext}^i_\Lambda(\mathcal{P}^{<\infty}(\Lambda\text{-mod}), C) = 0$ *for* $i \geq 1$ *("C is relatively* Ext-*injective in* $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$*").*

The upcoming supplement to Theorem I should be well-known. But since we could not locate a reference, we include the short argument.

Supplement II. *Suppose that* $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ *is contravariantly finite,* $T \in \Lambda\text{-mod}$ *a strong tilting module, and* $\mathcal{A}(E)$ *the minimal* $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$*-approximation of the basic injective cogenerator* E *of* Λ-mod*. Then*

$$\mathrm{add}(T) = \mathrm{add}(\mathcal{A}(E)).$$

Proof. Since, by Theorem I, $\mathrm{add}(T)$ consists of the objects in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ which are relatively Ext-injective in this category, the inclusion "$\supseteq$" follows from [**11**, Lemma 5.1].

For the reverse inclusion, suppose that U belongs to $\mathrm{add}(T)$, meaning that U is relatively Ext-injective in the category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$. To verify that U belongs to $\mathrm{add}(\mathcal{A}(E))$, let $\phi : \mathcal{A} = \mathcal{A}\big(E(U)\big) \to E(U)$ be a minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of the injective envelope $E(U)$ in Λ-mod. From $E(U) \in \mathrm{add}(E)$, we obtain $\mathcal{A} \in \mathrm{add}(\mathcal{A}(E))$. Moreover, the inclusion map $\iota : U \hookrightarrow E(U)$ factors through ϕ, say $\iota = \phi \circ \psi$ for a suitable monomorphism $\psi \in \mathrm{Hom}_A(U, \mathcal{A})$. Since $\mathcal{A}/\psi(U)$ has finite projective dimension, ψ splits due to relative Ext-injectivity of U in $\mathcal{P}^{<\infty}(A\text{-mod})$. This shows $U \in \mathrm{add}(\mathcal{A}(E))$ to be as required. $\square$

2.B. Dualities of subcategories of $\mathcal{P}^{<\infty}$-categories for arbitrary Λ.

Let Λ be a finite dimensional algebra, $T \in \Lambda\text{-mod}$ an arbitrary tilting module and $\widetilde{\Lambda} = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$. Denote by ${}^\perp({}_\Lambda T)$ the left perpendicular category of ${}_\Lambda T$, namely the full subcategory of Λ-mod consisting of the Λ-modules X with $\mathrm{Ext}^i_\Lambda(X, T) = 0$ for all $i \geq 1$; analogously, ${}^\perp(T_{\widetilde{\Lambda}})$ stands for the left perpendicular category of $T_{\widetilde{\Lambda}}$ in mod-$\widetilde{\Lambda}$. The upcoming duality is readily deduced from Miyashita's Theorem 3.5 in [**24**].

Reference Theorem III: Miyashita's duality. *In the above notation, the pertinent restrictions of the contravariant functors* $\mathrm{Hom}_\Lambda(-, T)$ *and* $\mathrm{Hom}_{\widetilde{\Lambda}}(-, T)$ *are inverse dualities*

$$\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \cap {}^\perp({}_\Lambda T) \;\longleftrightarrow\; \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda}) \cap {}^\perp(T_{\widetilde{\Lambda}}).$$

By the definition of ${}^{\perp}({}_{\Lambda}T)$ and ${}^{\perp}(T_{\widetilde{\Lambda}})$, the inverse dualities secured by this theorem are *strictly exact*, meaning that they take those sequences $0 \to X_1 \to X_2 \to X_3 \to 0$ (in the specified subcategories of Λ-mod, resp., mod-Λ') which are exact in the ambient full module categories to sequences of the same ilk.

Clearly, the inclusion "$\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \subseteq {}^{\perp}({}_{\Lambda}T)$" is tantamount to relative Ext-injectivity of the tilting module ${}_{\Lambda}T$ in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$. From Theorems I and III we thus infer that every tilting bimodule ${}_{\Lambda}T_{\widetilde{\Lambda}}$ which is strong on both sides yields a duality $\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \longleftrightarrow \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ as exhibited in the introduction. The converse provided by Corollary 2 below places additional emphasis on the pivotal role played by strong tilting modules with regard to dualities connecting categories of modules of finite projective dimension.

We supplement Miyashita's theorem by showing that strictly exact dualities between resolving subcategories of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{P}^{<\infty}(\text{mod-}\Lambda')$ are always afforded by tilting bimodules according to the blueprint of Theorem III. Recall that a subcategory $\mathcal{C}$ of Λ-mod is *resolving* if $\mathcal{C}$ is a full subcategory which contains the projectives and is closed under extensions and kernels of surjective homomorphisms in Λ-mod. Clearly, $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is a resolving subcategory of Λ-mod, as is the intersection $\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \cap {}^{\perp}M$ with the left perpendicular category of any left Λ-module M.

Theorem 1. *Let Λ and Λ' be finite dimensional algebras, and let $\mathcal{C} \subseteq \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{C}' \subseteq \mathcal{P}^{<\infty}(\text{mod-}\Lambda')$ be resolving subcategories of* Λ-mod *and* mod-Λ'*, respectively.*

Suppose $\mathcal{C}$ is dual to $\mathcal{C}'$ by way of strictly exact contravariant additive functors

$$F : \mathcal{C} \longrightarrow \mathcal{C}' \quad \text{and} \quad F' : \mathcal{C}' \longrightarrow \mathcal{C}$$

such that $F' \circ F$ and $F \circ F'$ are isomorphic to the pertinent identity functors.

Then there exists a tilting bimodule ${}_{\Lambda}T_{\Lambda'}$ such that:

- **(a)** *$F \cong \text{Hom}_{\Lambda}(-,T)\,|_{\mathcal{C}}$ and $F' \cong \text{Hom}_{\Lambda'}(-,T)\,|_{\mathcal{C}'}$;*
- **(b)** *$\mathcal{C} = \mathcal{P}^{<\infty}(\Lambda\text{-mod}) \cap {}^{\perp}({}_{\Lambda}T)$, and $\mathcal{C}' = \mathcal{P}^{<\infty}(\text{mod-}\Lambda') \cap {}^{\perp}(T_{\Lambda'})$;*
- **(c)** *The modules in $\mathcal{C}$ are precisely those left Λ-modules which have finite coresolutions by objects in $\text{add}({}_{\Lambda}T)$, and the modules in $\mathcal{C}'$ are those right Λ'-modules which have finite coresolutions by objects in $\text{add}(T_{\Lambda'})$.*

In particular, F and F' are "minimal dualities", in the sense that they do not induce any duality $\mathcal{C}_0 \longleftrightarrow \mathcal{C}_0'$ between proper resolving subcategories $\mathcal{C}_0$ of $\mathcal{C}$ and $\mathcal{C}_0'$ of $\mathcal{C}'$.

Addendum: Provided that $\mathcal{C}$ and $\mathcal{C}'$ are closed also under cokernels of injective morphisms in the ambient module categories, arbitrary inverse dualities $F : \mathcal{C} \longrightarrow \mathcal{C}'$ and $F' : \mathcal{C}' \longrightarrow \mathcal{C}$ are strictly exact. More strongly: Strict exactness of F and F' follows if $\mathcal{C}$ is closed under quotients C_1/C_2 with $C_i \in \mathcal{C}$ such that C_1/C_2 is isomorphic to a left ideal of Λ, and $\mathcal{C}'$ is closed under quotients C_1'/C_2' with $C_i' \in \mathcal{C}'$ such that C_1'/C_2' is isomorphic to a right ideal of Λ'.

Before proving Theorem 1, we state two immediate consequences. For the first, note that the categories $\mathcal{C} = \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{C}' = \mathcal{P}^{<\infty}(\text{mod-}\Lambda')$ satisfy the conditions spelled out in the addendum to Theorem 1.

Corollary 2. *Let Λ, Λ' be finite dimensional algebras. Then any pair of inverse dualities (F, F')*

$$\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \longleftrightarrow \mathcal{P}^{<\infty}(\text{mod-}\Lambda')$$

is isomorphic to a pair of functors of the form $\big(\text{Hom}_{\Lambda}(-,T), \text{Hom}_{\Lambda'}(-,T)\big)$ for some tilting bimodule ${}_{\Lambda}T_{\Lambda'}$ which is strong on both sides. In particular, potential existence of such a duality is restricted to algebras Λ and Λ' such that $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{P}^{<\infty}(\text{mod-}\Lambda')$ are contravariantly finite in Λ-mod *and* mod-Λ'*, respectively.*

Corollary 3. *Again, let Λ, Λ' be finite dimensional algebras. If there are resolving subcategories of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{P}^{<\infty}(\text{mod-}\Lambda')$, respectively, which are dual via strictly exact functors, then Λ and Λ' are derived equivalent.*

Proof of Theorem 1. In light of the fact that ${}_\Lambda\Lambda \in \mathcal{C}$ and $\Lambda'_{\Lambda'} \in \mathcal{C}'$, a theorem of Morita (see, e.g. [**1**, Theorem 23.5]) ensures that the functors F and F' are isomorphic to Hom-functors of the form $\text{Hom}_\Lambda(-,T)$ and $\text{Hom}_{\Lambda'}(-,T)$ for a balanced bimodule ${}_\Lambda T_{\Lambda'}$; namely, $T_{\Lambda'} = F({}_\Lambda\Lambda) \in \mathcal{C}'$ and ${}_\Lambda T = F'(\Lambda'_{\Lambda'}) \in \mathcal{C}$. Thus the restrictions $\text{Hom}_\Lambda(-,T)\,|_{\mathcal{C}}$ and $\text{Hom}_{\Lambda'}(-,T)\,|_{\mathcal{C}'}$ are mutually inverse dualities $\mathcal{C} \longleftrightarrow \mathcal{C}'$. In particular, $\text{Hom}_\Lambda(\mathcal{C},T) \subseteq \mathcal{C}'$ and $\text{Hom}_{\Lambda'}(\mathcal{C}',T) \subseteq \mathcal{C}$.

First we check the inclusion $\mathcal{C} \subseteq {}^\perp({}_\Lambda T)$, the inclusion $\mathcal{C}' \subseteq {}^\perp(T_{\Lambda'})$ being symmetric. That $\text{Ext}^1_\Lambda(\mathcal{C},T) = 0$ is immediate from strict exactness of F'. Since $\mathcal{C}$ is closed under syzygies, we infer that $\text{Ext}^i_\Lambda(\mathcal{C},T) = 0$ for all $i \ge 1$ which means $\mathcal{C} \subseteq {}^\perp({}_\Lambda T)$ as claimed.

We now prove (a) and (c) in tandem. By the preceding paragraph, $\text{Ext}^i_\Lambda(T,T) = 0$ for $i \ge 1$. Hence, to confirm the tilting status of the module ${}_\Lambda T$, it only remains to be shown that the left regular module ${}_\Lambda\Lambda$ has a finite $\text{add}({}_\Lambda T)$-coresolution, a requirement which is subsumed by (c). The latter condition will be verified next. For this purpose, let X be any object in $\mathcal{C}$ and

$$(\dagger) \qquad 0 \to P'_d \xrightarrow{f'_d} \cdots \xrightarrow{f'_2} P'_1 \xrightarrow{f'_1} P'_0 \xrightarrow{f'_0} F(X) \to 0$$

a projective resolution of $F(X)$ in mod-Λ'. Since $F(X)$ and P'_0 belong to $\mathcal{C}'$, so does the kernel of f'_0, and thus the exact sequence $0 \to \text{Ker}(f'_0) \to P'_0 \to F(X) \to 0$ belongs to $\mathcal{C}'$. An obvious induction now shows that all of the short exact sequences $0 \to \text{Ker}(f'_j) \to P'_{j-1} \to \text{Ker}(f'_{j-1}) \to 0$ belong to $\mathcal{C}'$, and therefore strict exactness of F' yields exactness in Λ-mod of the sequence $0 \to X \to F'(P'_0) \to \cdots \to F'(P'_d) \to 0$ induced from $(\dagger)$. Given that the right Λ'-modules P'_j belong to $\text{add}(\Lambda'_{\Lambda'})$, the left Λ-modules $F'(P'_j)$ belong to $\text{add}(F'(\Lambda'_{\Lambda'}) = \text{add}({}_\Lambda T)$ as required. That ${}_\Lambda T_{\Lambda'}$ is a tilting bimodule now follows from general tilting theory, due to balancedness of the bimodule T. Moreover, by symmetry, every object in $\mathcal{C}'$ has a finite $\text{add}(T_{\Lambda'})$-coresolution. That every left Λ-module (resp., right Λ'-module) with a coresolution as described belongs to $\mathcal{C}$ (resp., to $\mathcal{C}'$) is immediate from the fact that $\mathcal{C}$ and $\mathcal{C}'$ are resolving. Thus (a) and (c) are established.

To verify (b), we only need to address the inclusion $\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \cap {}^\perp({}_\Lambda T) \subseteq \mathcal{C}$; indeed, the reverse inclusion was already shown above, and the inclusion $\mathcal{P}^{<\infty}(\text{mod-}\Lambda') \cap {}^\perp(T_{\Lambda'}) \subseteq \mathcal{C}'$ is symmetric. So let X be an object of $\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \cap {}^\perp({}_\Lambda T)$. By Theorem III, the functors $\text{Hom}_\Lambda(-,T)$ and $\text{Hom}_{\Lambda'}(-,T)$ restrict to strictly exact inverse dualities G and G' between $\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \cap {}^\perp({}_\Lambda T)$ and $\mathcal{P}^{<\infty}(\text{mod-}\Lambda') \cap {}^\perp(T_{\Lambda'})$ (in fact, $(G,G') = (F,F')$, as will be seen a posteriori). Condition (c), applied to the pair (G,G'), yields an exact sequence $0 \to X \to T^{(1)} \xrightarrow{f_1} \cdots \xrightarrow{f_{m-1}} T^{(m)} \to 0$ with $T^{(j)} \in \text{add}({}_\Lambda T) \subseteq \mathcal{C}$. Once again using the fact that $\mathcal{C}$ is resolving, we infer $\text{Ker}(f_{m-1}) \in \mathcal{C}$, whence a downward induction yields $X \cong \text{Ker}(f_1) \in \mathcal{C}$ as required.

Finally, we justify the addendum. Let $F : \mathcal{C} \to \mathcal{C}'$ and $F' : \mathcal{C}' \to \mathcal{C}$ be arbitrary inverse dualities, and suppose that the weaker versions of the mentioned closure conditions for $\mathcal{C}$ and $\mathcal{C}'$ are satisfied. As in the proof of "(1) $\Longrightarrow$ (2)", F and F' are then restrictions of contravariant Hom-functors induced by a balanced bimodule ${}_\Lambda T_{\Lambda'}$ with ${}_\Lambda T = F(\Lambda'_{\Lambda'})$ and $T_{\Lambda'} = F'({}_\Lambda\Lambda)$. Clearly, strict exactness of F and F' amounts to $\text{Ext}^1_\Lambda(\mathcal{C},T) = 0$ and $\text{Ext}^1_{\Lambda'}(\mathcal{C}',T) = 0$.

To show $\text{Ext}^1_\Lambda(\mathcal{C},T) = 0$, let **(1)** $0 \to T \xrightarrow{f} C_1 \xrightarrow{g} C_2 \to 0$ be a sequence in $\mathcal{C}$ which is exact in Λ-mod; note that $C_2 \in \mathcal{C}$ implies $C_1 \in \mathcal{C}$, since $\mathcal{C}$ is closed under extensions in Λ-mod. Then $0 \to \text{Hom}_\Lambda(C_2,T) \to \text{Hom}_\Lambda(C_1,T) \to \text{Hom}_\Lambda(T,T) \cong \Lambda'_{\Lambda'}$ is exact in mod-Λ'; in particular, $\text{Hom}_\Lambda(g,T)$ is an injective homomorphism in $\mathcal{C}'$ and $L' := \text{Im}\big(\text{Hom}_\Lambda(f,T)\big)$ is isomorphic to a right ideal of Λ'. Since $\text{Ker}\big(\text{Hom}_\Lambda(f,T)\big) \cong \text{Hom}_\Lambda(C_2,T)$ belongs to $\mathcal{C}'$, our hypothesis on $\mathcal{C}'$ guarantees that

$L' \in \mathcal{C}'$. By construction, the sequence **(2)** $0 \to \mathrm{Hom}_\Lambda(C_2, T) \to \mathrm{Hom}_\Lambda(C_1, T) \to L' \to 0$ is exact in mod-Λ', whence it translates back into an exact sequence $0 \to \mathrm{Hom}_{\Lambda'}(L', T) \to C_1 \xrightarrow{g} C_2$ in Λ-mod. Hence both T and $\mathrm{Hom}_{\Lambda'}(L', T)$ are kernels of g in Λ-mod. This yields $T \cong \mathrm{Hom}_{\Lambda'}(L', T)$ in Λ-mod, and consequently $\mathrm{Hom}_\Lambda(T, T) \cong L'$ in mod-Λ' because $L' \in \mathcal{C}'$. In other words, $\Lambda'_{\Lambda'} \cong L'_{\Lambda'}$. Therefore, the sequence **(2)** splits, whence so does **(1)**. Thus, indeed, $\mathrm{Ext}^1_\Lambda(C, T) = 0$ for all $C \in \mathcal{C}$. That $\mathrm{Ext}^1_{\Lambda'}(\mathcal{C}', T) = 0$ follows from the symmetry of the setup. $\square$

We do not have an example of inverse dualites F, F' between resolving subcategories $\mathcal{C} \subseteq \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{C}' \subseteq \mathcal{P}^{<\infty}(\text{mod-}\Lambda')$ for which F or F' fails to be strictly exact. In general, relative epimorphisms in a resolving subcategory $\mathcal{C}$ of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ need not be surjections though. For instance, take Λ to be $KQ/\langle \beta\alpha \rangle$, where Q is the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, and let $\mathcal{C}$ be the full subcategory of $\mathcal{P}^{<\infty}(\Lambda\text{-mod}) = \Lambda\text{-mod}$ consisting of the projective modules. Then the map $f \in \mathrm{Hom}_\Lambda(\Lambda e_2, \Lambda e_1)$ with $f(e_2) = \alpha$ is a relative epimorphism in $\mathcal{C}$. Nonetheless, every duality between $\mathcal{C}$ and some resolving $\mathcal{C}' \subseteq \text{mod-}\Lambda'$ is easily seen to be strictly exact in this case.

Combining the above with results of Auslander-Reiten. The dual of [**4**, Theorem 5.5(b)] says that a resolving subcategory $\mathcal{C}$ of Λ-mod is contravariantly finite if and only if there exists a tilting module $T \in \mathcal{P}^{<\infty}$ Λ-mod such that $\mathcal{C}$ consists of the left Λ-modules with finite $\mathrm{add}(T)$-coresolutions. Combining this criterion with Theorem 1, one deduces: Whenever $T \in \Lambda$-mod is a tilting module, $\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \cap {}^\perp({}_\Lambda T)$ is contravariantly finite in Λ-mod. Moreover, a resolving subcategory $\mathcal{C}$ of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ admits a strictly exact duality with some resolving subcategory of a $\mathcal{P}^{<\infty}$-category of right modules (over some Λ') if and only if $\mathcal{C}$ is contravariantly finite.

2.C. Characterizations of strong tilting modules and finitistic dimensions.

Still, Λ stands for an arbitrary finite dimensional algebra. Let $T \in \Lambda$-mod be a tilting module. The purpose of this subsection is to provide alternative characterizations of strongness and to connect the finitistic dimensions of Λ and $\mathrm{End}_\Lambda(T)$ for strong T.

For $Y \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$, we define the *relative* Ext-*injective dimension* of Y as

$$\mathrm{rel\,idim}\, Y := \inf\{r \in \mathbb{N}_0 \mid \mathrm{Ext}^j_\Lambda(-, Y)|_{\mathcal{P}^{<\infty}(\Lambda\text{-mod})} = 0 \text{ for all } j \geq r+1\},$$

with the understanding that $\mathrm{rel\,idim}\, Y = \infty$ if the specified set is empty. By definition, $\mathrm{rel\,idim}\, Y = 0$ precisely when Y is relatively Ext-injective in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$. Moreover, to say that $T \in \Lambda$-mod is a *cogenerator for* $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ means that every object in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ embeds into a power of T. On the side, we point out that condition (2) of the upcoming proposition reveals contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ to be a weakened Gorenstein condition for Λ.

Proposition 4. *Let Λ be any finite dimensional algebra. For $T \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$, the following conditions are equivalent:*

(1) *T is a strong tilting module in the sense of Auslander-Reiten, i.e., T is a tilting module and every object in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ has a finite* $\mathrm{add}(T)$*-coresolution.*

(2) *T is a relatively* Ext*-injective cogenerator in the category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$, and all objects in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ have finite relative* Ext*-injective dimension.*

(3) *T is a tilting module which is relatively* Ext*-injective in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$.*

In case conditions **(1)** – **(3)** *are satisfied,*

$$\sup\{\mathrm{rel\,idim}\, Y \mid Y \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})\} = \mathrm{fin\,dim}\,\Lambda = \mathrm{Fin\,dim}\,\Lambda,$$

where $\mathrm{fin\,dim}\,\Lambda$ *and* $\mathrm{Fin\,dim}\,\Lambda$ *denote the left little and big finitistic dimensions of* Λ.

Proof. "(1) $\Longrightarrow$ (2)". That (1) forces T to be a relatively Ext-injective cogenerator for $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is immediate from the definiton of strongness and Reference Theorem I. To ascertain finiteness of the relative Ext-injective dimension of any object X in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$, let $0 \to X \to T^{(1)} \to \cdots \to T^{(m)} \to 0$ be an exact sequence with $T^{(j)} \in \operatorname{add}(T)$. In light of $\operatorname{rel\,idim} T^{(j)} = 0$ for all j, we find $\operatorname{rel\,idim} X \le m - 1$.

"(2) $\Longrightarrow$ (3)". Assume (2). To see that T is a tilting module, we only need to check that ${}_\Lambda\Lambda$ has a finite $\operatorname{add}(T)$-coresolution. Since T is a cogenerator for $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$, every object Y of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ has a coresolution (a priori infinite) by objects in $\operatorname{add}(T)$. Finiteness of the relative Ext-injective dimension of Y shows that one of the cokernels along such a coresolution has relative Ext-injective dimension 0, say the r-th, C_r. Since C_r embeds into an object in $\operatorname{add}(T) \subseteq \mathcal{P}^{<\infty}(\Lambda\text{-mod})$, we deduce $C_r \in \operatorname{add}(T)$, which yields a finite $\operatorname{add}(T)$-coresolution of Y as required.

"(3) $\Longrightarrow$ (1)". Assume (3), and let $\widetilde{\Lambda} = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. Then $\mathcal{C} := \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contained in ${}^\perp({}_\Lambda T)$, whence Theorem III entails that $\mathcal{C}$ is dual to $\mathcal{C}' := \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda}) \cap {}^\perp(T_{\widetilde{\Lambda}})$ by way of the restricted functors $\operatorname{Hom}_\Lambda(-,T)|_{\mathcal{C}}$ and $\operatorname{Hom}_{\widetilde{\Lambda}}(-,T)|_{\mathcal{C}'}$, both of which are strictly exact in this situation. Consequently, condition (1) follows from Theorem 1.

This proves the equivalences.

Now suppose that (1)-(3) are satisfied. Then $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite. In particular, $\operatorname{fin\,dim} \Lambda = \operatorname{Fin\,dim} \Lambda < \infty$; see [**18**]. Observe that, for any $X \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$, the projective dimension of X in Λ-mod equals the relative projective dimension in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$, meaning that this dimension is the smallest nonnegative integer k with the property that the restricted functor $\operatorname{Ext}^{k+1}_\Lambda(X,-)|_{\mathcal{P}^{<\infty}(\Lambda\text{-mod})}$ vanishes. Since the relative injective dimensions of the modules in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ are finite by (2), we infer

$$\begin{gathered}\operatorname{fin\,dim} \Lambda = \inf\{k \ge 0 \mid \operatorname{Ext}^{k+1}_\Lambda(X,-)|_{\mathcal{P}^{<\infty}(\Lambda\text{-mod})} = 0 \text{ for all } X \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})\} = \\ \inf\{k \ge 0 \mid \operatorname{Ext}^{k+1}_\Lambda(-,-), \text{ restricted to } \mathcal{P}^{<\infty}(\Lambda\text{-mod}) \times \mathcal{P}^{<\infty}(\Lambda\text{-mod}), \text{ equals } 0\} = \\ \sup\{\operatorname{rel\,idim} Y \mid Y \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})\}. \qquad \square\end{gathered}$$

Combining Corollary 2 and Proposition 4, we derive:

Corollary 5. *Suppose that $T \in \Lambda\text{-mod}$ is a strong tilting module, and let $\widetilde{\Lambda} = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. Then the coinciding left finitistic dimensions $\operatorname{fin\,dim} \Lambda$ and $\operatorname{Fin\,dim} \Lambda$ are smaller than or equal to the right little finitistic dimension of $\widetilde{\Lambda}$. In particular, the left finitistic dimensions of Λ coincide with the right finitistic dimensions of $\widetilde{\Lambda}$ whenever the tilting bimodule ${}_\Lambda T_{\widetilde{\Lambda}}$ is strong on both sides.* (For the final assertion, see also [**11**, Section 8].)

3. Setting the stage for truncated path algebras

In the sequel, Λ will denote a truncated path algebra, say

$$\Lambda = KQ/\langle \text{all paths of length } L+1 \rangle$$

for some quiver Q and some $L \ge 1$. We identify the vertices of Q with a full sequence of primitive idempotents $e_1, \dots, e_n$ of Λ. Recall that a vertex e of Q (alias a primitive idempotent of Λ) is *precyclic* in case there is an oriented path in Q which starts in e and ends on an oriented cycle; the vertex e is *postcyclic* if the dual is true, and *critical* if it is both pre- and postcyclic.

3.A. The intrinsic homology of Λ-Mod.

This homology is understood. We excerpt what will be needed in the sequel.

Set $\varepsilon = \sum_{\text{non-precyclic}} e_i$, and note that εM is a Λ-submodule of M for any $M \in \Lambda\text{-Mod}$. The following overview will be used throughout.

Reference Theorem IV. [**11**, Theorems 3.1, 4.1, 4.2, 5.3, and Corollary 7.2]

(1) $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$**-approximations.** *The category* $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ *is contravariantly finite, and, up to isomorphism, the minimal* (*right*) $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$*-approximation* $\mathcal{A}(M)$ *of* $M \in \Lambda\text{-mod}$ *is* $P/\varepsilon \operatorname{Ker}(f)$*, where* $f : P \to M$ *is a projective cover of* M*; in particular,* $\mathcal{A}(S_i) = \Lambda e_i/\varepsilon J e_i$ *for* $i \le n$*. Moreover,*

$$\operatorname{p\,dim} M < \infty \iff M/\varepsilon M \cong \bigoplus_{1 \le i \le n} \mathcal{A}(S_i)^{m_i} \quad \text{for suitable } m_i \ge 0.$$

(2) The basic strong tilting module. *The basic strong tilting module in* Λ-mod *is* $T = \bigoplus_{1 \le i \le n} T_i$*, where* $T_i = \mathcal{A}(S_i)$ *in case* e_i *is precyclic, and* $T_i = \mathcal{A}(E_i)$ *otherwise; here* $E_i = E(S_i)$ *denotes the injective envelope of* S_i *in* Λ-mod*. We provide structural detail on* T_i *in the two cases "*e_i *precyclic" and "*e_i *non-precyclic"* (referring to T_i and S_i as "precyclic", "non-precyclic" or "critical" in case e_i has the pertinent property):

• *Any precyclic* T_i *is a local module* (*namely* $\mathcal{A}(S_i)$) *all of whose simple composition factors are in turn precyclic.*

• *Now suppose that* T_i *is not precyclic. Then the socle of* T_i *contains precisely one non-precyclic simple direct summand, namely* S_i*. Moreover, the factor module* $T_i/\varepsilon T_i$ *is isomorphic to a direct sum of copies of precyclic* T_j*'s.*

Furthermore: The multiplicity of any critical T_j *as a direct summand of* $T_i/\varepsilon T_i$ *coincides with the number of paths of length* L *from* e_j *to* e_i*.*

Finally, every submodule $U \subseteq T_i$ *with* $U \not\subseteq JT_i$ *contains* S_i *in its socle.*

(3) Left-right symmetry. *Let* T *be as in* (2) *and* $\widetilde{\Lambda} = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$*. Then the following conditions are equivalent:*

(i) ${}_\Lambda T_{\widetilde{\Lambda}}$ *is strong also in* mod-$\widetilde{\Lambda}$.

(ii) *All precyclic vertices of* Q *are critical* (*equivalently,* Q *does not have a precyclic source*).

The modules $\mathcal{A}(S_i)$ and T_i of the theorem are determined up to isomorphism by their (tree) graphs. Conversely, these graphs are uniquely determined by the modules they represent. In light of the descriptions of the $\mathcal{A}(S_i)$ and T_i, the graphs may be effortlessly pinned down from the quiver and Loewy length of Λ; for illustrations, consult the upcoming example.

The strongly tilted algebra $\widetilde{\Lambda}$ is in turn a path algebra modulo relations, $\widetilde{\Lambda} = K\widetilde{Q}/\widetilde{I}$; see Observation 10 below. For the construction of $\widetilde{\Lambda}$, see [**17**].

Notational convention: We set $\widetilde{e}_i = \iota_i \circ \pi_i$, where $\pi_i : T \to T_i$ is the canonical projection, and $\iota_i : T_i \hookrightarrow T$ is the corresponding injection. The lineup of the e_i with the T_i as described in Theorem IV(2) thus brings the vertices of $\widetilde{Q}$ into a bijective correspondence with the vertices of Q.

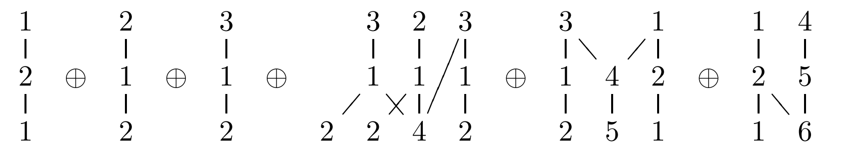
3.B. Reference example.

For our graphing conventions, we point to [**13**, Section 2]. The first stage of exploration of the following truncated algebra Λ was carried out in [**11**, Example 9.2]. It will be supplemented in several installments below. We record the earlier findings.

Let $\Lambda = KQ/\langle \text{all paths of length } 3 \rangle$, where Q is

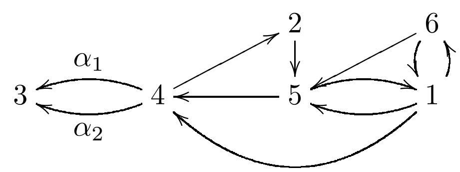
$$\begin{tikzcd}
3 \arrow[r] \arrow[d] & 4 \arrow[r] & 5 \arrow[d] \\
1 \arrow[ur] \arrow[r, bend left] & 2 \arrow[l, bend left] \arrow[r] & 6
\end{tikzcd}$$

Clearly, e_1, e_2 are the critical vertices of Q, while e_3 is pre- but not postcyclic, and e_4, e_5, e_6 are post- but not precyclic. The basic strong tilting module in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is $T = \bigoplus_{i=1}^{6} T_i$, pinned down, up to isomorphism, by the following graphs (cf. Theorem IV):

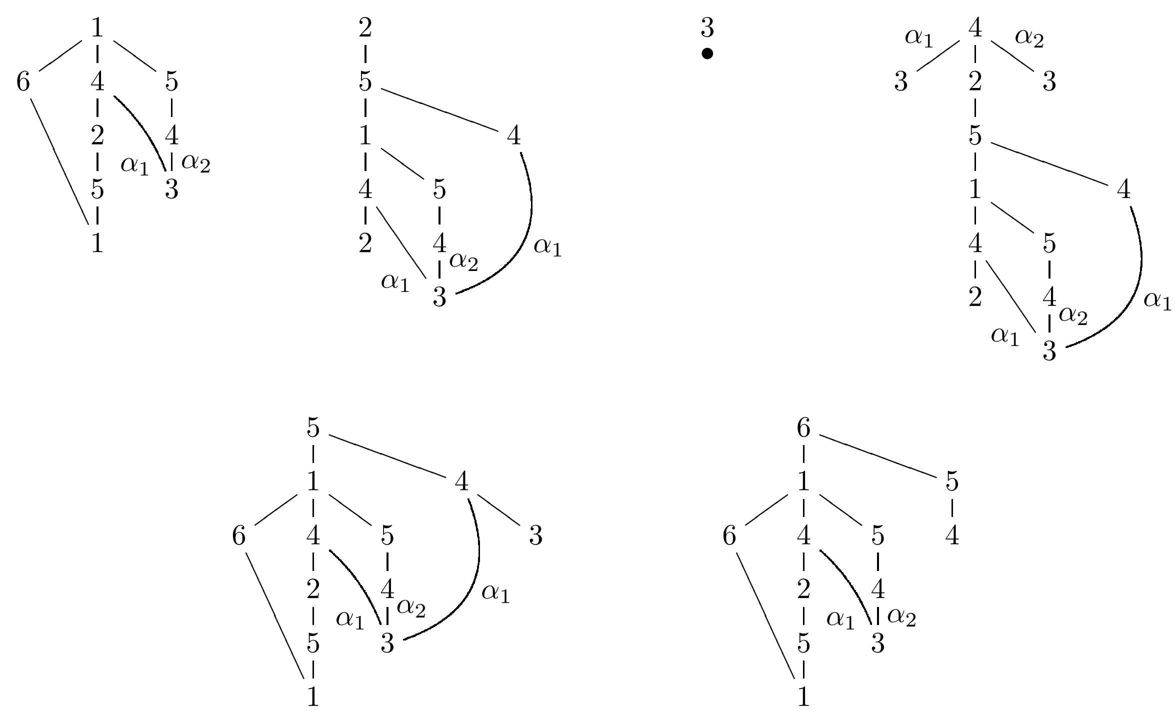


T_1 $\quad T_2$ $\quad T_3$ $\quad T_4 = \mathcal{A}(E_4)$ $\quad T_5 = \mathcal{A}(E_5)$ $\quad T_6 = \mathcal{A}(E_6)$

Then $\widetilde{\Lambda} = \mathrm{End}_\Lambda(T)^{\mathrm{op}} = K\widetilde{Q}/\widetilde{I}$, where the quiver $\widetilde{Q}^{\mathrm{op}}$ of $\mathrm{End}_\Lambda(T) \cong \widetilde{\Lambda}^{\mathrm{op}}$ is



and the indecomposable projective right $\widetilde{\Lambda}$-modules $\widetilde{e}_i\widetilde{\Lambda}$ are:



When combined with the additional information that $\widetilde{I}$ is generated by monomial relations and binomial relations of the form $\widetilde{p} - \widetilde{q}$ for paths $\widetilde{p}$ and $\widetilde{q}$ in $K\widetilde{Q}$, these graphs pin down the $\widetilde{e}_i\widetilde{\Lambda}$ for $i \le 6$ up to isomorphism. Note that in this example the Loewy length of $\widetilde{\Lambda}$ is 7.

From Theorem IV(3) we moreover know that the tilting bimodule ${}_\Lambda T_{\widetilde{\Lambda}}$ fails to be strong in mod-$\widetilde{\Lambda}$, since Q has a precyclic source, an issue that will be picked up in Sections 7, 8. Further examples illustrating the phenomena that may occur in the passage from Λ to $\widetilde{\Lambda}$ will follow in [**17**].

4. The strongly tilted category Mod-$\widetilde{\Lambda}$

Still $\Lambda = KQ/\langle \text{all paths of length } L+1\rangle$ for some $L \geq 1$. We pick up the notation of Section 3.A, simplifying it by writing $\mathcal{A}_i$ for the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation $\mathcal{A}(S_i)$ of S_i. In particular, the basic strong tilting object in Λ-mod is $T = \bigoplus_{1\leq i\leq n} T_i$ as specified in Theorem IV(2), and $\widetilde{\Lambda}$ is the strongly tilted algebra $\mathrm{End}_\Lambda(T)^{\mathrm{op}}$. This makes ${}_\Lambda T_{\widetilde{\Lambda}}$ a tilting bimodule which is strong on the left, but not on the right in general. By $\widetilde{J}$ we denote the Jacobson radical of $\widetilde{\Lambda}$.

Further conventions: Whenever $i, j \in \{1, \dots, n\}$, the maps in $\mathrm{Hom}_\Lambda(T_i, T_j)$ will be viewed as elements of $\mathrm{End}_\Lambda(T)$, and thus as elements of $\widetilde{\Lambda}$, the pertinent embedding $\mathrm{Hom}_\Lambda(T_i, T_j) \hookrightarrow \mathrm{End}_\Lambda(T)$ being $f \mapsto \iota_j \circ f \circ \pi_i$, where π_i and ι_j are the canonical projections and injections for the chosen decomposition of T. Given $f, g \in \mathrm{End}_\Lambda(T)$, we write $f \circ g$ for the standard composition in $\mathrm{End}_\Lambda(T)$, and $f * g = g \circ f$ for the product in $\widetilde{\Lambda}$ whenever there is a risk of ambiguity. If the context is unequivocal, we omit the composition symbol "$*$"; in particular, we will write $f\widetilde{\Lambda}$ for $f * \widetilde{\Lambda}$ when $f \in \widetilde{\Lambda}$.

Theorem III guarantees that the contravariant Hom-functors $\mathrm{Hom}_\Lambda(-, T)$ and $\mathrm{Hom}_{\widetilde{\Lambda}}(-, T)$ induce inverse dualities

$$\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \longleftrightarrow \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda}) \cap {}^{\perp}(T_{\widetilde{\Lambda}}).$$

Since the righthand category is properly contained in $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ in general, this duality does not a priori permit us to transfer information from $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ to $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$. The analysis of $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ through the lens of Q and L additionally hinges on the following **refined partition of the vertices e_i of Q**, and on the synchronized partition of the vertices of $\widetilde{Q}$.

We assume the n vertices $e_1, \dots, e_n$ to be ordered as follows:

- $e_1, \dots, e_r$ are the *critical* vertices of Q, i.e., those that are both pre- and postcyclic.
- $e_{r+1}, \dots, e_m$ are the *precyclic vertices which fail to be postcyclic.*
- $e_{m+1}, \dots, e_s$ are the vertices which are *postcyclic, but not precyclic*; and
- $e_{s+1}, \dots, e_n$ are the vertices which are *neither pre- nor postcyclic.*

Theorem IV(2) provided us with a natural bijection $\{e_1, \dots, e_n\} \leftrightarrow \{T_1, \dots, T_n\}$. Recall that our choice of a supplementary bijection $\{T_1, \dots, T_n\} \leftrightarrow \{\text{vertices of } \widetilde{\Lambda}\}$ is as follows:

The preferred primitive idempotents $\widetilde{e}_i$ of $\widetilde{\Lambda}$, alias vertices of $\widetilde{Q}$:

$$\widetilde{e}_i = \iota_i \circ \pi_i : \ T \twoheadrightarrow \ T_i \ \hookrightarrow T, \quad \text{as introduced at the end of 3.A.}$$

The idempotents μ and $\widetilde{\mu}$. We set $\mu = \sum_{1\leq i\leq r} e_i$ and $\widetilde{\mu} = \sum_{1\leq i\leq r} \widetilde{e}_i = \sum_{e_i \text{ critical}} \widetilde{e}_i$. An idempotent $\widetilde{e}_i$ of $\widetilde{\Lambda}$ will be referred to as (*tilted*) *critical*, (*tilted*) *pre-* or *postcyclic* precisely when the corresponding idempotent e_i of Λ has the specified property. In fact, in the sequel we will drop the qualifier "tilted" in reference to criticality (etc.) of the $\widetilde{e}_i$, since we will not incur any danger of ambiguity; see however the caveat below. We moreover extend the use of the attributes "pre(post)cyclic" and "critical" from e_i to the indecomposable projective modules $\Lambda e_i \in \Lambda$-mod and $\widetilde{e}_i\widetilde{\Lambda} \in \text{mod-}\widetilde{\Lambda}$, as well as to the simple modules $S_i = \Lambda e_i/Je_i \in \Lambda$-mod and $\widetilde{S}_i = \widetilde{e}_i\widetilde{\Lambda}/\widetilde{e}_i\widetilde{J} \in \text{mod-}\widetilde{\Lambda}$. Note that, for $1 \leq k \leq r$, all simple composition factors of T_k are critical, i.e., $\mu\big(\bigoplus_{k=1}^r T_k\big) = \bigoplus_{k=1}^r T_k$ (cf. Reference Theorem IV(2)).

Caveat: In general, the status of an idempotent $\widetilde{e}_i$ (such as tilted precyclic, critical, etc.) is not in line with the position of the corresponding vertex relative to oriented cycles of the quiver $\widetilde{Q}$ of $\widetilde{\Lambda}$; in fact, even for a non-precyclic vertex e_i of Q, the vertex $\widetilde{e}_i$ of $\widetilde{Q}$ will typically lie on multiple oriented cycles of $\widetilde{Q}$; see Example 3.B.

4.A. The endofunctors ∇ and Δ, and the critical core.

We introduce three endofunctors, ∇, Δ and $\mathcal{C}$, of Mod-$\widetilde{\Lambda}$ as follows: Let $\widetilde{M} \in$ Mod-$\widetilde{\Lambda}$.

• $\nabla\widetilde{M} := \widetilde{M}\,\widetilde{\mu}\,\widetilde{\Lambda}$. Thus $\nabla\widetilde{M}$ is the unique smallest $\widetilde{\Lambda}$-submodule of $\widetilde{M}$ with the property that $\big(\widetilde{M}/\nabla\widetilde{M}\big)\widetilde{\mu} = 0$. Note that the top of $\nabla\widetilde{M}$ is a sum of critical simples.

• $\Delta\widetilde{M} = \Delta(\widetilde{M}) := \{x \in \widetilde{M} \mid x\widetilde{\Lambda}\widetilde{\mu} = 0\} = \operatorname{ann}_{\widetilde{M}}(\widetilde{\Lambda}\widetilde{\mu})$ is the unique largest $\widetilde{\Lambda}$-submodule of $\widetilde{M}$ with $(\Delta\widetilde{M})\widetilde{\mu} = 0$. In particular, the socle of $\widetilde{M}/(\Delta\widetilde{M})$ is a sum of critical simples.

• $\widetilde{\Delta} := \Delta(\widetilde{\Lambda}_{\widetilde{\Lambda}})$ is a two-sided ideal of $\widetilde{\Lambda}$, referred to as the *noncritical ideal*. Clearly, $\widetilde{M}\cdot\widetilde{\Delta} \subseteq \Delta\widetilde{M}$ with equality holding in case $\widetilde{M}$ is projective.

• $\mathcal{C}(\widetilde{M}) := \nabla\widetilde{M}/\Delta(\nabla\widetilde{M})$ is called the *critical core* of $\widetilde{M}$. Clearly, $\widetilde{M}\widetilde{\mu}$ embeds in $\mathcal{C}(\widetilde{M})$.

By definition, both the top and the socle of $\mathcal{C}(\widetilde{M})$ are direct sums of critical simples in mod-$\widetilde{\Lambda}$. In fact, if $\widetilde{M} \in$ mod-$\widetilde{\Lambda}$, then $\mathcal{C}(\widetilde{M})$ is the unique subfactor of $\widetilde{M}$ which has maximal dimension relative to the conditions that top and socle of $\mathcal{C}(\widetilde{M})$ be sums of critical simples. On the other hand, typically $\mathcal{C}(\widetilde{M})\widetilde{\mu} \subsetneqq \mathcal{C}(\widetilde{M})$, i.e., $\mathcal{C}(\widetilde{M})$ has also noncritical composition factors in general. In our reference example, the latter is the case when $\widetilde{M} = \widetilde{e}_i\widetilde{\Lambda}$ for $i \neq 3$.

The critical cores of the projective $\widetilde{\Lambda}$-modules will play a pivotal role in the sequel.

Proposition 6. The critical cores of the $\widetilde{e}_i\widetilde{\Lambda}$.

(a) *For $1 \le i \le r$: $\mathcal{C}_i := \mathcal{C}(\widetilde{e}_i\widetilde{\Lambda}) = \widetilde{e}_i\widetilde{\Lambda}/\widetilde{e}_i\widetilde{\Delta}$.*

(b) *For $r+1 \le i \le m$: $\mathcal{C}(\widetilde{e}_i\widetilde{\Lambda}) = 0$.*

(c) *For $m+1 \le i \le s$: $\mathcal{C}(\widetilde{e}_i\widetilde{\Lambda}) = \bigoplus_{k=1}^r \mathcal{C}_k^{\mathfrak{m}_{ik}}$, where $\mathfrak{m}_{ik}$ is the number of paths of length L from the vertex e_k of Q to the vertex e_i.*

(d) *For $s+1 \le i \le n$: $\mathcal{C}(\widetilde{e}_i\widetilde{\Lambda}) = 0$.*

In particular, the right $\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$-module $\widetilde{\Lambda}\widetilde{\mu}$ is isomorphic to $\bigoplus_{k=1}^r (\mathcal{C}_k\widetilde{\mu})^{\tau_k} \cong \bigoplus_{k=1}^r (\widetilde{e}_k\widetilde{\Lambda}\widetilde{\mu})^{\tau_k}$ with $\tau_k = 1 + \sum_{i=m+1}^s \mathfrak{m}_{ik}$.

Proof. For (a) it suffices to note that $i \le r$ implies $\nabla(\widetilde{e}_i\widetilde{\Lambda}) = \widetilde{e}_i\widetilde{\Lambda}$, and $\Delta(\widetilde{e}_i\widetilde{\Lambda}) = \widetilde{e}_i\widetilde{\Delta}$.

(b) Suppose $i \in \{r+1, \dots, m\}$. Then $T_i = \mathcal{A}_i$ has a non-postcyclic top. We deduce that $\widetilde{e}_i\widetilde{\Lambda}\widetilde{e}_k = \operatorname{Hom}_\Lambda(T_i, T_k) = 0$ for $k \le r$, since all simple composition factors of such T_k's are critical. In other words, $\widetilde{e}_i\widetilde{\Lambda}\widetilde{\mu} = 0$, which amounts to vanishing of $\nabla(\widetilde{e}_i\widetilde{\Lambda})$ and hence of $\mathcal{C}(\widetilde{e}_i\widetilde{\Lambda})$.

(c) Fix an index $i \in \{m+1, \dots, s\}$. Then e_i is postcyclic but not precyclic; in particular, $T_i = \mathcal{A}(E(S_i))$. By Theorem IV of Section 3.A, we thus obtain

$$T_i/\varepsilon T_i \;\cong\; \bigoplus_{k=1}^{r} T_k^{\mathfrak{m}_{ik}} \oplus \bigoplus_{k=r+1}^{m} T_k^{\mathfrak{n}_{ik}} \qquad \text{for suitable exponents } \mathfrak{n}_{ik}.$$

Since all composition factors of $\bigoplus_{j=1}^r T_j$ are critical, $\operatorname{Hom}_\Lambda(\bigoplus_{k=r+1}^m T_k^{\mathfrak{n}_{ik}}, \bigoplus_{j=1}^r T_j) = 0$, which yields K-space isomorphisms

$$\widetilde{e}_i\widetilde{\Lambda}\widetilde{\mu} \cong \operatorname{Hom}_\Lambda(T_i, \bigoplus_{j=1}^r T_j) \cong \operatorname{Hom}_\Lambda(T_i/\varepsilon T_i, \bigoplus_{j=1}^r T_j) \cong \bigoplus_{k=1}^r \operatorname{Hom}_\Lambda(T_k, \bigoplus_{j=1}^r T_j)^{\mathfrak{m}_{ik}}.$$

Clearly, the rightmost of the above spaces is just $\bigoplus_{k=1}^r (\widetilde{e}_k \widetilde{\Lambda} \widetilde{\mu})^{\mathfrak{m}_{ik}}$. Under the obvious identifications, we therefore obtain

$$\nabla(\widetilde{e}_i \widetilde{\Lambda}) = (\widetilde{e}_i \widetilde{\Lambda} \widetilde{\mu}) \widetilde{\Lambda} = \bigoplus_{k=1}^r \nabla(\widetilde{e}_k \widetilde{\Lambda})^{\mathfrak{m}_{ik}} = \bigoplus_{k=1}^r (\widetilde{e}_k \widetilde{\Lambda})^{\mathfrak{m}_{ik}} \ ,$$

and our claim follows from part (a) and the definition of the critical core.

(d) Next let $i \in \{s+1, \dots, n\}$. Given that e_i fails to be postcyclic, all simple summands of $\mathrm{top}(T_i) = \mathrm{top}\big(\mathcal{A}(E(S_i))\big) = \mathrm{top}(E(S_i))$ are nonpostcyclic. In particular, $\mathrm{top}(T_i)$ does not contain any critical simple summand. Given that, for critical e_k, all simple composition factors of $T_k = \mathcal{A}_k$ are in turn critical, we again infer $\widetilde{e}_i \widetilde{\Lambda} \widetilde{e}_k = 0$, for $k \le r$. As in the proof of (b), we conclude $\mathcal{C}(\widetilde{e}_i \widetilde{\Lambda}) = 0$. $\square$

Return to the Reference Example in 3.B. The critical cores of the $\widetilde{e}_i \widetilde{\Lambda}$ are as follows: $\mathcal{C}(\widetilde{e}_1 \widetilde{\Lambda})$ is the quotient $\widetilde{e}_1 \widetilde{\Lambda}/U$ by the uniserial module U with composition factors $(\widetilde{S}_5, \widetilde{S}_4, \widetilde{S}_3)$, and $\mathcal{C}(\widetilde{e}_2 \widetilde{\Lambda})$ is the uniserial module with consecutive composition factors $(\widetilde{S}_2, \widetilde{S}_5, \widetilde{S}_1, \widetilde{S}_4, \widetilde{S}_2)$. Moreover, $\mathcal{C}(\widetilde{e}_3 \widetilde{\Lambda}) = 0$, $\mathcal{C}(\widetilde{e}_4 \widetilde{\Lambda}) \cong \mathcal{C}(\widetilde{e}_2 \widetilde{\Lambda})$, and $\mathcal{C}(\widetilde{e}_5 \widetilde{\Lambda}) \cong \mathcal{C}(\widetilde{e}_6 \widetilde{\Lambda}) \cong \mathcal{C}(\widetilde{e}_1 \widetilde{\Lambda})$. Observe that the critical cores $\mathcal{C}(\widetilde{e}_i \widetilde{\Lambda})$ for $i = 4, 5, 6$ are neither sub- nor factor modules of the corresponding $\widetilde{e}_i \widetilde{\Lambda}$, but are properly sandwiched between two submodules $0 \ne U_i \subseteq V_i \subsetneqq \widetilde{e}_i \widetilde{\Lambda}$ with the property that all simple composition factors of U_i and $\widetilde{e}_i \widetilde{\Lambda}/V_i$ are non-critical.

4.B. The simples in $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$.

Proposition 9 below covers one implication of Theorem A of the introduction. The converse will be established in Corollary 15 of Section 6. We leave the straightforward proof of the lemma to the reader.

Lemma 8. Homomorphisms among the T_i.

(a) *If $i \le m$, then $\mathrm{Hom}_\Lambda(T_i, T_u) = \mathrm{Hom}_\Lambda(T_i, JT_u)$ for all $u \in \{1, \dots, n\} \setminus \{i\}$.*

(b) *The radical of $\widetilde{\Lambda}$ is $\widetilde{J} = \{f \in \widetilde{\Lambda} \mid \widetilde{e}_u f \widetilde{e}_u$ is not an isomorphism for $1 \le u \le n\}$.*

(c) *If e_i is a precyclic source of Q, then $\widetilde{S}_i = \widetilde{e}_i \widetilde{\Lambda}$.*

Proposition 9. *Every noncritical simple right $\widetilde{\Lambda}$-module has finite projective dimension.*

Proof. Let $\widetilde{S}_i = \widetilde{e}_i \widetilde{\Lambda}/\widetilde{e}_i \widetilde{J}$ for some noncritical idempotent e_i. First consider the case where e_i is not precyclic, i.e., $i \in \{m+1, \dots, n\}$. Then $\mathrm{p\,dim}_\Lambda S_i < \infty$ (a special case of [**12**, Theorem 2.6]). Recall from Section 2 that the functors $\mathrm{Hom}_\Lambda(-, T)$ and $\mathrm{Hom}_{\widetilde{\Lambda}}(-, T)$ induce inverse dualities

$$\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \longleftrightarrow \ \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda}) \cap {}^{\perp}(T_{\widetilde{\Lambda}}).$$

Therefore, $\mathrm{Hom}_\Lambda(S_i, T)$ has finite projective dimension as a right $\widetilde{\Lambda}$-module, and our claim will follow once we have verified that $\mathrm{Hom}_\Lambda(S_i, T) \cong \widetilde{S}_i$. But S_i occurs precisely once in the socle of T; in fact, this single copy of S_i belongs to $\mathrm{soc}\, T_i$ (Theorem IV). Consequently, $\dim \mathrm{Hom}_\Lambda(S_i, T) = 1$ and $\mathrm{Hom}_\Lambda(S_i, T) * \widetilde{e}_i = \widetilde{e}_i \circ \mathrm{Hom}_\Lambda(S_i, T) \ne 0$ as required.

Now suppose that $i \in \{r+1, \dots, m\}$; in particular, $T_i = \mathcal{A}_i = \Lambda e_i/\varepsilon J e_i$. Since e_i fails to be postcyclic, every path p in Q that ends in e_i is a terminal subpath of one of the finitely many paths $p_1, \dots, p_t$ which start in some source of Q and end in e_i; in other words, $p\,p' = p_u$ for some path p' and $u \le t$. We set

$$\partial(i) := \max\{\mathrm{length}(p_u) \mid 1 \le u \le t\}$$

and show $\operatorname{p\,dim} \widetilde{S}_i < \infty$ by induction on $\partial(i) \geq 0$.

If $\partial(i) = 0$, then e_i is a source of Q, whence $\widetilde{S}_i = \widetilde{e}_i \widetilde{\Lambda}_i$ is projective by part (c) of Lemma 8. So assume $\partial(i) \geq 1$. We will prove that the radical $\widetilde{e}_i \widetilde{J}$ of $\widetilde{e}_i \widetilde{\Lambda}$ has no critical simple composition factors and that, for all composition factors $\widetilde{S}_j$ with $j \in \{r+1, \dots, m\}$, we have $\partial(j) < \partial(i)$. The induction hypothesis combined with the first part of the proof will then ensure that all simple composition factors of $\widetilde{e}_i \widetilde{J}$ have finite projective dimension, whence finiteness of $\operatorname{p\,dim} \widetilde{S}_i$ will follow.

To verify the two claims stated in the preceding paragraph, we note that $\widetilde{e}_i \widetilde{\Lambda} \widetilde{e}_k = \operatorname{Hom}_\Lambda(T_i, T_k) = 0$ for $k \leq r$, since T_k has only critical composition factors in this situation, whereas S_i is noncritical. Now let $j \in \{r+1, \dots, m\}$, and suppose that $\widetilde{S}_j$ is a composition factor of $\widetilde{e}_i \widetilde{J}$. In light of Lemma 8, we infer $0 \neq \widetilde{e}_i \widetilde{J} \widetilde{e}_j \subseteq \operatorname{Hom}_\Lambda(T_i, JT_j)$. Hence $e_i JT_j \neq 0$, meaning that S_i is a composition factor of JT_j. But the simple composition factors S_u of JT_j with $u \in \{r+1, \dots, m\}$ correspond to indices u with $\partial(u) > \partial(j)$, and we conclude $\partial(i) > \partial(j)$ as desired. □

Return to the Reference Example in 3.B. By Proposition 9, the simple right $\widetilde{\Lambda}$-modules $\widetilde{S}_i$ for $i = 3, 4, 5, 6$ have finite projective dimension.

As we will see in Corollary 15, the findings in our example carry over to the general case: The critical simples in mod-$\widetilde{\Lambda}$ always have infinite projective dimension.

Observation 10. The basic algebra $\widetilde{\Lambda}$ is again a path algebra modulo relations, irrespective of the base field K. Indeed, in the proof of Proposition 9 we saw that, for $i \geq m+1$, the simple right $\widetilde{\Lambda}$-modules $\widetilde{S}_i$ are 1-dimensional over K. The same is true for $i \leq m$, since Lemma 8 yields K-vector space equalities

$$\widetilde{e}_i \widetilde{\Lambda} = \operatorname{Hom}_\Lambda(T_i, T) = K\widetilde{e}_i \oplus \operatorname{Hom}(T_i, JT) = K\widetilde{e}_i \oplus \widetilde{e}_i \widetilde{J}.$$

.

5. Auxiliaries: Annihilators of key objects in mod-$\widetilde{\Lambda}$

The upcoming information targets the annihilators of the critical cores $\mathcal{C}(\widetilde{e}_i \widetilde{\Lambda})$ and those of the indecomposable injective right $\widetilde{\Lambda}$-modules.

Lemma 11. **(a)** *Let $k \in \{1, \dots, r\}$, i.e., $\widetilde{e}_k$ is critical. There exists a sequence of indices $k_0 = k$, $k_1, \dots, k_L$ in $\{1, \dots, r\}$, combined with homomorphisms $f_u \in \operatorname{Hom}_\Lambda(T_{k_{u-1}}, JT_{k_u}) \subseteq \widetilde{\Lambda}$ for $1 \leq u \leq L$, such that the product $f_1 * \cdots * f_L$ in $\widetilde{\Lambda}$ is nonzero.* (Again $*$ denotes multiplication in $\widetilde{\Lambda}$.)
(b) *Let $l \in \{1, \dots, r\}$. There exists a sequence of indices $l_0 = l$, $l_1, \dots, l_L$ in $\{1, \dots, r\}$, combined with homomorphisms $g_u \in \operatorname{Hom}_\Lambda(T_{l_u}, JT_{l_{u-1}}) \subseteq \widetilde{\Lambda}$ for $1 \leq u \leq L$, such that the product $g_L * \cdots * g_1$ in $\widetilde{\Lambda}$ is nonzero.*

Proof. We verify (a) and leave the dual argument to the reader.

Set $k_0 = k$, and for each $v \leq r$, let $x_v = e_v + \varepsilon J e_v$ be the obvious generator for the local module T_v. We will inductively choose indices $k_1, \dots, k_L$ in $\{1, \dots, r\}$, next to maps $f_1, \dots f_L$ induced by arrows $\alpha_u : e_{k_u} \to e_{k_{u-1}}$ in Q such that $f_u(x_{k_{u-1}}) = \alpha_u x_{k_u}$; as we will argue, any such choice will entail that $f_1 * \cdots * f_u = f_u \circ \cdots \circ f_1$ is nonzero. Let $0 \leq \kappa < L$, and suppose $k = k_0, k_1, \dots, k_\kappa \in \{1, \dots, r\}$, maps $f_1, \dots f_\kappa$, and arrows $\alpha_1, \dots \alpha_\kappa$ with $\operatorname{start}(\alpha) = e_{k_{u-1}}$, $\operatorname{end}(\alpha) = e_{k_u}$ for $1 \leq u \leq \kappa$ are as described. Then $f_\kappa \circ \cdots \circ f_1(x_k) = \alpha_1 \cdots \alpha_\kappa x_\kappa \neq 0$. Indeed: Whenever $p = p\, e_\kappa$ is a path of length $\leq L$ in Q, which is contingent to precyclic vertices *only*, we have $p\, x_{k_\kappa} \in J^{\operatorname{length}(p)} T_{k_\kappa} \setminus J^{\operatorname{length}(p)+1} T_{k_\kappa}$ by the structure of the T_j for critical e_j (cf. Theorem IV); so, in particular, $\alpha_1 \cdots \alpha_\kappa x_{k_\kappa} \in J^\kappa T_{k_\kappa} \setminus J^{\kappa+1} T_{k_\kappa}$.

To complete the induction step, we observe that criticality of the idempotent e_{k_κ} guarantees the existence of a postcyclic predecessor of e_{k_κ}; by this we mean a postcyclic vertex $\mathbf{e}_{k_{\kappa+1}}$ in Q, together with an arrow $\alpha_{\kappa+1}$ from $e_{k_{\kappa+1}}$ to e_{k_κ}. Then $e_{k_{\kappa+1}}$ is again critical, i.e., $k_{\kappa+1} \le r$. Define $f_{\kappa+1} \in \mathrm{Hom}_\Lambda(T_{k_\kappa}, T_{k_{\kappa+1}})$ via $x_{k_\kappa} \mapsto \alpha_{\kappa+1} x_{k_{\kappa+1}}$, clearly legitimate. In light of $\kappa + 1 \le L$, we obtain $f_{\kappa+1} \circ \cdots \circ f_1(x_k) \in J^{\kappa+1}T_k \setminus J^{\kappa+2}T_k$ by a repeat of the argument concluding the previous paragraph. This proves part (a). $\square$

Proposition 12. *Again, let $k, l \le r$.*

(a) *If $f_1, \dots, f_L$ are as in part* (a) *of* Lemma 11 *and $\mathcal{C}_k$ is the critical core of $\widetilde{e}_k\widetilde{\Lambda}$ as before, then*

$$\mathcal{C}_k\, f_1 * \cdots * f_L \neq 0.$$

In particular, $\mathcal{C}_k$ is not annihilated by $(\widetilde{\mu}\widetilde{J}\widetilde{\mu})^L$.

(b) *The injective envelope $\widetilde{E}_l = \widetilde{E}(\widetilde{S}_l)$ of $\widetilde{S}_l$ in* mod-$\widetilde{\Lambda}$ *is not annihilated by $(\widetilde{\mu}\widetilde{J}\widetilde{\mu})^L$ either.*

Proof. We use the notational setup of Lemma 11.

(a) Recall that $\mathcal{C}_k = \widetilde{e}_k\widetilde{\Lambda}/\widetilde{e}_k\widetilde{\Delta}$, where $\widetilde{\Delta}$ is the noncritical ideal of $\widetilde{\Lambda}$; that is,

$$\widetilde{\Delta} = \{\phi \in \widetilde{\Lambda} \mid \phi * \widetilde{\Lambda} * \widetilde{\mu} = 0\} = \{\phi \in \mathrm{End}_\Lambda(T) \mid \widetilde{\mu} \circ f \circ \phi = 0 \text{ for all } f \in \mathrm{End}_\Lambda(T)\}.$$

In particular, as a right ideal, $\widetilde{\Delta}$ has no critical simple composition factors in mod-$\widetilde{\Lambda}$. To see that $\mathcal{C}_k \cong \mathrm{Hom}_\Lambda(T_k, T)/\big(\mathrm{Hom}_\Lambda(T_k, T) * \widetilde{\Delta}\big)$ is not annihilated by $f_1 * \cdots * f_L$, we recall that we view the latter map as an element of $\widetilde{\Lambda}$ by identifying it with $\pi_{k_L} \circ f_L \circ \cdots \circ f_1 \circ \iota_{k_0}$, where the π_j and ι_j are the projections and injections corresponding to the decomposition $T = \bigoplus_{j=1}^n T_j$. Thus it suffices to check that $\pi_{k_L} \circ f_L \circ \cdots \circ f_1 \circ \iota_{k_0} \not\subseteq \widetilde{\Delta}$. But this is clear, because $\widetilde{\mu} \circ \pi_{k_L} = \pi_{k_L}$ in view of criticality of the index k_L.

(b) Let $g_1, \dots, g_L$ be as in part (b) of Lemma 11 and set $g := g_L * \cdots * g_1$. By construction, g is then a nonzero element of $(\widetilde{\mu}\widetilde{J}\widetilde{\mu})^L \subseteq \widetilde{\Lambda}$. Any map g with these properties provides us with at least one path $\widetilde{p}$ of length $\ge L$ from $\widetilde{e}_{l_L}$ to $\widetilde{e}_{l_0} = \widetilde{e}_l$ in $\widetilde{Q} \setminus \widetilde{I}$ such that $\widetilde{p}$ passes through $L+1$ critical vertices of $\widetilde{Q}$; in other words, $\widetilde{p} + \widetilde{I} \in (\widetilde{\mu}\widetilde{J}\widetilde{\mu})^L$. Since $\widetilde{p}$ is nonzero in $\widetilde{\Lambda}$, there exists an element $\widetilde{y} = \widetilde{y}\,\widetilde{e}_{l_L} \in \widetilde{E}_l$ such that $\widetilde{y}\,(\widetilde{p} + \widetilde{I})$ generates the socle $\widetilde{S}_l$ of $\widetilde{E}_l$. In particular, $\widetilde{E}_l(\widetilde{p} + \widetilde{I}) \neq 0$. $\square$

6. Theorems A, B. Structure of the projective $\widetilde{\Lambda}$-modules

If Q has at least one oriented cycle, meaning that μ and $\widetilde{\mu}$ are nonzero, the corner algebras $\mu\Lambda\mu$ and $\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$ are isomorphic. Indeed, these algebras are Morita equivalent by dint of the functors $\mathrm{Hom}_{\mu\Lambda\mu}(\mu T\widetilde{\mu}, -)$ and $\mu T\widetilde{\mu} \otimes_{\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}} -$. To see this, observe that μT is a finitely generated projective generator for $(\mu\Lambda\mu)$-Mod with $\mathrm{End}_{\mu\Lambda\mu}(\mu T)^{\mathrm{op}} \cong \widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$; projectivity of μT is due to the facts that $\mu\Lambda\mu$ has vanishing finitistic dimensions and $\mathrm{p\,dim}_{\mu\Lambda\mu}\, \mu T < \infty$; the generator property follows from the facts that $\mathrm{rank}\, K_0(\mu\Lambda\mu) = r$ and $\mu T = \bigoplus_{1 \le k \le r} T_k$ has r pairwise non-isomorphic indecomposable direct summands. More strongly: Since both of the algebras $\mu\Lambda\mu$ and $\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$ are basic, they are isomorphic.

However, the objects in the two ambient categories, Λ-mod and mod-$\widetilde{\Lambda}$, relate differently to this shared territory. For instance: If $M \in \Lambda$-mod, then finiteness of $\mathrm{p\,dim}_\Lambda M$ does not imply projectivity of the left $\mu\Lambda\mu$-module μM in general, nor does the converse hold (the algebra Λ in 3.B lends itself to easy counterexamples). This contrasts the situation we announced in Theorem B for mod-$\widetilde{\Lambda}$ versus mod-$\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$. In Theorem 13 below, we add the announced intrinsic description of the modules in $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ in terms of their critical cores. In fact, this criterion extends to the "big" category $\mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$. The result once more highlights the symmetrizing effect of passage from Λ to $\widetilde{\Lambda}$ as we will specify in a homological comparison following Corollary 15.

The indecomposable projective right $\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$-modules are $\widetilde{e}_k\widetilde{\Lambda}\widetilde{\mu}$ for $1 \le k \le r$. They all have Loewy length ≥ 2; indeed, in light of criticality of e_k, there exists a critical e_i such that $e_k J e_i \ne 0$, whence $\widetilde{e}_k\widetilde{J}\widetilde{e}_i \ne 0$. In the proof of the upcoming theorem, we will consider a second auxiliary algebra to analyze $\mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$, namely $\widetilde{\Lambda}/\widetilde{\Delta}$ (for the noncritical ideal $\widetilde{\Delta}$ see Section 4.A). Its indecomposable projective right modules $\widetilde{e}_i\widetilde{\Lambda}/\widetilde{e}_i\widetilde{\Delta}$ include the critical cores $\mathcal{C}_k = \mathcal{C}(\widetilde{e}_k\widetilde{\Lambda}) = \widetilde{e}_k\widetilde{\Lambda}/\widetilde{e}_k\widetilde{\Delta}$ for $k \le r$. The link between the two settings is provided by the fact that $\widetilde{e}_k\widetilde{\Lambda}\widetilde{\mu} = \mathcal{C}_k\widetilde{\mu}$ for $k \le r$.

Theorem 13. *As before, $\widetilde{\mu} = \sum_{k=1}^r \widetilde{e}_k$ denotes the sum of the critical idempotents in $\widetilde{\Lambda}$, and $\mathcal{C}_j = \mathcal{C}(\widetilde{e}_j\widetilde{\Lambda})$ is the critical core of $\widetilde{e}_j\widetilde{\Lambda}$. For any module $\widetilde{M} \in \text{Mod-}\widetilde{\Lambda}$, the following statements are equivalent:*

(1) $\operatorname{p\,dim} \widetilde{M}_{\widetilde{\Lambda}} < \infty$.

(2) *$\widetilde{M}\,\widetilde{\mu}$ is a projective right $\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$-module, i.e., $\widetilde{M}\,\widetilde{\mu}$ is a direct sum of copies of certain $\widetilde{e}_k\widetilde{\Lambda}\widetilde{\mu}$ with $k \le r$.*

(3) *The critical core $\mathcal{C}(\widetilde{M})$ is a direct sum of copies of $\mathcal{C}_1, \dots, \mathcal{C}_r$.*

Proof. "(1) $\Longrightarrow$ (2)". Assume (1) holds. We show by induction on $d := \operatorname{p\,dim} \widetilde{M}$ that the right $\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$-module $\widetilde{M}\widetilde{\mu}$ is projective, meaning that it is a direct sum of copies of $\widetilde{e}_k\widetilde{\Lambda}\widetilde{\mu}$ for certain $k \le r$.

If $d = 0$, we have $\widetilde{M} \cong \bigoplus_{j=1}^n (\widetilde{e}_j\widetilde{\Lambda})^{(\sigma_j)}$ for suitable cardinal numbers σ_j, and hence $\widetilde{M}\widetilde{\mu} \cong \bigoplus_{j=1}^n (\widetilde{e}_j\widetilde{\Lambda}\widetilde{\mu})^{(\sigma_j)}$. The proof of Proposition 6 shows that $\widetilde{e}_j\widetilde{\Lambda}\widetilde{\mu} = 0$ for $r+1 \le j \le m$ and $s+1 \le j \le n$. Therefore the argument backing Proposition 6(c) yields

$$\widetilde{M}\widetilde{\mu} = \bigoplus_{j=1}^{r} (\widetilde{e}_j\widetilde{\Lambda}\widetilde{\mu})^{(\sigma_j)} \oplus \bigoplus_{j=m+1}^{s} (\widetilde{e}_j\widetilde{\Lambda}\widetilde{\mu})^{(\sigma_j)} \;\cong\; \bigoplus_{j=1}^{r} (\widetilde{e}_j\widetilde{\Lambda}\widetilde{\mu})^{(\tau_j)}$$

for suitable cardinal numbers σ_j and τ_j, which confirms projectivity of $\widetilde{M}\widetilde{\mu}$ over $\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$.

Now suppose that $d \ge 1$, and let

$$0 \to \widetilde{P}_d \to \widetilde{P}_{d-1} \to \cdots \to \widetilde{P}_0 \to \widetilde{M} \to 0$$

be a projective resolution of $\widetilde{M}$ in $\text{Mod-}\widetilde{\Lambda}$. The epimorphism from $\widetilde{P}_0$ onto $\widetilde{M}$ is denoted by f_0. Since $- \otimes_{\widetilde{\Lambda}} \widetilde{\Lambda}\widetilde{\mu}$ is exact, the case $d = 0$ provides us with a projective resolution

$$0 \to \widetilde{P}_d\,\widetilde{\mu} \to \widetilde{P}_{d-1}\,\widetilde{\mu} \to \cdots \to \widetilde{P}_0\,\widetilde{\mu} \to \widetilde{M}\,\widetilde{\mu} \to 0$$

of $\widetilde{M}\widetilde{\mu}$ in $\text{Mod-}\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$. We decompose $\widetilde{P}_0\widetilde{\mu}$ in the form

$$\widetilde{P}_0\widetilde{\mu} = \widetilde{V} \oplus \widetilde{W} \;\text{ such that } f_0|_{\widetilde{V}} : \widetilde{V} \to \widetilde{M}\,\widetilde{\mu} \text{ is a } \widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}\text{-projective cover and } \widetilde{W} \subseteq \operatorname{Ker}(f_0|_{\widetilde{P}_0\widetilde{\mu}}).$$

Moreover, we denote the restriction of f_0 to $\widetilde{V}$ by f. If $\operatorname{Ker}(f) = 0$, then $\widetilde{M}\widetilde{\mu}$ is projective, and we are done. So suppose that $\operatorname{Ker}(f) \ne 0$. Since $\operatorname{Ker}(f_0)$ is a $\widetilde{\Lambda}$-module of projective dimension $d-1$ and $\operatorname{Ker}(f)$ is a $\mu\widetilde{\Lambda}\widetilde{\mu}$-direct summand of $\operatorname{Ker}(f_0)\,\widetilde{\mu}$, the induction hypothesis guarantees that $\operatorname{Ker}(f)$ is a projective $\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$-module. In view of nontriviality of $\operatorname{Ker}(f)$, Proposition 12(a) thus ensures that the Loewy length of $\mathrm{Ker}(f)$ over $\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$ is $L+1$. On the other hand, $\mathrm{Ker}(f) \subseteq (\widetilde{P}_0\widetilde{\mu})(\widetilde{\mu}\widetilde{J}\widetilde{\mu})$ by the choice of $\widetilde{V}$, whence the Loewy length of the kernel is at most L. This contradiction completes the argument.

"(2) $\Longrightarrow$ (3)". Clearly, (3) amounts to the condition that $\mathcal{C}(\widetilde{M})$ be projective as a right $\widetilde{\Lambda}/\widetilde{\Delta}$-module. Suppose this condition fails, and let $f : \widetilde{P} \to \mathcal{C}(\widetilde{M})$ be a projective cover of $\mathcal{C}(\widetilde{M})$ in Mod-$(\widetilde{\Lambda}/\widetilde{\Delta})$. We denote the Jacobson radical of $\widetilde{\Lambda}/\widetilde{\Delta}$ by $J(\widetilde{\Lambda}/\widetilde{\Delta})$ and the kernel of f by $\widetilde{K}$. Then: $\widetilde{K} \subseteq \widetilde{P} \cdot J(\widetilde{\Lambda}/\widetilde{\Delta})$. Since all simple summands in the top of $\mathcal{C}(\widetilde{M})$ are critical, we find that

$$\widetilde{P} \;\cong\; \bigoplus_{k=1}^{r} \bigl(\widetilde{e}_k\widetilde{\Lambda}/\widetilde{e}_k\widetilde{\Delta}\bigr)^{(\sigma_k)} \;=\; \bigoplus_{k=1}^{r} \mathcal{C}_k^{(\sigma_k)}$$

for suitable cardinal numbers σ_k. Again using flatness of the left ideal $\widetilde{\Lambda}\widetilde{\mu}$, we obtain an exact sequence

$$0 \longrightarrow \widetilde{K}\widetilde{\mu} \longrightarrow \widetilde{P}\widetilde{\mu} \longrightarrow \mathcal{C}(\widetilde{M})\widetilde{\mu} \longrightarrow 0$$

with $\widetilde{K}\widetilde{\mu} \subseteq \bigoplus_{k=1}^{r} (\widetilde{e}_k\widetilde{J}\widetilde{\mu})^{(\sigma_k)}$. The middle term, $\widetilde{P}\widetilde{\mu}$, of this exact sequence is projective over $\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$ by the comments preceding the theorem, and $\mathcal{C}(\widetilde{M})\widetilde{\mu} = \widetilde{M}\widetilde{\mu}$. Therefore the map $f\widetilde{\mu} : \widetilde{P}\widetilde{\mu} \longrightarrow \widetilde{M}\widetilde{\mu}$ induced by f is a projective cover of $\widetilde{M}\widetilde{\mu}$ in Mod-$\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$.

From the fact that $\widetilde{K} \neq 0$ we moreover deduce that $\widetilde{K}\widetilde{\mu} \neq 0$: Indeed, the equality $0 = \widetilde{P}\widetilde{\Delta} = \Delta(\widetilde{P})$ forces all simple summands in the socle of $\widetilde{P}$ to be critical; in other words, $\operatorname{soc} \widetilde{P} = (\operatorname{soc} \widetilde{P})\widetilde{\mu}$. This guarantees that $\widetilde{K}\widetilde{\mu} \neq 0$. Consequently, the $\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$-module $\widetilde{M}\widetilde{\mu}$ is not projective either, which proves the implication.

"(3) $\Longrightarrow$ (1)". Assume $\mathcal{C}(\widetilde{M})$ to be projective over $\widetilde{\Lambda}/\widetilde{\Delta}$. Since both the submodule $\Delta(\widetilde{M}\widetilde{\mu}\widetilde{\Lambda}) = \Delta\bigl(\nabla(\widetilde{M})\bigr)$ and the factor module $\widetilde{M}/\widetilde{M}\widetilde{\mu}\widetilde{\Lambda}$ of $\widetilde{M}$ have only noncritical composition factors and $\mathcal{C}(\widetilde{M}) = \widetilde{M}\widetilde{\mu}\widetilde{\Lambda}/\Delta(\widetilde{M}\widetilde{\mu}\widetilde{\Lambda})$ is projective, we invoke Proposition 9 to conclude that $\operatorname{p\,dim} \widetilde{M}_{\widetilde{\Lambda}}$ is indeed finite. $\square$

Next we complete the proof of Theorem A of the introduction.

Corollary 15. *For $j \in \{1, \dots, n\}$, the simple right $\widetilde{\Lambda}$-module $\widetilde{S}_j$ has finite projective dimension if and only if $\widetilde{S}_j$ is noncritical* (i.e., $r+1 \le j \le n$ in our numbering of the primitive idempotents).

Proof. We already saw in Proposition 9 that the noncritical simples in Mod-$\widetilde{\Lambda}$ have finite projective dimension. As for the critical ones: Whenever e_j is critical, the module $\widetilde{S}_j = \widetilde{S}_j\widetilde{\mu}$ fails to be projective over $\widetilde{\mu}\widetilde{\Lambda}\widetilde{\mu}$ because its Loewy length is 1; indeed, we invoke Proposition 12 to conclude that $\mathcal{C}(\widetilde{S}_j) = \widetilde{S}_j$ is not isomorphic to any $\mathcal{C}_k$ with $k \le r$. Consequently, Theorem 13 yields $\operatorname{p\,dim} \widetilde{S}_j = \infty$. $\square$

We compare the "diagnostic homological filtrations" of the modules in Λ-Mod and Mod-$\widetilde{\Lambda}$:

• Each $M \in \Lambda$-Mod has a unique submodule U (namely $U = \varepsilon M$) with the property that all simple composition factors of U have finite projective dimension, while those of $\operatorname{soc} M/U$ have infinite projective dimension. This submodule U gives rise to the criterion: $\operatorname{p\,dim}_\Lambda M < \infty$ if and only if M/U is a direct sum of copies of $\mathcal{A}_1, \dots, \mathcal{A}_m$.

• Every module $\widetilde{M} \in$ Mod-$\widetilde{\Lambda}$ contains a unique submodule chain $\widetilde{U} \subseteq \widetilde{V} \subseteq \widetilde{M}$ $\bigl($namely, $\widetilde{V} = \nabla\widetilde{M} \supseteq \widetilde{U} = \Delta(\nabla\widetilde{M})\bigr)$ such that $\widetilde{U}$ and $\widetilde{M}/\widetilde{V}$ have only composition factors of finite projective dimension, while socle and top of $\widetilde{V}/\widetilde{U}$ consist of simples with infinite projective dimension. This chain gives rise to the criterion: $\operatorname{p\,dim}_{\widetilde{\Lambda}} \widetilde{M} < \infty$ if and only if $\widetilde{V}/\widetilde{U}$ is a direct sum of copies of $\mathcal{C}_1, \dots, \mathcal{C}_k$.

Schematically, the relevant filtrations of the modules in Λ-Mod and Mod-$\widetilde{\Lambda}$ look as follows.

	$M \in \Lambda$-Mod
M/U	comp. factors of infinite $\operatorname{p\,dim}$
U	comp. factors of finite $\operatorname{p\,dim}$

$\widetilde{M} \in$ Mod-$\widetilde{\Lambda}$	
comp. factors of finite $\operatorname{p\,dim}$	$\widetilde{M}/\widetilde{V}$
critical core	$\widetilde{V}/\widetilde{U}$
comp. factors of finite $\operatorname{p\,dim}$	$\widetilde{U}$

As for the structure of the projective $\widetilde{\Lambda}$-modules, a more pronounced kinship between truncated path algebras Λ and hereditary algebras than is apparent in the category Λ-Mod surfaces in Mod-$\widetilde{\Lambda}$:

Corollary 16. Weak heredity property of $\widetilde{\Lambda}_{\widetilde{\Lambda}}$. *For any projective right $\widetilde{\Lambda}$-module $\widetilde{P}$, the submodule $\nabla(\widetilde{P}) = \widetilde{P}\widetilde{\mu}\widetilde{\Lambda}$ is again projective.*

Proof. Since the endofunctor ∇ commutes with direct sums and $\nabla(\widetilde{e}_k\widetilde{\Lambda}) = \widetilde{e}_k\widetilde{\Lambda}$ for critical $\widetilde{e}_k$, it suffices to address the special case where $\widetilde{P}$ is local with $\widetilde{P}/\widetilde{P}\widetilde{J}$ noncritical. As is immediate from the definition, $\widetilde{P}_0 := \nabla(\widetilde{P})$ is the unique (inclusion-)smallest among the submodules W of $\widetilde{P}$ with the property that all simple composition factors of $\widetilde{P}/W$ are noncritical. Thus we find $\widetilde{P}_0 \subseteq \widetilde{P}\widetilde{J}$ in the present situation. Given that $\widetilde{P}_0$ is generated by the elements of $\widetilde{P}$ which are normed by critical idempotents, all simple summands of its top have infinite projective dimension by Corollary 15.

Our restriction to a local projective module $\widetilde{P}$ with $\operatorname{p\,dim} \widetilde{P}/\widetilde{P}\widetilde{J} < \infty$ means that, up to isomorphism, $\widetilde{P} = \widetilde{e}_i\widetilde{\Lambda}$ for some noncritical idempotent $\widetilde{e}_i$. We verify that $\widetilde{P}_0 := \nabla(\widetilde{e}_i\widetilde{\Lambda})$ is projective: If $\widetilde{e}_i$ is non-postcyclic, then $\nabla(\widetilde{e}_i\widetilde{\Lambda}) = 0$ by Proposition 6(b)(d). Hence we may assume that $\widetilde{e}_i$ is post- but not precyclic. We pick up on the proof of Proposition 6(c), slightly simplifying the notation used there; this is legitimate since the index i is now fixed. Namely,

$$(\dagger) \qquad T_i/\varepsilon T_i \;\cong\; \bigoplus_{k=1}^{r} T_k^{\mathfrak{m}_k} \oplus \bigoplus_{k=r+1}^{m} T_k^{\mathfrak{n}_k} \qquad \text{for suitable exponents } \mathfrak{m}_k \text{ and } \mathfrak{n}_k.$$

In the argument for Proposition 6(c), we derived the $\widetilde{\Lambda}$-isomorphism

$$\nabla(\widetilde{e}_i\widetilde{\Lambda}) \cong \bigoplus_{k=1}^{r} (\widetilde{e}_k\widetilde{\Lambda})^{\mathfrak{m}_k},$$

which shows $\widetilde{P}_0$ to be projective also in this case. $\square$

Return to Example 3.B. Here $\nabla(\widetilde{e}_4\widetilde{\Lambda}) \cong \widetilde{e}_2\widetilde{\Lambda}$ and $\nabla(\widetilde{e}_5\widetilde{\Lambda}) \cong \nabla(\widetilde{e}_6\widetilde{\Lambda}) \cong \widetilde{e}_1\widetilde{\Lambda}$. In general, $\nabla(\widetilde{e}_j\widetilde{\Lambda})$ may be a direct sum of arbitrarily high powers of critical projectives $\widetilde{e}_k\widetilde{\Lambda}$; see [**17**].

7. Contravariant finiteness of $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$

To show that $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ is contravariantly finite, it suffices to establish $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$-approximations of the simples in mod-$\widetilde{\Lambda}$; this is due to Auslander-Reiten's Proposition 4.7(b) in [**4**]. Since $\operatorname{p\,dim}(\widetilde{S}_j)_{\widetilde{\Lambda}} < \infty$ for $j \geq r+1$, it is, in fact, enough to find $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$-approximations of the critical simples $\widetilde{S}_1, \dots, \widetilde{S}_r$.

Fix $k \le r$. We first describe our candidate for a minimal $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$-approximation $\phi_k : \widetilde{\mathcal{A}}_k \to \widetilde{S}_k$.

Definition 17 and remarks. Still $k \in \{1, \dots, r\}$.

Fix an injective envelope $\widetilde{E}(\mathcal{C}_k)$ of $\mathcal{C}_k = \widetilde{e}_k\widetilde{\Lambda}/\widetilde{e}_k\widetilde{\Delta}$ in mod-$\widetilde{\Lambda}$, and let $\widetilde{\mathcal{A}}_k \subseteq \widetilde{E}(\mathcal{C}_k)$ be a submodule which contains $\mathcal{C}_k$ and is maximal relative to the following two properties:

(i) The top element $a_k := \widetilde{e}_k + \widetilde{e}_k\widetilde{\Delta}$ of $\mathcal{C}_k$ belongs to $\widetilde{\mathcal{A}}_k \setminus \widetilde{\mathcal{A}}_k\widetilde{J}$;

(ii) $\widetilde{\mathcal{A}}_k$ is an object of $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$.

Existence of $\widetilde{\mathcal{A}}_k$ follows from the fact that $\mathcal{C}_k$ satisfies (i) and (ii).

In particular, $\widetilde{\mathcal{A}}_k$ is an essential extension of $\mathcal{C}_k$. In view of Theorem 13, condition (ii) entails that the critical core of $\widetilde{\mathcal{A}}_k$ equals $\mathcal{C}_k$. Indeed, finiteness of $\operatorname{p\,dim} \widetilde{\mathcal{A}}_k$ implies that $\mathcal{C}(\widetilde{\mathcal{A}}_k)$ is a direct sum of copies of $\mathcal{C}_l$'s for suitable $l \le r$ and thus is a projective right $\widetilde{\Lambda}/\widetilde{\Delta}$-module (see the comments preceding Theorem 13). Since the top of the indecomposable projective $\widetilde{\Lambda}/\widetilde{\Delta}$-submodule $\mathcal{C}_k = a_k\widetilde{\Lambda}$ of $\widetilde{\mathcal{A}}_k$ canonically embeds into that of $\widetilde{\mathcal{A}}_k$ and, a fortiori, into that of $\mathcal{C}(\widetilde{\mathcal{A}}_k)$, we deduce that $\mathcal{C}_k$ is a direct summand of $\mathcal{C}(\widetilde{\mathcal{A}}_k)$. Therefore essentiality of $\mathcal{C}_k$ in $\widetilde{\mathcal{A}}_k$ yields the equality $\mathcal{C}(\widetilde{\mathcal{A}}_k) = \mathcal{C}_k$.

Consequently, $\widetilde{\mathcal{A}}_k/\mathcal{C}_k$ has only noncritical simple composition factors. Given that $\mathcal{C}_k$ is a local module with top $\widetilde{S}_k$, we further conclude that

$$(\dagger) \qquad \dim \operatorname{top}(\widetilde{\mathcal{A}}_k)\, \widetilde{\mu} = \dim \operatorname{top}(\widetilde{\mathcal{A}}_k)\, \widetilde{e}_k = 1.$$

Clearly, there exists an epimorphism

$$(\ddagger) \qquad \phi_k : \widetilde{\mathcal{A}}_k \to \widetilde{S}_k \quad \text{with} \quad \phi_k(a_k) = \widetilde{e}_k + \widetilde{e}_k\widetilde{J} \in \widetilde{S}_k.$$

In view of ($\dagger$), specifying the value $\phi_k(a_k)$ fully determines the map ϕ_k. To see this, note that every top element of $\widetilde{\mathcal{A}}_k$ which is independent of a_k modulo $\widetilde{\mathcal{A}}_k\widetilde{J}$ is normed by an idempotent different from $\widetilde{e}_k$, and is therefore sent to zero by any homomorphism in $\operatorname{Hom}_{\widetilde{\Lambda}}(\widetilde{\mathcal{A}}_k, \widetilde{S}_k)$.

Next we address a strong uniqueness property of the module $\widetilde{\mathcal{A}}_k$ which will be required to confirm ϕ_k as the minimal approximation of $\widetilde{S}_k$.

Lemma 18. *Keep the notation of* Definition 17*.*

(a) *There is a unique submodule $\widetilde{\mathcal{A}}_k$ of $\widetilde{E}(\mathcal{C}_k)$ satisfying the conditions of* Definition 17*.*

(b) *Suppose that $\mathcal{C}_k \subseteq \widetilde{\mathcal{B}}_k$ is an essential extension with $a_k \notin \widetilde{\mathcal{B}}_k\widetilde{J}$ and $(\widetilde{\mathcal{B}}_k/\mathcal{C}_k)\widetilde{\mu} = 0$. Then $\widetilde{\mathcal{B}}_k$ has a unique maximal submodule $Z = Z(\widetilde{\mathcal{B}}_k)$ with $\widetilde{\mathcal{B}}_k/Z \cong S_k$; namely, $Z = \widetilde{\mathcal{B}}_k(1 - \widetilde{\mu})\widetilde{\Lambda} + \mathcal{C}_k\widetilde{J}$.*

(c) *Again, let $\widetilde{\mathcal{B}}_k$ be as in part (b). Then there exists a monomorphism $\rho : \widetilde{\mathcal{B}}_k \to \widetilde{\mathcal{A}}_k$ with $\rho(a_k) = a_k$ which sends $Z(\widetilde{\mathcal{B}}_k)$ to $\ker(\phi_k) = Z(\widetilde{\mathcal{A}}_k)$.*

Proof. (a) Let $\pi : \widetilde{E}(\mathcal{C}_k) \to \widetilde{E}(\mathcal{C}_k)/\mathcal{C}_k\widetilde{J}$ be the canonical map, and define

$$V := \pi^{-1}\Big(\Delta\big(\widetilde{E}(\mathcal{C}_k)/\mathcal{C}_k\widetilde{J}\big)\Big) \quad \text{(cf. Section 4.A for the endofunctor } \Delta \text{ of Mod-}\widetilde{\Lambda}\text{)}.$$

By construction, V is the unique largest submodule of $\widetilde{E}(\mathcal{C}_k)$ with the properties that (i') $\mathcal{C}_k\widetilde{J} \subseteq V$, and (ii') $V\widetilde{\mu} \subseteq \mathcal{C}_k\widetilde{J}$. Set $\mathcal{D}_k := V + \mathcal{C}_k$, and observe that V is a maximal submodule of $\mathcal{D}_k$ giving rise to a vector space decomposition $\mathcal{D}_k = V \oplus Ka_k$. Moreover, $\mathcal{D}_k$ has critical core $\mathcal{C}(\mathcal{D}_k) = \mathcal{C}_k$, which guarantees that $\mathcal{D}_k \in \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ due to Theorem 13. We will show that $\widetilde{\mathcal{A}}_k = \mathcal{D}_k$.

To that end, we first check that the submodule $W := \widetilde{\mathcal{A}}_k(1-\widetilde{\mu})\widetilde{\Lambda} + \mathcal{C}_k\widetilde{J}$ of $\widetilde{\mathcal{A}}_k$ has the following properties: $\widetilde{\mathcal{A}}_k = W \oplus Ka_k$ as a K-space, and W in turn satisfies (i') and (ii') above. Indeed, $W = \ker(\phi_k)$, and in light of $\widetilde{\mathcal{A}}_k\widetilde{\mu} \subseteq \mathcal{C}_k$ we see that $W\widetilde{\mu} \subseteq \mathcal{C}_k\widetilde{J}$. Consequently, $W \subseteq V$ due to the uniqueness condition enjoyed by V. Clearly, this implies $\widetilde{\mathcal{A}}_k \subseteq \mathcal{D}_k$. To obtain equality from the maximal choice of $\mathcal{A}_k$ under conditions (i) and (ii) of Definition 17, it thus suffices to check that $\mathcal{D}_k$ again satisfies these latter two conditions. That $\mathcal{D}_k$ has finite projective dimension has already been verified. Assume, to the contrary of (ii), that $a_k \in \mathcal{D}_k\widetilde{J}$. Then $\mathcal{C}_k = a_k\widetilde{\Lambda} \subseteq \mathcal{D}_k\widetilde{J} = V\widetilde{J} + \mathcal{C}_k\widetilde{J} \subseteq V$, which is incompatible with the fact that $a_k \notin V$ by construction. This shows that $\mathcal{D}_k$ indeed satisfies condition (ii), which entails the postulated equality $\widetilde{\mathcal{A}}_k = \mathcal{D}_k$.

(b) Let $g \in \mathrm{Hom}_{\widetilde{\Lambda}}(\widetilde{\mathcal{B}}_k, \widetilde{S}_k)$ be an epimorphism. From $\widetilde{S}_k = \widetilde{S}_k\widetilde{\mu}$, we obtain $g\big(\widetilde{\mathcal{B}}_k(1-\widetilde{\mu})\big) = 0$. In light of the equality $\widetilde{\mathcal{B}}_k = \widetilde{\mathcal{B}}_k(1-\widetilde{\mu})\widetilde{\Lambda} + \mathcal{C}_k$, we thus find the restriction of g to $\mathcal{C}_k$ to be nontrivial, and since $\mathcal{C}_k$ has a unique maximal submodule, its radical $\mathcal{C}_k\widetilde{J}$ coincides with the kernel of this restriction. This shows shows that $\ker(g) = \widetilde{\mathcal{B}}_k(1-\widetilde{\mu})\widetilde{\Lambda} + \mathcal{C}_k\widetilde{J}$ is indeed the only maximal submodule of $\widetilde{\mathcal{B}}_k$.

(c) By hypothesis, $\widetilde{\mathcal{B}}_k$ is an extension of $\mathcal{C}_k$ with $\operatorname{p\,dim} \widetilde{\mathcal{B}}_k/\mathcal{C}_k < \infty$, and hence $\widetilde{\mathcal{B}}_k$ belongs to $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ because $\mathcal{C}_k$ does. Since our hypotheses also guarantee that $a_k \in \widetilde{\mathcal{B}}_k \setminus \widetilde{\mathcal{B}}_k\widetilde{J}$, we find that $\widetilde{\mathcal{B}}_k$ satisfies conditions (i) and (ii) of Definition 17.

Extend the inclusion map $\mathcal{C}_k \subseteq \widetilde{E}(\mathcal{C}_k)$ to a homomorphism $\rho : \widetilde{\mathcal{B}}_k \to \widetilde{E}(\mathcal{C}_k)$. Clearly, ρ is a monomorphism because $\mathcal{C}_k$ is essential in $\widetilde{\mathcal{B}}_k$ and $\mathrm{Ker}(\rho) \cap \mathcal{C}_k = 0$. Therefore the essential extension $\rho(\widetilde{\mathcal{B}}_k) \subseteq \widetilde{E}(\mathcal{C}_k)$ of $\mathcal{C}_k$ again satisfies conditions (i) and (ii) of Definition 17, and consequently part (a) of the lemma ensures that $\rho(\widetilde{\mathcal{B}}_k) \subseteq \widetilde{\mathcal{A}}_k$. By construction, ρ restricts to the identity on $\mathcal{C}_k$; in particular $\rho(a_k) = a_k$ and $\rho(\mathcal{C}_k\widetilde{J}) = \mathcal{C}_k\widetilde{J}$. Since evidently $\rho\big(\widetilde{\mathcal{B}}(1-\widetilde{\mu})\big) \subseteq \widetilde{\mathcal{A}}(1-\widetilde{\mu})$, we conclude that $\rho\big(Z(\widetilde{\mathcal{B}}_k)\big) \subseteq Z(\widetilde{\mathcal{A}}_k) = \ker(\phi_k)$ is as required. $\square$

We are now in a position to prove the main result of this section.

Theorem 19. *For $k \in \{1, \dots, r\}$, the homomorphism $\phi_k : \widetilde{\mathcal{A}}_k \to \widetilde{S}_k$ introduced in* Definition 17 *is a minimal $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$-approximation of $\widetilde{S}_k$.*

In particular, $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ is contravariantly finite in mod-$\widetilde{\Lambda}$.

Proof. We recall that the noncritical simples in mod-$\widetilde{\Lambda}$ have finite projective dimension. Therefore the final claim follows from the first.

Let $k \le r$. To show that ϕ_k is a $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$-approximation of $\widetilde{S}_k$, suppose $\widetilde{M}$ is a right $\widetilde{\Lambda}$-module of finite projective dimension and $f \in \mathrm{Hom}_{\widetilde{\Lambda}}(\widetilde{M}, \widetilde{S}_k)$ an epimorphism. Then f factors through the quotient $\widetilde{M}/\Delta(\widetilde{M})$, the latter being again an object in $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$. So we may assume $\Delta(\widetilde{M}) = 0$. This adjustment makes $\widetilde{M}$ a $(\widetilde{\Lambda}/\widetilde{\Delta})$-module and f a $(\widetilde{\Lambda}/\widetilde{\Delta})$-homomorphism. In particular, $\mathcal{C}(\widetilde{M})$ is a submodule of $\widetilde{M}$ in this situation.

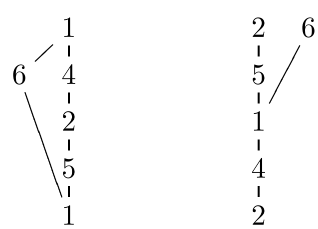
Theorem 13 yields a decomposition $\mathcal{C}(\widetilde{M}) = \bigoplus_{i=1}^{r} \bigoplus_{l=1}^{u_i} C_{il}$ with $u_i \ge 0$ and $C_{il} \cong \mathcal{C}_i$ for $l \le u_i$. Since all critical composition factors of $\widetilde{M}$ are subfactors of $\mathcal{C}(\widetilde{M})$, we find that the restriction of f to $\bigoplus_{l=1}^{u_k} C_{kl}$ is again an epimorphism onto $\widetilde{S}_k$; in particular $u_k \ge 1$. Moreover, we note that the C_{il} with $i \ne k$ belong to the kernel of f, given that each C_{il} is a local module with top $\widetilde{S}_i$. We further observe that it is harmless to assume $f(C_{k1}) = \widetilde{S}_k$ and $f(C_{kl}) = 0$ for $2 \le l \le u_k$; indeed, the C_{kl} are indecomposable projective over $\widetilde{\Lambda}/\widetilde{\Delta}$, which permits us to suitably shift the given decomposition

$\bigoplus_{l=1}^{u_k} C_{kl}$ to one that accommodates the specified condition. If $D := \bigoplus_{l=2}^{u_k} C_{kl} \oplus \bigoplus_{i\neq k} \bigoplus_{l=1}^{u_i} \mathcal{C}_{il}$, we thus have $f(D) = 0$. In trying to find a factorization of f through ϕ_k, we may therefore factor D out of $\widetilde{M}$. Since $\widetilde{M}/D$ in turn has finite projective dimension by Theorem 13, this reduces our argument to the situation where $\mathcal{C}(\widetilde{M}) = C_{k1} \cong \mathcal{C}_k$.

Write C for C_{k1}. By construction, f maps C onto $\widetilde{S}_k$. Now choose $U \subseteq \widetilde{M}$ to be maximal with the property that $U \cap C = 0$. Since $\widetilde{M}\widetilde{\mu} = C\widetilde{\mu}$, we find that $U\widetilde{\mu} = 0$. In particular, $U \subseteq \mathrm{Ker}(f)$ and $\mathrm{p\,dim}\, \widetilde{M}/U < \infty$ by Proposition 9. Our choice of U further guarantees that the embedding $\mathrm{in} : C \hookrightarrow \widetilde{M}/U$ is an essential extension with $\mathrm{in}(C) \not\subseteq (\widetilde{M}/U)\widetilde{J}$; indeed, given any top element a of C, we have $f(a) \neq 0$ in $\widetilde{S}_k$, whence $\mathrm{in}(a)$ is a top element of $\widetilde{M}/U$ normed by $\widetilde{e}_k$. Clearly, we do not lose generality by identifying $\mathrm{in}(C)$ with $\mathcal{C}_k$. This identification makes $\widetilde{\mathcal{B}}_k := \widetilde{M}/U$ an essential extension of $\mathcal{C}_k$ with $a_k \notin \widetilde{\mathcal{B}}_k\widetilde{J}$. As in the comments following Definition 17, one obtains $C = \mathcal{C}_k = \mathcal{C}(\widetilde{\mathcal{B}}_k)$. Since this implies $(\widetilde{\mathcal{B}}_k/\mathcal{C}_k)\widetilde{\mu} = 0$, Lemma 18(c) provides us with a monomorphism $g : \widetilde{\mathcal{B}}_k \to \widetilde{\mathcal{A}}_k$ which sends $a_k \in \widetilde{\mathcal{B}}_k$ to $a_k \in \widetilde{\mathcal{A}}_k$ and maps $Z(\widetilde{\mathcal{B}}_k)$ to $Z(\widetilde{\mathcal{A}}_k) = \ker(\phi_k)$. But, by Lemma 18(b), $Z(\widetilde{\mathcal{B}}_k)$ is the kernel of the epimorphism $\overline{f} : \widetilde{\mathcal{B}}_k \to \widetilde{S}_k$ induced by f. We conclude that $\overline{f}$ factors through ϕ_k: Indeed, if $\overline{f}(a_k) = \kappa \cdot (\widetilde{e}_k + \widetilde{e}_k\widetilde{J})$ for some scalar κ, then $\overline{f} = \phi_k \circ (\kappa \cdot g)$. But this implies that also f factors through ϕ_k and thus shows ϕ_k to be a $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of $\widetilde{S}_k$ as claimed.

To verify minimality of ϕ_k as a $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$-approximation of $\widetilde{S}_k$, we argue that $\widetilde{\mathcal{A}}_k$ is indecomposable. This is a consequence of the facts that $\widetilde{\mathcal{A}}_k$ is an essential extension of the local (hence indecomposable) module $\mathcal{C}_k = \widetilde{\mathcal{A}}_k\, \widetilde{\mu}\widetilde{\Lambda}$ as follows. Indeed, if $\widetilde{\mathcal{A}}_k = A_1 \oplus A_2$, then $\mathcal{C}_k = A_1\, \widetilde{\mu}\widetilde{\Lambda} \oplus A_2\, \widetilde{\mu}\widetilde{\Lambda}$, and consequently one of the two latter summands is zero, say $A_2\, \widetilde{\mu}\widetilde{\Lambda} = 0$. Then $a_k \in A_1$, which means $\mathcal{C}_k \subseteq A_1$. Now essentiality of $\mathcal{C}_k$ in $\widetilde{\mathcal{A}}_k$ yields $A_2 = 0$. In view of [**4**, Proposition 1.1], the postulated minimality of ϕ_k is therefore automatic. $\square$

Return to the Reference Example in 3.B. The minimal approximations $\widetilde{\mathcal{A}}_i$ of the critical simple right $\widetilde{\Lambda}$-modules $\widetilde{S}_1$ and $\widetilde{S}_2$ are determined by their graphs, namely:



8. Iterated strong tilting. Theorem D

In light of Theorem I of Section 2.A, Theorem 19 guarantees the existence of strong tilting objects in mod-$\widetilde{\Lambda}$. From Supplement II to this theorem we moreover know that the basic strong tilting module $\widetilde{T} \in \text{mod-}\widetilde{\Lambda}$ belongs to

$$\mathrm{add}\Big(\bigoplus_{i=1}^{n} \widetilde{\mathcal{A}}\big(\widetilde{E}(\widetilde{S}_i)\big)\Big),$$

where $\widetilde{E}(-)$ again denotes $\widetilde{\Lambda}$-injective envelopes, and $\widetilde{\mathcal{A}}(-)$ assigns to every object in mod-$\widetilde{\Lambda}$ (the isomorphism class of) its minimal $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$-approximation. In fact, we know $\widetilde{T}_{\widetilde{\Lambda}}$ to be the direct sum of one copy of each of the n indecomposable objects in this category.

Towards an analysis of the left module ${}_{\widetilde{\widetilde{\Lambda}}}\widetilde{T}$, where $\widetilde{\widetilde{\Lambda}} = \mathrm{End}_{\widetilde{\Lambda}}(\widetilde{T})$, we explore the socles of the $\widetilde{\mathcal{A}}\big(\widetilde{E}(\widetilde{S}_i)\big)$. Clearly, the injective $\widetilde{\Lambda}$-module $\widetilde{E}(\widetilde{S}_i)$ embeds into its approximation $\widetilde{\mathcal{A}}\big(\widetilde{E}(\widetilde{S}_i)\big)$ precisely when it belongs to $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$, i.e., precisely when $\widetilde{E}(\widetilde{S}_i)$ coincides with its minimal approximation; this fails in general. On the other hand, the following weakened statement holds.

Lemma 20. *The simple right $\widetilde{\Lambda}$-module $\widetilde{S}_i$ is contained in the socle of $\widetilde{\mathcal{A}}(\widetilde{E}(\widetilde{S}_i))$.*

Proof. Fix i, and abbreviate $\widetilde{\mathcal{A}}(\widetilde{E}(\widetilde{S}_i))$ to $\widetilde{\mathcal{A}}$.

Case 1. Let $i \in \{1, \dots, r\}$. We first show that $\Delta(\widetilde{\mathcal{A}}) = 0$. To see this, we note that $\Delta(\widetilde{E}(\widetilde{S}_i)) = 0$, because the socle $\widetilde{S}_i$ of $\widetilde{E}(\widetilde{S}_i)$ is critical. Given that Δ is an endofunctor, we deduce that the approximation $\phi : \widetilde{\mathcal{A}} \to \widetilde{E}(\widetilde{S}_i)$ factors through $\widetilde{\mathcal{A}}/\Delta(\widetilde{\mathcal{A}})$. Due to the fact that the quotient $\widetilde{\mathcal{A}}/\Delta(\widetilde{\mathcal{A}})$ again has finite projective dimension, we therefore obtain $\Delta(\widetilde{\mathcal{A}}) = 0$ from minimality of $\widetilde{\mathcal{A}}$. In other words, the socle of $\widetilde{\mathcal{A}}$ consists of critical simples, i.e., $\operatorname{soc}\widetilde{\mathcal{A}} = (\operatorname{soc}\widetilde{\mathcal{A}})\widetilde{\mu}$; in particular, the critical core $\mathcal{C}(\widetilde{\mathcal{A}})$ is a submodule of $\widetilde{\mathcal{A}}$.

Now we use Proposition 12(b) to infer that $\widetilde{S}_i$ is contained in the socle of $\widetilde{\mathcal{A}}$. Indeed, this corollary tells us that $\widetilde{E}(\widetilde{S}_i)$ is not annihilated by $(\widetilde{\mu}\widetilde{J}\widetilde{\mu})^L$; neither is $\widetilde{\mathcal{A}}$, given that ϕ is a surjection. Recall, moreover, that $(\widetilde{\mu}\widetilde{J}\widetilde{\mu})^{L+1} = 0$ by Lemma 11. Consequently, $\widetilde{E}(\widetilde{S}_i)(\widetilde{\mu}\widetilde{J}\widetilde{\mu})^L$ equals the socle $\widetilde{S}_i = \widetilde{S}_i\widetilde{\mu}$ of $\widetilde{E}(\widetilde{S}_i)$, and $\widetilde{\mathcal{A}}\,(\widetilde{\mu}\widetilde{J}\widetilde{\mu})^L \subseteq \operatorname{soc}\widetilde{\mathcal{A}}\,\widetilde{\mu} = \operatorname{soc}\widetilde{\mathcal{A}}$. In light of the equality $\phi\big(\widetilde{\mathcal{A}}\,(\widetilde{\mu}\widetilde{J}\widetilde{\mu})^L\big) = \widetilde{S}_i$, we thus conclude that $\operatorname{soc}\widetilde{\mathcal{A}}$ indeed contains a copy of $\widetilde{S}_i$.

Case 2. Let $i \in \{r+1, \dots, n\}$. By Proposition 9, the simple $\widetilde{S}_i$ then has finite projective dimension in mod-$\widetilde{\Lambda}$. Thus, if $\phi : \widetilde{\mathcal{A}} \to \widetilde{E}(\widetilde{S}_i)$ is a $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$-approximation, the embedding $\widetilde{S}_i \hookrightarrow \widetilde{E}(\widetilde{S}_i)$ factors through ϕ. This yields a monomorphism $\widetilde{S}_i \hookrightarrow \widetilde{\mathcal{A}}$ in the present case as well. $\square$

We proceed to Theorem D stated in the introduction.

Theorem 21. *Let $\widetilde{T}_{\widetilde{\Lambda}}$ be a basic strong tilting module in $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ and $\widetilde{\widetilde{\Lambda}} = \mathrm{End}_{\widetilde{\Lambda}}(\widetilde{T})$. Then the $\widetilde{\widetilde{\Lambda}}$-$\widetilde{\Lambda}$ tilting bimodule $\widetilde{T}$ is strong also as a left $\widetilde{\widetilde{\Lambda}}$-module, and the strongly tilted algebra $\mathrm{End}_{\widetilde{\widetilde{\Lambda}}}(\widetilde{T})^{\mathrm{op}}$ is isomorphic to $\widetilde{\Lambda}$.*

Proof. By [**4**, Proposition 6.5] (which the authors attribute to Auslander and Green), the first claim is equivalent to the requirement that all simple right $\widetilde{\Lambda}$-modules embed into $\widetilde{T}_{\widetilde{\Lambda}}$. For a short alternative proof of this fact based on Miyashita's duality, we refer to [**11**, Proposition 7.1]. Thus Lemma 20 provides what we need. The final assertion follows from balancedness of the bimodule ${}_{\widetilde{\widetilde{\Lambda}}}T_{\widetilde{\Lambda}}$. $\square$

In particular, Theorem 21 yields mutually inverse dualities

$$\mathrm{Hom}_{\widetilde{\Lambda}}(-,\widetilde{T}) :\ \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda}) \quad\longleftrightarrow\quad \mathcal{P}^{<\infty}(\widetilde{\widetilde{\Lambda}}\text{-mod})\ : \mathrm{Hom}_{\widetilde{\widetilde{\Lambda}}}(-,\widetilde{T}),$$

which permit us to pivot the structural information garnered for the objects of $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ to those of $\mathcal{P}^{<\infty}(\widetilde{\widetilde{\Lambda}}\text{-mod})$.

References

1. F. W. Anderson and K.R. Fuller, *Rings and categories of modules, second edition*, Springer, New York, Berlin, 1992.

2. L. Angeleri-Huegel, D. Happel, and H.Krause (eds.), *Handbook of tilting theory*, LMS Lecture Note Series, vol. 332, Cambridge Univ. Press, 2007.

3. M.Auslander, M.I. Platzeck and I. Reiten, *Coxeter functors without diagrams*, Trans. Amer. Math. Soc. **250** (1979), 1-46.

4. M. Auslander and I. Reiten, *Applications of contravariantly finite subcategories*, Adv. in Math. **86** (1991), 111–152.

5. M. Auslander and S.O. Smalø, *Preprojective modules over artin algebras*, J. Algebra **66** (1980), 61-122.

6. I.N. Bernstein, I.M. Gel'fand and V.A. Ponomarev, *Coxeter functors and Gabriel's Theorem*, Uspehi Math. Nauk **28** (1973), 19-33.

7. K. Bongartz, *Tilted algebras*, Representations of algebras (Puebla 1980), Lecture Notes in Math. 903, Springer, Berlin, 1981, pp. 26-38.

8. S. Brenner and M.C.R. Butler, *Generalizations of the Bernstein-Gel'fand-Ponomarev reflection functors*, Representation Theory II, Lecture Notes in Math. 832, Springer, Berlin, 1980, pp. 103-169.

9. W. Burgess and B. Huisgen-Zimmermann, *Approximating modules by modules of finite projective dimension*, J. Algebra **178** (1995), 48-81.

10. E. Cline, B. Parshall and L. Scott, *Derived categories and Morita theory*, J. Algebra **104** (1986), 397 - 409.

11. A. Dugas and B. Huisgen-Zimmermann, *Strongly tilting truncated path algebras*, Manuscripta math. **134** (2011), 225-257.

12. A. Dugas, B. Huisgen-Zimmermann and J. Learned, *Truncated path algebras are homologically transparent*, Models, Modules and Abelian Groups (R. Goebel and B.Goldsmith eds.), de Gruyter, Berlin, 2008, pp. 445-461.

13. K.R. Goodearl and B. Huisgen-Zimmermann, *Closures in module varieties and irreducible components*, Algebra and Number Theory **12** (2018), 379-410.

14. D. Happel, *On the derived category of a finite dimensional algebra*, Comm. Math. Helv. **62** (1987), 339-389.

15. D. Happel and C. M. Ringel, *Tilted algebras*, Trans. Amer. Math. Soc. **274** (1982), 399-433.

16. D. Happel and L. Unger, *On a partial order of tilting modules*, Algebras and Rep. Theory **8** (2005), 147-156.

17. B. Huisgen-Zimmermann and M. Saorín, *Tilting, contravariant finiteness, and dualities*, in preparation.

18. B. Huisgen-Zimmermann and S.O. Smalø, *A homological bridge between finite and infinite dimensional representations*, Algebras and Representation Theory **1** (1998), 169-188.

19. B. Huisgen-Zimmermann and S.O. Smalø, *The homology of string algebras I*, J. reine angew. Math. **580** (2005), 1-37.

20. K. Igusa, S.O. Smalø and G. Todorov, *Finite projectivity and contravariant finiteness*, Proc. Amer. Math. Soc. **109** (1990), 937-941.

21. O. Iyama and X. Zhang, *Tilting modules over Auslander-Gorenstein algebras*, arXiv:1801.04738.

22. B. Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), 63-102.

23. S. König and A. Zimmermann, *Derived equivalences for group rings*, Lecture Notes in Math. 1685, Springer, 1998.

24. Y. Miyashita, *Tilting modules of finite projective dimension*, Math. Z. **193** (1986), 113-146.

25. J. Rickard, *Morita theory for derived categories*, J. London Math. Soc. **39** (1989), 436-456.

Department of Mathematics, University of California, Santa Barbara, CA 93106
E-mail address: `birge@math.ucsb.edu`