Quasi-Duo Rings and Stable Range Descent

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Abstract

In a recent paper, the first author introduced a general theory of corner rings in noncommutative rings that generalized the classical theory of Peirce decompositions. This theory is applied here to the study of the stable range of rings upon descent to corner rings. A ring is called *quasi-duo* if every maximal 1-sided ideal is 2-sided. Various new characterizations are obtained for such rings. Using some of these characterizations, we prove that, if a quasi-duo ring R has stable range $\leq n$, the same is true for any semisplit corner ring of R. This contrasts with earlier results of Vaserstein and Warfield, which showed that the stable range can increase unboundedly upon descent to (even) Peirce corner rings.

§1. Introduction

In a recent work [La_5], the first author introduced a general theory of corner rings by defining a subring S of a ring R to be a *corner ring* (or simply a corner of R) if $R = S \oplus C$ for an additive subgroup $C \subseteq R$ such that $SC \subseteq C$ and $CS \subseteq C$. (Any such C is called a *complement* of S in R. Note that the identity of S may be different from that of R.) The best known class of examples is given by the Peirce corners $R_e := eRe$ [Pe], where e is any idempotent in R. (It is shown in [La_5: (2.7)] that R_e has a *unique* complement $C_e := eRf \oplus fRe \oplus fRf$.) In general, if a corner ring S has identity e, then S is a unital corner of its "associated Peirce corner" R_e; for more details, see [La_5: (5.1)]. We say S is a *split corner* if it has an *ideal* complement in R, and S is a *semisplit corner* if it is a split corner in its associated Peirce corner. (Obviously, all Peirce corners are semisplit.) Criteria for split and semisplit corners were given in [La_5: (5.4), (5.8)].

In [La_5], the first author developed many of the basic facts on corner rings and their complements, and proved various theorems on the preservation of ring-theoretic properties by corner rings (or at least by semisplit corner rings). In this sequel to [La_5], we apply the corner ring theory to the study of the stable range of rings. In the standard literature on the stable range (see, e.g. [Va_1], [Wa]), it is shown that stable range $\leq n$ is a Morita invariant property of rings only for $n = 1$, and not for other n's. Specifically, if a ring R has stable range n with $n \geq 2$, its Peirce corner rings R_e (even for full idempotents e) may have *arbitrarily large* stable range. In attempting to understand this phenomenon, we found that one "obstruction" to the preservation of stable range upper bounds lies in the lack of symmetry between left and right unimodularity in an arbitrary ring R. The main purpose of this paper is essentially to report this finding.

After a short preparatory section on unimodularity sequences in general corner rings, we introduce in §3 the class of *right quasi-duo rings*: rings in which maximal right ideals are ideals. We characterize these as rings in which left unimodularity implies right unimodularity. Using this new viewpoint on right quasi-duo rings, we re-examine in §4 the problem of determining the subclass of right quasi-duo rings in certain classes of rings, including right primitive rings, semilocal rings, von Neumann regular rings, π-regular rings, and 1-sided self-injective rings. This work led to the question whether a ring R is (right) quasi-duo if $R/\mathrm{rad}(R)$ is a subdirect product of division rings: this question is answered negatively in §5.

In §6, we prove the main result (6.3), to the effect that, for any (right and left) quasi-duo ring R, the stable range of any semisplit corner ring of R is bounded by the stable range of R. This theorem provides an interesting contrast to the well known results of Vaserstein and Warfield mentioned above. Thus, the more general viewpoint of corners introduced in [La_5] has enabled us to get new results even within the classical framework of Peirce corner rings.

Much (if not all) of this paper can be read independently of [La_5]. Throughout the note, R denotes a ring with an identity element $1 = 1_R$, and by the word "subring", we shall always mean a subgroup $S \subseteq R$ that is closed under multiplication (hence a ring in its own right), but with an identity element possibly different from 1_R. If 1_R happens to be in S (so it is also the identity of S), we'll say that S is a *unital subring* of R. Other general notations and conventions in this paper follow closely those used in [La_1], [La_3], [La_5], and [Fa].

§2. Unimodularity in Corner Rings

A sequence $r_1, \ldots, r_n$ in a ring R is said to be *left unimodular* if the r_i's generate R as a left ideal, and *right unimodular* if the r_i's generate R as a right ideal (that is, respectively, $Rr_1 + \cdots + Rr_n = R$, and $r_1R + \cdots + r_nR = R$). We begin by proving a result that establishes a basic connection between the left unimodular sequences in R and those in a general corner ring of R.

(2.1) Theorem. *Let S be a corner of R with identity e, and let $f = 1 - e$. For any $s_1, \ldots, s_n \in S$, the following are equivalent:*

(1) $\{s_1, \ldots, s_n\}$ *is left unimodular in* S*;*

(2) $\{s_1 + f, \ldots, s_n + f\}$ *is left unimodular in* R*;*

(3) $\exists\, b_1, \ldots, b_n \in Rf$ *such that* $\{s_1 + b_1, \ldots, s_n + b_n\}$ *is left unimodular in* R*.*

(4) $\forall\, b_2, \ldots, b_n \in fR$, $\{s_1 + f, s_2 + b_2, \ldots, s_n + b_n\}$ *is left unimodular in* R*.*

Proof. (4) $\Rightarrow$ (2) $\Rightarrow$ (3) are tautologies. For (3) $\Rightarrow$ (1), assume there exists an equation $\sum_{i=1}^n r_i(s_i + b_i) = 1$, where $b_i \in Rf$ and $r_i \in R$. Fix a complement C for S, and recall from [La$_5$: (5.1)(3)] that C must contain $C_e := eRf \oplus fRe \oplus fRf$. Writing $r_i = t_i + c_i$ with $t_i \in S$ and $c_i \in C$, we have

$$\sum_{i=1}^{n} (t_i s_i + c_i s_i + r_i b_i) = 1 = e + f, \tag{2.2}$$

Here, $t_i s_i \in S$, $c_i s_i \in C$, and

$$r_i b_i \in r_i R f \subseteq Rf = eRf \oplus fRf \subseteq C_e \subseteq C.$$

Thus, (2.2) implies that $\sum_{i=1}^n t_i s_i = e$, and thus $\sum_{i=1}^n S s_i = S$. Finally, for (1) $\Rightarrow$ (4), fix an equation $\sum_{i=1}^n s_i' s_i = e$, where $s_i' \in S$, and take any $b_2, \ldots, b_n \in fR$. Setting $b_1 = f$, we have, for all i, $s_i' b_i \in (s_i' e)(fR) = 0$, and $b_1 s_1 = f(es_1) = 0$. Thus,

$$(s_1' + b_1)(s_1 + b_1) + \sum_{i=2}^{n} s_i'(s_i + b_i) = s_1' s_1 + \cdots + s_n' s_n + b_1^2 = e + f = 1.$$

Therefore, $\sum_{i=1}^n R(s_i + b_i) = R$ (with $b_1 = f$), as desired. $\square$

This result will be crucial for proving our main theorem (6.3).

§3. Right Quasi-Duo Rings and Unimodularity

A ring R is said to be *right duo* (resp. *right quasi-duo*) if every right ideal (resp. maximal right ideal) of R is an ideal. Obviously, right duo rings are right quasi-duo. Other examples of right quasi-duo rings include, for instance, commutative rings, local rings, rings in which every nonunit has a (positive) power that is central, endomorphism rings of uniserial modules, and power series rings and rings of upper triangular matrices over any of the afore-mentioned rings (see [Yu]). Also, it is easy to see that

(3.0). *If a ring* R *is right duo (resp. right quasi-duo), so is any factor ring of* R*.*

The condition that maximal right ideals be two-sided has appeared (without a name) in the investigation of Burgess and Stephenson [BS] on rings all of whose Pierce stalks are local (and quite possibly even earlier). The name "right quasi-duo" for this condition was coined by H.-P. Yu, who initiated the first substantial study of right quasi-duo rings in [Yu]. This study was continued in the work of other authors, in [Ki], [KKJ], [KL], etc. But surprisingly, some very natural characterizations of such rings seemed to have so far escaped notice. In this section, we begin by offering some of these new characterizations of right quasi-duo rings, which are in terms of left and right unimodular sequences.

For any integer $n \geq 1$, let "right D_n" denote the following condition on a ring R:

$$\forall\, r_i \in R, \quad Rr_1 + \cdots + Rr_n = R \Longrightarrow r_1R + \cdots + r_nR = R. \tag{3.1}$$

("Left D_n" is defined similarly.) For $n = 1$, this amounts to the classical notion of Dedekind-finiteness (and is therefore left-right symmetric). For $n \geq 2$, these conditions turn out to be all equivalent to R being right quasi-duo, as the following theorem shows. Note that the conditions (2) through (7) in this theorem amount to characterizations of "right quasi-duo" in terms of *first-order statements* on the ring $(R, +, \times)$.

Theorem 3.2. *For any ring R, the following statements are equivalent:*

(1) *R is right quasi-duo.*
(2) *R satisfies right D_n for all n.*
(3) *R satisfies right D_n for some $n \geq 2$.*
(4) *For some $n \geq 2$, any subset $\{x_1, y_1, \ldots, x_{n-1}, y_{n-1}\} \subseteq R$ satisfies*

$$x_1R + \cdots + x_{n-1}R + (y_1x_1 + \cdots + y_{n-1}x_{n-1} - 1)R = R. \tag{3.3}$$

(5) (3.3) *holds for all $n \geq 2$ and all $x_1, y_1, \ldots, x_{n-1}, y_{n-1} \in R$.*
(6) *For any $x, y \in R$, $xR + (yx - 1)R = R$.*
(7) *For any finite set $S \subseteq R$, $RSR = R \Longrightarrow SR = R$.*

(In particular, right quasi-duo rings are always Dedekind-finite.)

Proof. (1) $\Rightarrow$ (2). Let $x_1, \ldots, x_n \in R$ be such that $x_1R + \cdots + x_nR \neq R$. Then $x_1R + \cdots + x_nR$ is contained in a maximal right ideal $\mathfrak{m}$ of R. By (1), $\mathfrak{m}$ is an ideal. Since $x_i \in \mathfrak{m}$ for all i, it follows that $Rx_1 + \cdots + Rx_n \subseteq \mathfrak{m}$, and so $Rx_1 + \cdots + Rx_n \neq R$.

(2) $\Rightarrow$ (3) is a tautology.

(3) $\Rightarrow$ (4). Suppose R satisfies right D_n for a fixed $n \geq 2$. Given x_i, $y_i \in R$ ($1 \leq i \leq n-1$), the sequence

$$\{x_1, \ldots, x_{n-1}, y_1x_1 + \cdots + y_{n-1}x_{n-1} - 1\} \tag{3.4}$$

is always left unimodular, in view of

$$y_1 \cdot x_1 + \cdots + y_{n-1} \cdot x_{n-1} - (y_1x_1 + \cdots + y_{n-1}x_{n-1} - 1) = 1. \tag{3.5}$$

Since R satisfies right D_n, it follows that the sequence (3.4) is right unimodular, as desired.

(4) $\Rightarrow$ (6) is clear, upon setting $x_i = y_i = 0$ for all $i \geq 2$ in (3.3).
(6) $\Rightarrow$ (1). If (1) fails, there would exist a maximal right ideal $\mathfrak{m} \subset R$ that is not an ideal. Take an element $y \in R$ such that $y\,\mathfrak{m} \nsubseteq \mathfrak{m}$. Since $y\,\mathfrak{m}$ is also a right ideal, we have $\mathfrak{m} + y\,\mathfrak{m} = R$, so there exists an equation $1 = m + yx$ where $m, x \in \mathfrak{m}$. This leads to $xR + (yx-1)R = xR + mR \subseteq \mathfrak{m} \subsetneq R$, so (6) does not hold.

The argument for (3) $\Rightarrow$ (4) also shows (2) $\Rightarrow$ (5), and (5) $\Rightarrow$ (4) is a tautology. This proves the equivalence of (1) through (6). Finally, (7) is just a slight reformulation of (2). $\square$

As an immediate application of the first-order characterization (3.2)(6) for right quasi-duo rings, we have the following result.

(3.6) Corollary. (1) *If R is a direct product of the rings $\{R_i : i \in I\}$, then R is right quasi-duo iff each R_i is.*

(2) *If R is a subdirect product of a finite family of rings $\{R_i : 1 \leq i \leq n\}$, then R is right quasi-duo iff each R_i is.*

(3) *If $\{R_i : i \in I\}$ is a directed system of right quasi-duo rings, then the direct limit of these rings is also right quasi-duo.*

(4) *The ultraproduct of any family of right quasi-duo rings is right quasi-duo.*

Proof. (3) is clear from the characterization (3.2)(6), since every pair of elements x, y in the direct limit "comes from" some ring R_i in the system. A similar reasoning gives the "if" part of (1), and this (together with (3.0)) implies (4). The "only if" parts in (1) and (2) likewise follow from (3.0).

The last step is now to prove the "if" part in (2). For this, we may induct on n, and reduce the consideration to the key case $n = 2$. In this case, for convenience, we represent R_i in the form R/A_i, where A_1, A_2 are ideals in R with intersection (0). Consider any $x, y \in R$. Since R/A_i is right quasi-duo, there exists an equation $xr_i + (yx-1)s_i = 1 + a_i$, where $a_i \in A_i$. and $r_i, s_i \in R$ (for $i = 1, 2$). Therefore,

$$(xr_1 + (yx-1)s_1 - 1)(xr_2 + (yx-1)s_2 - 1) = a_1a_2 \in A_1A_2 \subseteq A_1 \cap A_2 = 0.$$

Writing $t = xr_2 + (yx-1)s_2 - 1$, the LHS above has the form

$$xr_1t + (yx-1)s_1t - (xr_2 + (yx-1)s_2 - 1) \in 1 + xR + (yx-1)R.$$

This gives $xR + (yx-1)R = R$ (for all $x, y \in R$), as desired. $\square$

(3.7) Remark. The proof given in the last paragraph (for the "if" part of (2)) depended heavily on working with finite subdirect products of right quasi-duo rings (for which we can carry out an induction). In fact, the "if" part in (2) fails to hold in general for *infinite* (or even countable) subdirect products. Examples to this effect will be given in §5.

The right D_n conditions do not seem to have been explicitly stated before. However, some related notions have previously appeared in the literature. In extending a result of Herstein and Small [HS], Lenagan [Le] considered the property (3.1) for *pairwise commuting* elements $\{r_i\}$ in R. In parallel to our definition of "right D_n", let us say that R satisfies *right* S_n if (3.1) holds for all pairwise commuting elements $\{r_i\} \subseteq R$. (The letter "S" comes from "Schur": see [HS].) This is a weak version of our right D_n property. In [HR], Handelman and Raphael studied right S_2 rings, and applied this notion to their work on pseudo rank functions on von Neumann regular rings. In [Le], it is shown that Artinian (in fact semilocal) rings are "Schur rings"; that is, they are left/right S_n for all n.[1] Later, Burgess and Menal [BM] showed that strongly π-regular rings are also Schur rings. (I thank K. Goodearl for bringing some of this literature to my attention.) In this paper, our main focus will be on the right D_n property (that is, the right quasi-duo property, in case $n \geq 2$); results on the right S_n properties will be given only if they can be obtained by the same methods as in the D_n case.

The following theorem guarantees the descent of the right D_n and right S_n properties to *arbitrary* corner rings. The proofs of results such as (2.1) and (3.8) demonstrate rather clearly the utility of the notions of general corner rings and their complements.

(3.8) Theorem. *For a fixed $n \geq 1$, if R satisfies right D_n (resp. right S_n), then so does any corner ring S of R. (Thus, corners of Dedekind-finite rings remain Dedekind-finite, and corners of right quasi-duo rings remain right quasi-duo.)*

Proof. Let e be the identity of S, and $f = 1 - e$ as in (2.1). For convenience, we fix a complement C for S in R. Say $\sum_{i=1}^n Ss_i = S$, where $s_i \in S$. Using (2.1), we have $\sum_{i=1}^n R(s_i + f) = R$. Since R satisfies right D_n, we have $\sum_{i=1}^n (s_i + f)R = R$. Applying now the right analogue of (2.1), we see that $\sum_{i=1}^n s_i S = S$, so $\{s_1, \ldots, s_n\}$ is right unimodular in S. This shows that S satisfies right D_n. The same proof works in the right S_n case, for, if the s_i's are pairwise commuting elements in the above, then so are the elements $\{s_i + f\}$, since $s_i = es_ie$ implies that $s_if = fs_i = 0$. The parenthetical statement now follows from Th. (3.2). $\square$

§4. Special Classes of Right Quasi-Duo Rings

Within certain classes of rings, it is often possible to characterize the right quasi-duo rings by other more familiar conditions. In this section, we'll recapitulate some known results in this direction, with self-contained proofs wherever possible. For the ring-theoretic terms used below, we refer the reader to Facchini's book [Fa]. Let us begin with the class of right primitive rings.

[1] Caution on terminology: the Schur rings discussed here are not to be confused with the Schur rings over groups in algebraic combinatorics.

(4.1) Proposition. [Yu] *A right primitive ring R is right quasi-duo iff R is a division ring.*

Proof. ("Only if" part.) Let S be a faithful simple right R-module. Then $S \cong R/\mathfrak{m}$ for a suitable maximal right ideal $\mathfrak{m} \subseteq R$. If R is right quasi-duo, $\mathfrak{m}$ is an ideal. But then $S \cdot \mathfrak{m} = 0$, so $\mathfrak{m} = 0$, which implies that R is a division ring. □

(4.2) Remarks. (1) Since simple rings are primitive, (4.1) applies to show that (left or right) quasi-duo simple rings are division rings. (2) If "right quasi-duo" is replaced by "right duo", the proposition (4.1) was known much earlier; see, for instance, Lemma 3.1 in [Ch].

(4.3) Proposition. [Yu] (1) *A ring R is right quasi-duo iff $R/\mathrm{rad}\,(R)$ is.*
(2) *If R is right quasi-duo, then $R/\mathrm{rad}(R)$ is a subdirect product of division rings; in particular, $R/\mathrm{rad}(R)$ is a reduced ring.*

Proof. (1) is clear from (3.0) and the fact that any maximal right ideal contains $\mathrm{rad}(R)$. For (2), assume R is right quasi-duo. Then each maximal right ideal $\mathfrak{m}_i \subset R$ is an ideal, so $R/\mathfrak{m}_i$ is a division ring. Since $\bigcap_i \mathfrak{m}_i = \mathrm{rad}(R)$, it follows that $R/\mathrm{rad}(R)$ is a subdirect product of the division rings $\{R/\mathfrak{m}_i\}$. □

(4.4) Remark. The reducedness of $R/\mathrm{rad}(R)$ in (2) can also be seen directly from the characterizing property (3.2)(6) of a right quasi-duo ring R, as follows. We may assume, without loss of generality, that $\mathrm{rad}(R) = 0$. It suffices to show in this case that $x^2 = 0 \Rightarrow x = 0$. Assuming that $x^2 = 0$, we have for any $y \in R$,

$$(yx - 1)x = yx^2 - x = -x,$$

so $x \in (yx - 1)R$. Thus, (3.2)(6) implies that

$$R = xR + (yx - 1)R = (yx - 1)R.$$

Since R is Dedekind-finite, this gives $1 - yx \in \mathrm{U}(R)$ (for all $y \in R$). Thus, $x \in \mathrm{rad}(R) = 0$. For more discussions on (4.3)(2), see §5.

(4.5) Corollary. *Let $R = \mathbb{M}_m(k)$, where k is a nonzero ring and $m \geq 2$. Then R is not right (or left) quasi-duo. (In particular, "right quasi-duo" is not a Morita invariant property of rings.)*

Proof. This follows from (4.3)(2) since $R/\mathrm{rad}(R) \cong \mathbb{M}_m(k/\mathrm{rad}(k))$ is not reduced. □

Note that, while the reducedness of $R/\mathrm{rad}(R)$ is a necessary condition for R to be right quasi-duo, it is not a sufficient condition in general. For instance, take any right primitive domain R that is not a division ring (e.g. the free algebra $R = \mathbb{Q}\langle x, y\rangle$).

Then $R/\mathrm{rad}(R) = R$ is reduced, but R is not right quasi-duo by (4.1). However, for certain classes of rings, it can be shown that the reducedness of $R/\mathrm{rad}(R)$ is also *sufficient* for R to be right (and left) quasi-duo. We quote the following result from [Yu: (4.1)], a part of which is based on [BS]; see also [St: Lemma 4.10].

(4.6) Theorem. *Let R be a ring such that $\overline{R} := R/\mathrm{rad}(R)$ is an exchange ring. Then the following statements are equivalent:*

(1) *R is right quasi-duo;*

(2) *R is quasi-duo (that is, left and right quasi-duo);*

(3) *$\overline{R}$ is a reduced ring;*

(4) *$\overline{R}$ is abelian (that is, idempotents are central in $\overline{R}$).*

(4.7) Corollary *Let R be a ring such that $\overline{R} := R/\mathrm{rad}(R)$ is π-regular. Then the statements* (1)–(4) *in* (4.6) *are equivalent; moreover, they are also equivalent to each of the following:*

(5) *$\overline{R}$ is strongly regular;*

(6) *$\overline{R}$ is a duo ring.*

Proof. Since a π-regular ring is an exchange ring, (4.6) applies to give the equivalence of (1)–(4). (5) $\Rightarrow$ (6) is well-known, and (6) $\Rightarrow$ (2) is clear from (4.3)(1). Thus, we are done if we can show that (3) $\Rightarrow$ (5). This implication is a part of [LH: Lemma 4]; here is an ad hoc proof. Assume that $\overline{R}$ is reduced and π-regular, and consider any $x \in \overline{R}$. By results of Azumaya in [Az], $\overline{R}$ is strongly π-regular, and in fact $x^n = x^{n+1}r$ for some $n \geq 1$ and some element $r \in \overline{R}$ commuting with x (see also [Di], [Hi], or [La_2: Ex. (23.6)]). Therefore,

$$\begin{aligned}(x - x^2r)^n &= [x(1 - xr)]^n = x^n(1 - xr)^n \\ &= x^n(1 - xr)(1 - xr)^{n-1} \\ &= (x^n - x^{n+1}r)(1 - xr)^{n-1} = 0.\end{aligned}$$

Since $\overline{R}$ is reduced, we have $x - x^2r = 0$. Since this holds for every $x \in \overline{R}$, the ring $\overline{R}$ is strongly regular. $\square$

Since quite a few classes of rings satisfy the hypotheses of (4.6) and (4.7), these two results apply well to the determination of quasi-duo rings within those classes of rings. For sample applications, we have the following.

(4.8) Corollary. (1) *A semilocal ring R is right (left) quasi-duo iff $R/\mathrm{rad}(R)$ is a finite direct product of division rings.*

(2) *A von Neumann regular ring R is right (left) quasi-duo iff R is duo, iff R is strongly regular.*

(3) *A one-sided self-injective ring R is right (left) duo iff $R/\mathrm{rad}(R)$ is reduced, iff $R/\mathrm{rad}(R)$ is strongly regular.*

For further consequences of (4.6) and (4.7), recall that a ring R is 2-*primal* if every nilpotent element of R lies in the lower nil radical (or prime radical) $\mathrm{Nil}_*(R)$. From (4.6) and (4.7), we can deduce the following result.

(4.9) Corollary. (1) [BS] *Any abelian exchange ring is quasi-duo.*

(2) *Any right quasi-duo π-regular ring is strongly π-regular.*

(3) *Any 2-primal π-regular ring is quasi-duo (and therefore strongly π-regular).*

Proof. (1) Let R be an abelian exchange ring. It is well-known that the factor ring $\overline{R} = R/\mathrm{rad}(R)$ is also an exchange ring. Thus, to see that R is quasi-duo, it suffices (by (4.6)) to check that $\overline{R}$ is abelian. Let $\overline{e}$ be any idempotent in $\overline{R}$. By [Ni], $\overline{e}$ can be lifted to an idempotent $e \in R$. By assumption, e is central in R; this implies that $\overline{e}$ is central in $\overline{R}$, as desired.

(2) Let R be any right quasi-duo ring R. We claim that *any (von Neumann) regular element $x \in R$ lies in x^2R.* To see this, write $x = xyx$, where $y \in R$. By the property (3.2)(6), we have an equation $1 = xr + (yx - 1)r'$, for suitable $r, r' \in R$. Left multiplying this equation by x and noting that $x(yx - 1) = 0$, we get $x = x^2r$. Our claim gives a quick alternative way to prove that a right quasi-duo regular ring is strongly regular, but it shows more. If R is right quasi-duo and π-regular, then for any $x \in R$, x^n is regular for some $n \geq 1$. By the claim above, we have $x^n \in x^{2n}R$. This shows that the ring R is strongly π-regular, as desired. □

(3) Let R be a 2-primal π-regular ring. We claim that $R/\mathrm{rad}(R)$ is reduced. To see this, let $x \in R$ be such that $x^n \in \mathrm{rad}(R)$ for some $n \geq 1$. Then $(x^n)^m = 0$ for some $m \geq 1$ (since the Jacobson radical of a π-regular ring is nil). But then $x \in \mathrm{Nil}_*(R) \subseteq \mathrm{rad}(R)$ since R is 2-primal. Having shown that $R/\mathrm{rad}(R)$ is reduced, we see from (4.7) that R is quasi-duo. □

(4.10) Remark. The fact that any 2-primal π-regular ring R is strongly π-regular was first noted in Hirano's paper [Hi] (see (2) $\Rightarrow$ (6) in his Theorem 1). Here, in (4.9)(3), we have added the conclusion that R is also quasi-duo.

To close this section, we point out a connection between right quasi-duo rings and another class of rings called right Kasch rings: recall that a ring R is *right Kasch* if every simple right R-module is isomorphic to a minimal right ideal of R (see [La$_3$: §8C]).

Proposition 4.11. *If a right Kasch ring R is semicommutative (that is, $\forall\ a, b \in R$, $ab = 0 \in R \Rightarrow aRb = 0$), then R is right quasi-duo.*

Proof. Let $\mathfrak{m}$ be any maximal right ideal in R. Since R is right Kasch, $\mathfrak{m}$ has the form $\mathrm{ann}_r(a)$ for some $a \in R$ (where $\mathrm{ann}_r(a)$ denotes the right annihilator of a). For any $m \in \mathfrak{m}$, we have $am = 0$, and so $aRm = 0$. This implies that $Rm \subseteq \mathrm{ann}_r(a) = \mathfrak{m}$, which proves that $\mathfrak{m}$ is an ideal of R. □

Corollary 4.12. *Let R be a ring such that $\overline{R} := R/\mathrm{rad}(R)$ is a right Kasch ring. Then R is right quasi-duo iff $\overline{R}$ is reduced.*

Proof. The "only if" part is true (without the Kasch assumption on $\overline{R}$) by (4.3)(2). Conversely, if $\overline{R}$ is reduced, then it is easily seen to be semicommutative. If $\overline{R}$ is also right Kasch, then (4.11) implies that $\overline{R}$ is right quasi-duo, and therefore so is R by (4.3)(1). □

§5. Subdirect Products of Division Rings

In connection with the two results (3.6)(2) and (4.3)(2), the following question concerning subdirect products arises naturally:

Question 5.1. Does the converse of Yu's result (4.3)(2) hold? More precisely, *if a ring R is such that $R/\mathrm{rad}(R)$ is a subdirect product of division rings, does it follow that R is right quasi-duo?*

There is a very good reason for asking this question. If the answer to this question is "yes", it would follow immediately from this and Yu's result (4.3) that a ring R is left quasi-duo iff it is right quasi-duo! Such a statement has never been proved or disproved in the literature.

If $\overline{R} = R/\mathrm{rad}(R)$ is an exchange ring or a right Kasch ring, we see from (4.6) and (4.12) that even the weaker assumption that $\overline{R}$ is reduced would imply that R is (left and right) quasi-duo. Another positive case for (5.1) is when R has only a finite number of simple right modules (up to isomorphisms). In this case, if $\{\pi_j : \overline{R} \to D_j\}$ is a representation of $\overline{R}$ as a subdirect product of the division rings $\{D_j\}$, then each $I_j = \ker(\pi_i)$ is the annihilator of a simple right R-module, so (by our assumption) there is only a finite number of distinct I_j 's. Thus, we can re-express $\overline{R}$ as a subdirect product of finitely many D_j 's, and then invoke (3.6)(2) and (4.3)(1) to conclude that $\overline{R}$, and R, are (left and right) quasi-duo rings. (A similar argument would also have worked if $R/\mathrm{rad}(R)$ is finitely cogenerated as a right R-module: see [La$_3$: (19.1)].)

Since any subdirect product of division rings is Jacobson-semisimple (see, e.g. [La$_2$: Ex. 4.12A]), Question (5.1) amounts to asking if any such subdirect product is right quasi-duo. Unfortunately, this has a negative answer in general, as the following example shows.

(5.2) Example. Let $R = \mathbb{Q}\langle x, y\rangle$ with the relation $xy + yx = 0$, and for any $a, b < 0$ in $\mathbb{Q}$, let $D_{a,b}$ be the rational quaternion division algebra generated by i, j with the relations $i^2 = a$, $j^2 = b$, and $ij = -ji$. (Of course, these generators depend on a, b.) Let $\pi_{a,b} : R \to D_{a,b}$ be the $\mathbb{Q}$-algebra surjection defined by $x \mapsto i$ and $y \mapsto j$. Then $\ker(\pi_{a,b})$ is the ideal $(x^2 - a,\ y^2 - b) \subseteq R$. We will show below that $\bigcap_{a,b<0} \ker(\pi_{a,b}) = 0$. Thus, R is a subdirect product of countably many division rings $D_{a,b}$'s. But R also maps onto the split quaternion algebra $\mathbb{M}_2(\mathbb{Q})$ (e.g. by $x \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $y \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$), so by (4.5) and (3.0), R is *not* left/right quasi-duo (and, in particular, not left/right duo).

Let $I_{a,b} = \ker(\pi_{a,b}) \subseteq R$. To see that $\bigcap_{a,b<0} I_{a,b} = 0$, first note that x^2 is central in R, and that R has a natural $\mathbb{Z}_2$-grading, in which the monomials are graded by their x-degrees (mod 2):

$$R = R_0 \oplus R_1, \quad \text{where } R_0 = \mathbb{Q}\,[x^2, y], \text{ and } R_1 = xR_0. \tag{5.3}$$

With respect to this grading, the ideals $I_{a,b}$ are homogeneous, so for each polynomial $f = g + xh \in R$ (g, $h \in R_0$) and any $a, b < 0$, we have $f \in I_{a,b}$ iff g, $h \in I_{a,b}$ (noting that $\pi_{a,b}(x) = i \neq 0$ and $D_{a,b}$ is a division ring). Therefore, we have

$$I_{a,b} = (I_{a,b})_0 \oplus x(I_{a,b})_0, \quad \text{where } (I_{a,b})_0 = I_{a,b} \cap R_0,$$

so it suffices for us to check that $\bigcap_{a,b<0} (I_{a,b})_0 = 0$. By an easy gradation argument, we have

$$(I_{a,b})_0 = R_0(x^2 - a) + R_0(y^2 - b).$$

Thus, our problem is now reduced to one in commutative algebra: if we write $u = x^2$, $v = y$, and identify R_0 with the usual polynomial ring $\mathbb{Q}\,[u, v]$, our job is to show that, if a polynomial $g(u, v)$ lies in $(u - a,\ v^2 - b)$ for all $a, b < 0$ in $\mathbb{Q}$, then $g = 0$. If $g \neq 0$, write $g(u, v) = g_n(v)u^n + \cdots + g_0(v)$ with $g_n(v) \neq 0$. Then (working over $\mathbb{C}$), $g_n\big(\sqrt{b}\big) \neq 0$ for some rational $b < 0$, and so $g\big(u, \sqrt{b}\big) \neq 0 \in \mathbb{C}\,[u]$. But then $g\big(a, \sqrt{b}\big) \neq 0$ for some rational $a < 0$, which contradicts $g \in (u - a,\ v^2 - b)$.

(5.4) Remark. Clearly, the argument above would have worked if the subdirect product is formed over any set J of pairs of negative rationals $\{(a, b)\}$ such that there are infinitely many second coordinates b, and for each second coordinate b, there are infinitely many a's such that $(a, b) \in J$. Thus, we could have taken, for instance, $J = \{(-m^2, -n^2) : \ m, n \in \mathbb{N}\}$. In this case, all the subdirect factors involved are isomorphic to Hamilton's quaternion division algebra $D_{-1,-1}$ over the rationals. From this, it follows that *a subdirect product of copies of a given division ring need not be (right or left) quasi-duo.*

In conclusion, we should perhaps also point out that, if we were only interested in finding an example of a subdirect product of right quasi-duo rings that is not right quasi-duo, an easier construction with a much more routine proof would have sufficed, as follows.

(5.5) Example. Let σ be a non-identity automorphism of a field k, and let $R = k[x,\, \sigma]$ be the skew polynomial ring over k defined by the twist law $ax = x\sigma(a)$ (for all $a \in k$). If $a \in k$ is such that $\sigma(a) \neq a$, then

$$a(x - 1) = x\,\sigma(a) - a = (x - 1)\,\sigma(a) + (\sigma(a) - a) \notin (x - 1)R, \tag{5.6}$$

so the maximal right ideal $(x-1)R$ is not an ideal of R. This shows that R is not a right quasi-duo ring. However, for each $n \geq 1$, $x^n R$ is an ideal of R such that $R/x^n R$ is a local (and hence right quasi-duo) ring. Since $\bigcap_{n\geq 1} x^n R = 0$, R is a subdirect product of the right quasi-duo rings $\{R/x^n R : n \geq 1\}$, though R itself is not right quasi-duo.

§6. Stable Range Descent in Quasi-Duo Rings

For the convenience of the reader, let us first recall the definition of the stable range of rings introduced by H. Bass [Ba]. This notion was used by Bass for the study of the stability properties of linear groups in algebraic K-theory, but later it became an important notion in ring theory in its own right. (For a brief survey on this, see [La$_4$: §§8-9].)

(6.1) Definition. We say that an integer $n \geq 1$ is *in the stable range* of a ring R (or that R has stable range $\leq n$) if, for any right unimodular sequence $r_1, \ldots, r_{n+1}$ in R, there exist elements $x_1, \ldots, x_n \in R$ such that $\sum_{i=1}^{n} (r_i + r_{n+1} x_i) R = R$.

It is straightforward to see that, if n is in the stable range for R, then so is any larger integer. We can thus define the stable range of R to be the smallest integer n in the stable range of R. (If no such n exists, the stable range of R is taken to be ∞.) This should have been called the "right" stable range of R, but it will be harmless to suppress the adjective "right" since the right and left stable ranges of a ring turn out to be equal according to a result of Vaserstein [Va$_1$: Th. 2] and Warfield [Wa: Th. (1.6)].

An interesting phenomenon in the study of stable range is that, upon the passage to matrix rings, the stable range generally decreases. The precise statement of this, due to Vaserstein [Va$_1$: Th. 2] (see also [Wa: (1.12)]) is as follows.

(6.2) Theorem. *If a ring k has stable range n, then the matrix ring $R = \mathbb{M}_m(k)$ has stable range $1 + \lceil \frac{n-1}{m} \rceil$, where $\lceil x \rceil$ (the ceiling function[2] on x) denotes the smallest integer $\geq x$.*

According to this result, if k has stable range 1 or 2, then this stable range is preserved by $\mathbb{M}_m(k)$ (for any m), but if k has (finite) stable range ≥ 3, then $\mathbb{M}_m(k)$ has stable range 2 for sufficiently large m. In particular, this shows that the stable range of a ring R (if bigger than 1) is not a Morita invariant, and not inherited by Peirce corner rings $R_e \subseteq R$ even when e is a *full* idempotent. In view of this, it would seem that there is nothing more to be said about the behavior of the stable range upon a descent to Peirce corner rings. In developing the general theory of

[2]The stable range formulas in [Va$_1$] and [Wa] were both given in the form $1 - [-(n-1)/m]$, in terms of the "greatest integer" function. We feel, however, that the expression using the ceiling function is easier and more natural.

corners, however, we were led to the following question: *if R has stable range $\leq n$, what really is the obstruction to showing that a Peirce corner R_e has stable range $\leq n$?* In grappling with this question in the special case where R is a matrix ring $\mathbb{M}_m(k)$ and e is the matrix unit e_{11}, we realized eventually that the main trouble stems from the fact that matrix rings do not satisfy the D_n properties (for $n \geq 2$), which we have shown in (4.5).

Fortuitously, it turns out that the assumption of the D_n property is sufficient for us to get positive results on the preservation of stable range upper bounds by Peirce corner rings. In the following, we shall state and prove this more generally for all semisplit corner rings.

(6.3) Theorem. *Assume that a ring R is left and right quasi-duo. If R has (right) stable range $\leq n$, then any semisplit corner $S \subseteq R$ also has (right) stable range $\leq n$.*

Proof. (1) We first treat the case where S is a Peirce corner of R, say $S = R_e$ where $e = e^2 \in R$. As usual, we write $f = 1 - e$. To show that S has (right) stable range $\leq n$, we start with an equation $\sum_{i=1}^{n+1} s_i S = S$, where $s_i \in S$. By the right analogue of (2.1) ((1) $\Rightarrow$ (4)), the sequence $\{s_1 + f, \ldots, s_n + f, s_{n+1}\}$ is right unimodular in R. Since R has (right) stable range $\leq n$, it follows that

$$\sum_{i=1}^{n} (s_i + f + s_{n+1} r_i) R = R \quad \text{for suitable } r_i \in R.$$

Now the *left* D_n property on R implies that

$$\sum_{i=1}^{n} R(s_i + f + s_{n+1} r_i) = R.$$

Using the Peirce decomposition for the r_i's, we have

$$\begin{aligned} s_i + f + s_{n+1} r_i &= s_i + f + s_{n+1}(e r_i e + e r_i f + f r_i e + f r_i f) \\ &= s_i + s_{n+1}(e r_i e) + s_{n+1} e\,(e r_i f + f r_i e + f r_i f) + f \\ &= s_i + s_{n+1}(e r_i e) + (s_{n+1} e r_i + 1) f. \end{aligned}$$

Thus, upon setting $b_i = (s_{n+1} e r_i + 1) f$, we have $\sum_{i=1}^{n} R(s_i + s_{n+1} t_i + b_i) = R$, where $t_i := e r_i e \in R_e = S$ $(1 \leq i \leq n)$. Since each $b_i \in Rf$, it follows from (2.1) ((3) $\Rightarrow$ (1)) that $\sum_{i=1}^{n} S(s_i + s_{n+1} t_i) = S$. But by (3.8), the *right* D_n property of R implies that of S. Thus, it follows that $\sum_{i=1}^{n} (s_i + s_{n+1} t_i) S = S$, which is what we want!

(2) If S is any semisplit corner in R, then it is a split corner of its associated Peirce corner R_e, where e is the identity of S. By (1) above, we know R_e has (right) stable range $\leq n$. Since S is a split corner of R_e, it has (by definition) an ideal complement in R_e, and hence S is isomorphic to factor ring of R_e. Thus, it follows from [Ba: (4.1)] that S also has (right) stable range $\leq n$. $\square$

(6.4) Corollary. *If a ring R has (right) stable range* 1*, then so does any semisplit corner $S \subseteq R$.*

Proof. According to an observation of I. Kaplansky [Ka] (see also [La_4: (8.3)]), the fact that R has (right) stable range 1 implies that R is Dedekind-finite; that is, R satisfies (left and right) D_1. In the proof of Theorem 6.3, what we needed in treating the case of stable range n was the assumption of the (left and right) D_n property. Therefore, the work we did in Theorem 6.3 applies here for $n = 1$ to give the desired conclusion. $\square$

In the case of Peirce corners, the above corollary was first discovered by Vaserstein [Va_2: Th. (2.8)]. Vaserstein's proof of (6.4) in this case was based on an ad hoc argument with stable range 1 since (6.4) was perceived to be a special result not generalizable to higher stable range. Here, we are able to view (6.4) essentially as a consequence of (6.3) for rings of *arbitrary* stable range n.

As for the "semisplit" assumption on S in (6.3) and (6.4), it is of interest to point out that the theorem *does not* hold in general without this assumption. The following shows a typical example of the failure of stable range preservation under nonsplit unital corners.

(6.5) Example. Let R be the ring of all algebraic integers. Then, R is a free $\mathbb{Z}$-module of countable rank, having $\mathbb{Z}$ as a direct summand. (I thank Bjorn Poonen for suggesting the following proof of this fact. Express R as an ascending union of a chain $\mathbb{Z} = R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$, where each R_i is the full ring of algebraic integers in a number field. Each R_i is a free abelian group of finite rank, and is a $\mathbb{Z}$-direct summand of R_{i+1}, so a $\mathbb{Z}$-basis of R_i can be extended to one for R_{i+1}. Starting with the basis $\{1\}$ on $\mathbb{Z}$, we can thus extend it to a countable $\mathbb{Z}$-basis for R.) Since $S := \mathbb{Z}$ is a $\mathbb{Z}$-direct summand of R, it is a unital corner of R. Now, according to a classical theorem of Skolem [Sk], R is a *ring with many units*; that is, every primitive polynomial in $R[x]$ represents a unit in R.[3] In particular, R *has stable range* 1. (For, if $aR + bR = R$, then $a + bx$ is a primitive polynomial in $R[x]$, and Skolem's result implies that $a + br \in \mathrm{U}(R)$ for some $r \in R$.) An ad hoc proof for the fact that R has stable range 1 has also appeared in [Va_2: Ex. (1.2)]. However, the unital corner $\mathbb{Z}$ of R has stable range 2, not 1, so the conclusion of (6.3) (for descent from R to S) fails, even for $n = 1$. (Of course, in this example, S is *not* a (semi)split corner of R.)

To conclude this section, we would like to mention another recent result on the stable range of Peirce corner rings obtained by Ara and Goodearl. In [AG], these authors proved that, if e is a full idempotent in any ring R (that is, $e = e^2$ and $ReR = R$), then the stable range of eRe is $\geq$ that of R. However, this result does

[3]For more information on commutative rings with many units, see, e.g. [McD: pp. 336-339].

not further enhance ours (or vice versa), since our Th. 6.3 is proved for quasi-duo rings, and in such rings the only full idempotent is 1 (see (7.4)(A) below).

§7. Properties of Right Quasi-Duo Rings, and Open Questions

In this section, we'll record several special properties of right quasi-duo rings that have not been noted before in the literature. We start with a property of an "elementwise" nature.

(7.1) Proposition. *Let R be a right quasi-duo ring. If $x, z \in R$ are such that $x+z \in Rxz$, then $Rx = Rz$.*

Proof. Say $x+z = yxz$, where $y \in R$. Then $x = (yx-1)z$, so the property (3.2)(6) of a right quasi-duo ring yields

$$R = xR + (yx-1)R = (yx-1)R. \tag{7.2}$$

Since R is Dedekind-finite (by the last conclusion in (3.2)), this implies that $yx-1 \in \mathrm{U}(R)$. From $x = (yx-1)z$, it follows that $Rx = Rz$. $\square$

(7.3) Proposition. *For any ring R, consider the following two statements:*

(1) *R is right quasi-duo;*

(2) *For any elements y_i, z_i $(1 \le i \le n)$ and $u \in R$, we have $\sum_{i=1}^{n} y_i u z_i \in \mathrm{U}(R) \Rightarrow u \in \mathrm{U}(R)$.*

We have always (1) $\Rightarrow$ (2), *while* (2) $\Rightarrow$ (1) *holds if every maximal right ideal of R is principal (e.g. if R is a principal right ideal ring).*

Proof. (1) $\Rightarrow$ (2). If $\sum_{i=1}^{n} y_i u z_i \in \mathrm{U}(R)$, then $RuR = R$. Applying the condition (3.2)(7), we have then $uR = R$. Since R is Dedekind-finite, it follows that $u \in \mathrm{U}(R)$.

(2) $\Rightarrow$ (1) (assuming that every maximal right ideal of R is principal). We prove (1) by checking the condition (3.2)(6). Let $x, y \in R$. If $xR + (yx-1)R \ne R$, it is contained in some maximal right ideal $\mathfrak{m}$, which, by assumption, can be written as uR for some $u \in R$. Thus, $x = ur$ and $yx - 1 = us$, for some $r, s \in R$. Now $1 = yx - us = yur - us$, so (2) implies that $u \in \mathrm{U}(R)$, which is a contradiction. $\square$

(7.4) Remarks. (A) It follows from (1) $\Rightarrow$ (2) above that, in any right quasi-duo ring R, the only full idempotent is 1.

(B) From (7.3), we also see that: *if R is a principal ideal ring, then R is left quasi-duo iff it is right quasi-duo.* (This follows since the condition (2) in (7.3) is obviously left-right symmetric.) $\square$

The next result shows that, in a manner of speaking, a right quasi-duo ring is rather "rich with units".

(7.5) Theorem. *Let R be a right quasi-duo ring with elements $e_1, \ldots, e_n$ such that $e_1 + \cdots + e_n = 1$, and let $u_1, \ldots, u_n \in \mathrm{U}(R)$. Then:*

(1) *$u_1e_1R + \cdots + u_ne_nR = R$, and*

(2) *if the e_i's are mutually orthogonal idempotents, then $u_1e_1R + \cdots + u_ne_nR$ is a direct sum (equal to R), and we have $u_1e_1 + \cdots + u_ne_n \in \mathrm{U}(R)$.*

Proof. Since R satisfies right D_n, (1) follows from the fact that

$$Ru_1e_1 + \cdots + Ru_ne_n = Re_1 + \cdots + Re_n = R.$$

Now assume that the e_i's are mutually orthogonal idempotents. Let $\varphi_i : e_iR \to u_ie_iR$ be the isomorphism given by left multiplication by the unit u_i. Then $\varphi_1, \ldots, \varphi_n$ define a right R-module epimorphism

$$\varphi : R = e_1R \oplus \cdots \oplus e_nR \longrightarrow u_1e_1R + \cdots + u_ne_nR = R. \tag{7.6}$$

Since R_R is projective, this epimorphism φ splits, and hence $R \cong \ker(\varphi) \oplus R$. But R is Dedekind-finite, so we have $\ker(\varphi) = 0$. This means that φ in (7.6) is an isomorphism. Thus, $\sum_{i=1}^{n} u_ie_iR$ is a *direct* sum (equal to R). Since $\mathrm{End}(R_R) \cong R$, φ must be the left multiplication by some *unit* u on R_R. Thus, we have $ue_i = u_ie_i$ for all $i \le n$. Adding these equations, we get

$$u_1e_1 + \cdots + u_ne_n = u(e_1 + \cdots + e_n) = u \in \mathrm{U}(R),$$

as desired. □

In the special case where R is a right duo ring, it is known that R is abelian ([La$_2$: Ex. (22.4A)]). Thus, in the case (2) above (where the e_i's are mutually orthogonal idempotents), we have $u_ie_iR = u_iRe_i = Re_i = e_iR$ for all i. Then R is the direct product $e_1R \times \cdots \times e_nR$, and the conclusions in (7.5) in this case are immediate. In the above, however, the result (7.5) was proved more generally for right quasi-duo rings, where idempotents need no longer be central.

We close with a couple of open questions. The first question is the expected one concerning the left-right symmetry (or the lack thereof) of the quasi-duo notion. We have already briefly encountered this question in §5, although the existence of examples of the type (5.2) leaves us practically clueless as to its answer.

(7.7) Question. *Does there exist a right quasi-duo ring that is not left quasi-duo?*

Note that such a ring R must have the following properties: (1) it has an infinite number of distinct simple right modules; (2) $R/\mathrm{rad}(R)$ is not an exchange ring or a right Kasch ring. For instance, for all of the rings we dealt with in (4.6) and its corollaries, right and left quasi-duo did turn out to be equivalent conditions.

In (4.1), we showed that a right primitive right quasi-duo ring must be a division ring. What happens if, in this statement, we replace the "right primitive" condition by "left primitive"? This led us to the second

(7.8) Question. *Does there exist a left primitive, right quasi-duo ring that is not a division ring?*

It turns out (7.7) and (7.8) are essentially "equivalent" questions, as follows.

(7.9) Remark. *If the answer to* (7.7) *is "yes", then the answer to* (7.8) *is also "yes", and conversely.* To see this, assume there exists a ring R that is right quasi-duo but not left quasi-duo. Then, it has a maximal left ideal $\mathfrak{m}$ that is not an ideal. Let C be the largest ideal of R that is contained in $\mathfrak{m}$. Then R/C is a left primitive and right quasi-duo ring. But it is not a division ring since it has a nonzero maximal left ideal $\mathfrak{m}/C$. Thus, R/C provides an example for (7.8). Conversely, assume there exists a left primitive, right quasi-duo ring S that is not a division ring. Then S cannot be left quasi-duo, according to (the left analogue of) (4.1), so S would provide an example for (7.7). Note that, if S does exist, then it would also provide an example of a ring that is left primitive but not right primitive (again by (4.1)). Thus, it may be said that the "degree of complexity" of the questions (7.7) and (7.8) is "higher" than that of finding a left primitive but not right primitive ring. Of course, an even more challenging problem would be that of finding (if it is possible) a right duo ring that is not left quasi-duo.

The last question is prompted by the work in §5. In light of Example 5.2, it is of interest to raise the following

(7.10) Question. *Which rings can be represented as subdirect products of division rings, and when is such a subdirect product right quasi-duo?*

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