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# Truncated path algebras are homologically transparent

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**Abstract.** It is shown that path algebras modulo relations of the form $\Lambda = KQ/I$, where $Q$ is a quiver, $K$ a coefficient field, and $I \subseteq KQ$ the ideal generated by all paths of a given length, can be readily analyzed homologically, while displaying a wealth of phenomena. In particular, the syzygies of their modules, and hence their finitistic dimensions, allow for smooth descriptions in terms of $Q$ and the Loewy length of $\Lambda$. The same is true for the distributions of projective dimensions attained on the irreducible components of the standard parametrizing varieties for the modules of fixed $K$-dimension.

**Key words.** Representation theory of associative algebras and quivers, homological dimensions, algebraic varieties parametrizing representations.

**AMS classification.** 16G10, 16G20, 16E10.

## 1 Introduction and notation

The problem of opening up general access roads to the finitistic dimensions of a finite dimensional algebra $\Lambda$, given through quiver and relations, is quite challenging. This is witnessed, for instance, by the fact that the longstanding question "Is the (left) little finitistic dimension of $\Lambda$,

$$\operatorname{fin\,dim} \Lambda = \sup\{\operatorname{p\,dim} M \mid M \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})\},$$

always finite?" (Bass 1960) has still not been settled. Here $\operatorname{p\,dim} M$ is the projective dimension of a module $M$, and $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ denotes the category of finitely generated (left) $\Lambda$-modules of finite projective dimension.

In [1], Babson, the second author, and Thomas showed that truncated path algebras of quivers are particularly amenable to geometric exploration, while nonetheless displaying a wide range of interesting phenomena. This led the authors of the present paper to the serendipitous discovery that the same is true for the homology of such algebras. By a *truncated path algebra* we mean an algebra of the form $KQ/I$, where $KQ$ is the path algebra of a quiver $Q$ with coefficients in a field $K$ and $I \subseteq KQ$ is the ideal generated by all paths of a fixed length $L+1$. In particular, truncated path algebras are monomial algebras. In this case, the finitistic dimensions are known to be finite (see [6]). Our goal here is to show how much more is true in the truncated scenario.

Roughly, our three main results (Theorems 2.6, 3.2, 3.6) show the following for a truncated path algebra $\Lambda$:

• The little and big finitistic dimensions of $\Lambda$ coincide and can be determined through a straightforward computation from $Q$ and $L$. Moreover, from a minimal amount of

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structural data for a $\Lambda$-module $M$, namely the radical layering

$$\mathbb{S}(M) = \big(J^l M / J^{l+1} M\big)_{0 \le l \le L}$$

(or, alternatively, any "skeleton" of $M$), one can determine the syzygies and projective dimension of $M$ in a purely combinatorial fashion. (See Theorems 2.2, 2.6, and the first part of Theorem 3.2 for finer information.)

• The "generic projective dimension" of any irreducible component $\mathcal{C}$ of one of the classical module varieties (see beginning of Section 3) is readily obtainable from graph-theoretic data as well. So is the full spectrum of values of the function $\operatorname{p\,dim}$ attained on the class of modules parametrized by $\mathcal{C}$. In particular, it turns out that the supremum of the finite values among the generic finitistic dimensions of the various irreducible components equals $\operatorname{fin\,dim} \Lambda$. (See Theorems 3.2 and 3.6 for detail.)

The picture emerging from the main theorems will be supplemented in a sequel, where it will be shown that the category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in the full category of finitely generated $\Lambda$-modules, whenever $\Lambda$ is a truncated path algebra.

**Conventions.** We fix a positive integer $L$. Throughout, $\Lambda$ denotes a truncated path algebra of Loewy length $L+1$, that is, $\Lambda = KQ/I$, where $K$ is a field, $Q$ a quiver, and $I$ the ideal generated by all paths of length $L+1$. The Jacobson radical $J$ of $\Lambda$ satisfies $J^{L+1} = 0$ by construction. A (*nonzero*) *path in* $\Lambda$ is the $I$-residue of a path in $KQ \setminus I$, that is, the $I$-residue of a path $p$ in $KQ$ of length at most $L$; so, in particular, any path in $\Lambda$ is a *nonzero* element of $\Lambda$ under this convention. Clearly, the paths in $\Lambda$ form a $K$-basis for $\Lambda$. Due to the fact that $I$ is homogeneous with respect to the path-length grading of $KQ$, defining the *length* of such a path $p + I$ to be that of $p$, yields an unambiguous concept of length for the elements of this basis. A distinguished role is played by the paths $e_1, \dots, e_n$ of length zero in $\Lambda$: They constitute a full set of orthogonal primitive idempotents, which is in obvious one-to-one correspondence with the vertices of $Q$. We will identify each $e_i$ with the corresponding vertex, and whenever we refer to a primitive idempotent in $\Lambda$, we will mean one of the $e_i$. Then the left ideals $\Lambda e_i$ and their radical factors $S_i = \Lambda e_i / J e_i$, for $1 \le i \le n$, constitute full sets of isomorphism representatives for the indecomposable projective and simple left $\Lambda$-modules, respectively.

Finally, we say that a path $p$ in $\Lambda$ or in $KQ$ is an *initial subpath* of a path $q$ if there is a path $p'$ with $q = p'p$; here the product $p'p$ stands for "$p'$ after $p$."

## 2 The standard homological dimensions of $\Lambda$-Mod

The (left) *big finitistic dimension* of $\Lambda$ is the supremum, $\operatorname{Fin\,dim} \Lambda$, of the projective dimensions of all left $\Lambda$-modules of finite projective dimension; for the *little finitistic dimension* consult the introduction. We start by recording some prerequisites established in [1]. As was shown in [1], the well-known fact that all *second* syzygies of modules over a monomial algebra are direct sums of cyclic modules generated by paths of positive length (see [7] and [2]), can be improved for truncated path algebras so as to cover *first* syzygies as well. In particular, this makes the big and little finitistic dimensions of $\Lambda$ computable from a finite set of cyclic test modules.

More sharply: Given any left $\Lambda$-module $M$, we can explicitly pin down a decomposition of the syzygy $\Omega^1(M)$ into cyclics. This description of $\Omega^1(M)$ relies on a *skeleton of* $M$. Roughly speaking, this is a path basis for $M$ with the property that the path lengths respect the radical layering, $(J^l M/J^{l+1}M)_{0\le l\le L}$. The concept of a skeleton, defined in [1] in full generality, can be significantly simplified for a truncated path algebra $\Lambda$.

**Definition 2.1 (Skeleton of a $\Lambda$-module $M$).** Fix a projective cover $P$ of $M$, say $P = \bigoplus_{r\in R} \Lambda z_r$, where each $z_r$ is one of the primitive idempotents in $\{e_1, \dots, e_n\}$, tagged with a place number $r$ (the index set $R$ may be infinite). A *path of length* $l$ in $P$ is any element $pz_r \in P$, where $p$ is a path of length $l$ in $\Lambda$ which starts in $z_r$ (in particular, the paths in $P$ are again nonzero). Identify $M$ with an isomorphic factor module of $P$, say $M = P/C$.

(a) A *skeleton of* $M = P/C$ is a set $\sigma$ of paths in $P$ such that for each $l \le L$, the residue classes $q + J^l M$ of the paths $q$ of length $l$ in $\sigma$ form a $K$-basis for $J^l M/J^{l+1}M$. Moreover, we require that $\sigma$ be closed under initial subpaths, that is, if $q = p'pz_r \in \sigma$, then $pz_r \in \sigma$.

(b) A path $q$ in $P \setminus \sigma$ is called $\sigma$-*critical* if it is of the form $q = \alpha p z_r$, where $\alpha$ is an arrow and $pz_r$ a path in $\sigma$.

In particular, the definition entails that, for any skeleton $\sigma$ of $M = P/C$, the full set of residue classes $\{q + C \mid q \in \sigma\}$ forms a basis for $M$. Furthermore, it is easily checked that every $\Lambda$-module $M$ has at least one skeleton, and only finitely many when $M$ is finitely generated (as long as we keep the projective cover $P$ fixed).

**Theorem 2.2. Known Facts.** [1, Lemma 5.10] *If $M$ is any nonzero left $\Lambda$-module with skeleton $\sigma$, then*

$$\Omega^1(M) \cong \bigoplus_{q\ \sigma\text{-critical}} \Lambda q.$$

*In particular, $\Omega^1(M)$ is isomorphic to a direct sum of cyclic left ideals generated by nonzero paths of positive length in $\Lambda$.*

*Consequently,* $\operatorname{Fin\,dim}\Lambda = \operatorname{fin\,dim}\Lambda = s+1$, *where*

$$s = \max\{\operatorname{p\,dim}\Lambda q \mid q \text{ a path of positive length in } \Lambda \text{ with } \operatorname{p\,dim}\Lambda q < \infty\},$$

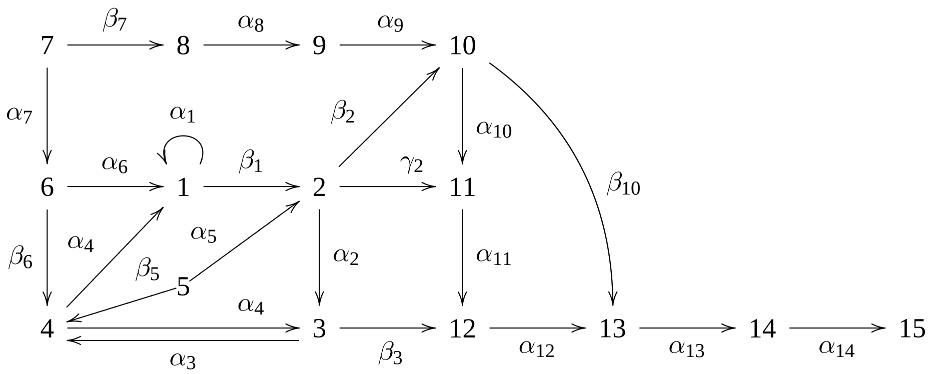
*provided that the displayed set is nonempty, and $s = -1$ otherwise.* □

We briefly point out another nice consequence concerning Auslander's notion of the representation dimension of an algebra. In [9] Ringel shows that any algebra with only finitely many indecomposable torsionless modules up to isomorphism has representation dimension at most 3. Since the above theorem classifies the indecomposable torsionless $\Lambda$-modules as those modules isomorphic to $\Lambda q$ for some nonzero path $q$, of which there are only finitely many, we obtain the following.

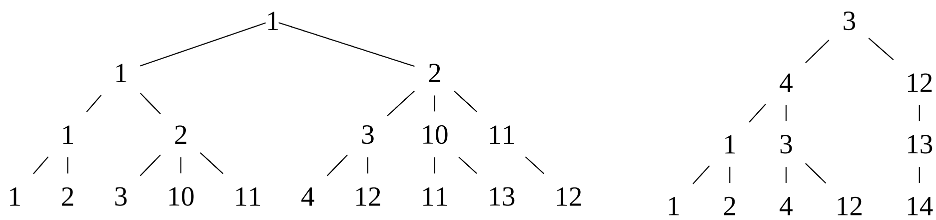
**Corollary 2.3.** *The representation dimension of any truncated path algebra is at most 3.*

We illustrate the results on finitistic dimension with an example which will accompany us throughout.

**Example 2.4.** Let $\Lambda = KQ/I$ be the truncated path algebra of Loewy length $L+1 = 4$ based on the following quiver $Q$.

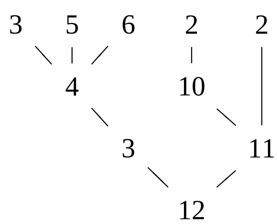



Then the indecomposable projective left $\Lambda$-modules $\Lambda e_1$ and $\Lambda e_3$ have the following layered and labeled graphs (in the sense of [7] and [8]):

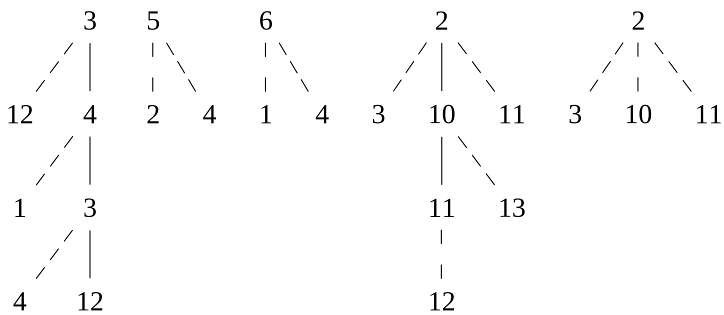


If $P = \Lambda z_1$ with $z_1 = e_i$ in the notation of Definition 2.1, each of the modules $\Lambda e_i$ has a unique skeleton, which can be read off the graph: It is the set of all initial subpaths of the edge paths in the graph, read from top to bottom. The skeleton of $\Lambda e_1$, for instance, consists of the paths $z_1 = e_1$ of length zero in $P$, the paths $\alpha_1 z_1$, $\beta_1 z_1$ of length 1, the paths $\alpha_1^2 z_1$, $\beta_1\alpha_1 z_1$, $\alpha_2\beta_1 z_1$, $\gamma_2\beta_1 z_1$, $\beta_2\beta_1 z_1$ of length 2, together with all edge paths of length 3.

For a sample application of Theorem 2.2, we consider the module $M$ determined by the following graph:



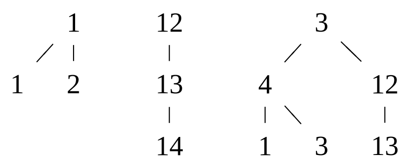
A projective cover of $M$ is $P = \Lambda e_3 \oplus \Lambda e_5 \oplus \Lambda e_6 \oplus (\Lambda e_2)^2$, where $z_1 = e_3, z_2 = e_5$ and so on. A skeleton $\sigma$ of $M$ (in this case there are several), together with the $\sigma$-critical paths is communicated by the following graph, in which the solid and dashed edges play different roles, as explained below:




As above, the paths in $\sigma$ correspond to the intial subpaths of the solidly drawn edge paths, including all paths of length zero – e.g., $\beta_4\alpha_3z_1$, $z_3$ and $\alpha_{10}\beta_2z_4$. The $\sigma$-critical paths are all the paths in the graph (again read from top to bottom) which terminate in a dashed edge; for instance, $\alpha_3\beta_4\alpha_3z_1$ and $\alpha_5z_2$ are $\sigma$-critical. Since $\Omega^1(M) \cong \bigoplus_{q\ \sigma\text{-critical}} \Lambda q$, we find this syzygy to be the direct sum

$$\Lambda\beta_3 \oplus \Lambda\alpha_4\alpha_3 \oplus \Lambda\alpha_3\beta_4\alpha_3 \oplus \Lambda\alpha_5 \oplus \Lambda\beta_5 \oplus \Lambda\alpha_6 \oplus \Lambda\beta_6 \oplus \Lambda\alpha_2 \oplus \cdots .$$

The graphs of $\Lambda\alpha_4\alpha_3$, $\Lambda\beta_3$, and $\Lambda\alpha_2$ are respectively




The main result of this section provides the projective dimensions of the building blocks for the syzygies of arbitrary $\Lambda$-modules; compare with Theorem 2.2.

**Definition 2.5.** Let $l$ be a nonnegative integer $\le L$, and $c$ any nonnegative integer. We define

$$l\text{-deg}(c) = \left[\frac{c}{L+1}\right] + \left[\frac{c+l}{L+1}\right].$$

Here $[x]$ stands for the largest integer smaller than or equal to $x$. Moreover, we set $l$-deg$(\infty) = \infty$.

The $l$-degree defines a nondecreasing function $\mathbb{N} \cup \{0, \infty\} \to \mathbb{N} \cup \{0, \infty\}$ for any $l \le L$. Moreover, for $0 \le l \le l' \le L$ and arbitrary $c \in \mathbb{N} \cup \{0\}$, the difference $l'$-deg$(c)$ – $l$-deg$(c)$ belongs to the set $\{0, 1\}$. This observation will entail the final claim of the upcoming theorem, once the first – displayed – equality is established.

**Theorem 2.6.** *Suppose $q \in \Lambda$ is a path of length $l > 0$ in $\Lambda$ (i.e., the $I$-residue of a path of length at most $L$ in $KQ$) with terminal vertex $e$. Let $c = c(e)$ be the supremum of the lengths of the paths in $KQ$ starting in $e$. Then*

$$\operatorname{p\,dim} \Lambda q \;=\; l\text{-deg}(c).$$

*In particular,* $\operatorname{p\,dim} \Lambda q < \infty$ *if and only if* $c(e) < \infty$ *(meaning that there is no path starting in $e$ and terminating on an oriented cycle).*

*Moreover, if $q'$ is another path in $\Lambda$ that ends in $e$ such that* $L \geq \text{length}(q') \geq \text{length}(q) \geq 1$, *then*

$$\operatorname{p\,dim} \Lambda q \leq \operatorname{p\,dim} \Lambda q' \leq 1 + \operatorname{p\,dim} \Lambda q.$$

In Example 2.4, $c(e_7)$ is infinite, for instance, while $c(e_{10}) = 5$; the latter shows that $\operatorname{p\,dim}(\Lambda \alpha_9 \alpha_8 \beta_7) = 3\text{-deg}(5) = 3$. The argument backing Theorem 2.6 is purely combinatorial, the intuitive underpinnings being of a graphical nature. We start with two definitions setting the stage. The first is clearly motivated by the statement of Theorem 2.6.

**Definition 2.7.** We call a vertex $e$ of the quiver $Q$ (alias a primitive idempotent of $\Lambda$) *cyclebound* in case there is a path from $e$ to a vertex lying on an oriented cycle. In case $e$ is cyclebound, we also call the simple module $\Lambda e / J e$ cyclebound.

Next, we consider the following partial order on the set of paths in $KQ$. Namely, given paths $p$ and $p'$ in $KQ$, we define

$$p' \leq p \quad \Longleftrightarrow \quad p' \text{ is an initial subpath of } p;$$

recall that the latter amounts to the existence of a path $p''$ with the property that $p = p''p'$. Hence, any two paths which are comparable have the same starting point, and $e \leq p$ for any path $p$ starting in the vertex $e$. Clearly, this partial order induces a partial order on the set of paths in $\Lambda$.

Finally, we introduce a class of modules, which will turn out to tell the full homological story of $\Lambda$. The left ideals of the form $\Lambda q$ – the basic building blocks of all syzygies of $\Lambda$-modules – are among them.

**Definition 2.8 (Tree modules and branches. Comments).** Any module $\mathcal{T}$ of the form $\mathcal{T} \cong \Lambda e / V$, where $e$ is a vertex of $Q$ and $V = \sum_{v \in \mathfrak{V}} \Lambda v$ is generated by some set $\mathfrak{V}$ of paths of positive length in $\Lambda e$ (possibly empty), will be called a *tree module with root* $e$. In particular, $\Lambda e$ is a tree module with root $e$, the unique candidate of maximal dimension among the tree modules with root $e$, in fact; the simple module $\Lambda e / J e$ is the tree module with root $e$ that has minimal positive dimension.

The terminology is motivated by the fact that the *graphs* of tree modules are trees "growing downwards" from their roots. Note that tree modules are determined up to isomorphism by their graphs.

Given a tree module $\mathcal{T}$ as above, let $b_1, \dots, b_r \in \Lambda$ be the maximal paths in $\Lambda e$ – in the above partial order – which are not contained in $V$. The $b_i$ are uniquely determined by the isomorphism class of $\mathcal{T}$ and are called the *branches* of $\mathcal{T}$. Conversely, if we know $\mathcal{T}$ to be a tree module, then the branches of $\mathcal{T}$ pin $\mathcal{T}$ down up to isomorphism.

If $\mathcal{T} \cong \Lambda e/Je$ is the simple tree module with root $e$, then $e$ is the only branch of $\mathcal{T}$. By contrast, if $\mathcal{T} = \Lambda e/V$ is a nonsimple tree module, then all branches of $\mathcal{T}$ have positive length. Moreover, it is straightforward to see that $\mathcal{T}$ has a basis of the following form:

$$\{e + V\} \cup \{q + V \mid q \text{ is an initial subpath of positive length of one of } b_1, \dots b_r\},$$

where $b_1, \dots, b_r$ are the branches of $\mathcal{T}$. If we pull back this basis to a set of paths in the projective cover $\Lambda e$ of $\mathcal{T}$, then $\sigma$ is a skeleton of $\mathcal{T}$ in the sense of Definition 2.1 (the only one).

Apart from $M$, all the modules displayed in Example 2.4 are tree modules. Their branches are precisely the maximal edge paths in their graphs, read from top to bottom. The proof of the next lemma is straightforward and we leave it to the reader.

**Lemma 2.9.** *Whenever $q$ is a path in $\Lambda$ ending in $e$, not necessarily of positive length, the cyclic left ideal $\Lambda q$ is a tree module with root $e$. More precisely: If $l = \text{length}(q)$, let $b_1, \dots, b_r$ be the maximal candidates among the paths of length $\le L - l$ starting in $e$. Then $\Lambda q = \Lambda e/V$, where*

$$V = \Omega^1(\Lambda q) = \bigoplus_{\beta \text{ an arrow},\, i \le r} \Lambda \beta b_i \,,$$

*and the $b_i$ are the branches of $\Lambda q$.*

*In particular, if $l > 0$, then* $\operatorname{p\,dim} \Lambda q < \infty$ *if and only if $e$ is non-cyclebound.* □

Combined with Theorem 2.2, Lemma 2.9 shows that all syzygies of $\Lambda$-modules are direct sums of tree modules. Contrasting the final statement for $l > 0$, we see that, for the path $q = e$ of length zero, $\Lambda q = \Lambda e$ is projective, irrespective of the positioning of $e$ in $Q$. As for the other extreme: By Lemma 2.9, the simple module $S = \Lambda e/Je$ has infinite projective dimension precisely when it is cyclebound. In Example 2.4, the vertices $e_1, \dots, e_7$ are cyclebound, while $e_8, \dots, e_{15}$ are not. Hence $S_1, \dots, S_7$ are precisely the simple modules of infinite projective dimension.

Note that the only potential branches $b_i$ of length $< L - l$ of a tree module $\Lambda q$ as in Lemma 2.9 end in a sink of the quiver $Q$.

*Proof of Theorem 2.6.* As in the statement of the theorem, let $q$ be a path of positive length $l \le L$ in $\Lambda$, which ends in the vertex $e$. In light of the remark preceding Theorem 2.6, we only need to show the equality $\operatorname{p\,dim} \Lambda q = l\text{-deg}(c)$, where $c = c(e)$ is the supremum of the lengths of the paths in $KQ$ starting in $e$. If $e$ is cyclebound, this equality follows from Lemma 2.9. So let us assume that $e$ is non-cyclebound – meaning $c < \infty$ – and induct on $c$. If $c \le L - l$, all of the branches of the tree module $\Lambda q$ end in sinks of the quiver $Q$. We infer that $\Lambda q \cong \Lambda e$ in that case, whence $\operatorname{p\,dim} \Lambda q = 0 = l\text{-deg}(c)$.

Now suppose $c > L - l$, and assume that $\operatorname{p\,dim} \Lambda p' = l'\text{-deg}(c(e'))$ for all paths $p'$ of length $l' \le L$ in $\Lambda$ that end in a non-cyclebound vertex $e'$ of $Q$ with $c(e') < c$. Using the notation of Lemma 2.9, we obtain $\Omega^1(\Lambda q) = \bigoplus_{\beta,\, i \le r} \Lambda \beta b_i$, where the $b_i$ are the branches of the tree module $\Lambda q$ and the $\beta$ are arrows. Since the lengths of the $b_i$ are bounded from above by $L - l \le L - 1$, the paths in $KQ$ of the form $\beta b_i$ where $\beta$ is an arrow, have length at most $L$; therefore each of them gives rise to a path in $\Lambda$. By the definition of $c$, there exists a path $u$ of length $c$ in $KQ$ which starts in the vertex $e$, and by the definition of the branches of $\Lambda q$, there exists an index $j$ such that $b_j$ is an initial subpath of $u$. Necessarily, length$(b_j) = L - l$, because length$(u) > L - l$. In fact, $c > L - l$ guarantees that $u = u'\beta_j b_j$ in $KQ$ for some arrow $\beta_j$ and a suitable path $u'$ of length $c' =$ length$(u) - (L - l) - 1 = c - (L - l) - 1 \leq c - 1$. Since $u$ starts in the non-cyclebound vertex $e$, the terminal vertex of $\beta_j b_j$ – call it $e'$ – is again non-cyclebound. Moreover, the maximality property of $u$ entails that $c' = c(e')$ is the maximal length of a path in $KQ$ starting in $e'$. Therefore, our induction hypothesis guarantees that $\operatorname{p\,dim} \Lambda\beta_j b_j = (L - l + 1)$-deg$(c')$. This degree in turn equals

$$\left[\frac{c'}{L+1}\right] + \left[\frac{c' + L - l + 1}{L+1}\right] = \left[\frac{c + l - (L+1)}{L+1}\right] + \left[\frac{c}{L+1}\right] = l\text{-deg}(c) - 1;$$

the final equality follows from $\frac{c+l-(L+1)}{L+1} = \frac{c+l}{L+1} - 1$. Analogous applications of the induction hypothesis, combined with the basic properties of the degree function, yield $\operatorname{p\,dim} \Lambda\beta b_i \leq \operatorname{p\,dim} \Lambda\beta_j b_j$ for any path $\beta b_i$ appearing in the decomposition of $\Omega^1(\Lambda q)$. We conclude that $\operatorname{p\,dim} \Lambda q = 1 + \operatorname{p\,dim} \Lambda\beta_j b_j = l$-deg$(c)$ as required. ∎

The following dichotomy for the finitistic dimension of $\Lambda$ results from a combination of Theorems 2.2 and 2.6 with Lemma 2.9.

**Corollary 2.10.** *Suppose that $S_1, \dots, S_t$ are precisely the non-cyclebound simple left $\Lambda$-modules. Then either*

$$\operatorname{fin\,dim} \Lambda = \max_{1 \leq i \leq t} \operatorname{p\,dim} S_i \quad \textit{or} \quad \operatorname{fin\,dim} \Lambda = 1 + \max_{1 \leq i \leq t} \operatorname{p\,dim} S_i\ ,$$

*and*

$$\max_{1 \leq i \leq m} \operatorname{p\,dim} S_i = 1 + 1\text{-deg}(m-1),$$

*where $m$ is the maximum of the lengths of the paths in $Q$ starting in one of the vertices $e_1, \dots, e_t$.* ∎

Both options for $\operatorname{fin\,dim} \Lambda$ occur in concrete instances (see below); of course, the smaller value equals the global dimension whenever the quiver $Q$ is acyclic. For the decision process in specific instances, combine Theorems 2.2 and 2.6. To contrast Corollary 2.10 with the homology of more general algebras: Recall that arbitrary natural numbers occur as finitistic dimensions of monomial algebras all of whose simple modules have infinite projective dimension. So the corollary again attests to the degree of simplification that occurs when the paths factored out of $KQ$ have uniform length.

**Example 2.4 revisited.** With the aid of Corollary 2.10, the finitistic dimension of $\Lambda$ can, in a first step, be computed up to an error of 1, through a simple count. Here $m = 7$, and $L = 3$, whence the maximum of the projective dimensions of the non-cyclebound simple modules (here $S_8, \dots, S_{15}$) is $1 + 1$-deg$(6) = 3$.

To obtain the precise value of the finitistic dimension, we further observe: The arrow $\beta_7$ ends in the vertex $e_8$ with maximal finite length $c(e_8) = 7$ of departing paths, and hence $\operatorname{p\,dim}(\Lambda e_7/\Lambda\beta_7) = 1 + 1$-deg$(7) = 4$. Consequently, $\operatorname{Fin\,dim} \Lambda = \operatorname{fin\,dim} \Lambda = 4$.

## 3 Generic behavior of the homological dimensions

Recall that, for any finite dimensional algebra $\Delta$ and $d \in \mathbb{N}$, the following affine variety $\mathbf{Mod}_d(\Delta)$ parametrizes the $d$-dimensional $\Delta$-modules: Let $a_1, \dots, a_r$ be a set of algebra generators for $\Delta$ over $K$. For instance, if $\Delta$ is a path algebra modulo relations, then the primitive idempotents (alias vertices of the quiver), together with the (residue classes in $\Delta$ of the) arrows constitute such a set of generators. For $d \in \mathbb{N}$,

$$\mathbf{Mod}_d(\Delta) =$$
$$\{(x_i) \in \prod_{1 \le i \le r} \mathrm{End}_K(K^d) \mid \text{ the } x_i \text{ satisfy all relations satisfied by the } a_i\}.$$

As is well-known, the isomorphism classes of $d$-dimensional (left) $\Lambda$-modules are in one-to-one correspondence with the orbits of $\mathbf{Mod}_d(\Lambda)$ under the $\mathrm{GL}_d$-conjugation action. Indeed, the orbits coincide with the fibres of the map from $\mathbf{Mod}_d(\Delta)$ to the set of isomorphism classes of $d$-dimensinal left $\Delta$-modules, which maps a point $x$ to the class of $K^d$, endowed with the $\Delta$-multiplication $a_i v = x_i(v)$. If $\mathcal{C}$ is a subvariety of $\mathbf{Mod}_d(\Delta)$, we refer to the modules represented by the points in $\mathcal{C}$ as *the modules in* $\mathcal{C}$.

It is, moreover, a standard fact that the homological dimensions of the $d$-dimensional modules, such as $\mathrm{p\,dim}$, are generically constant on any irreducible component of $\mathbf{Mod}_d(\Delta)$ (for a proof, see [4, Lemma 4.3] or [10, Theorem 5.3], where the result is attributed to Bongartz). In fact, it is known that, given any irreducible subvariety $\mathcal{C}$ of $\mathbf{Mod}_d(\Delta)$, there exists a dense open subset $U \subseteq \mathcal{C}$ such that the function $\mathrm{p\,dim}$ is constant on $U$. Moreover, this *generic projective dimension on* $\mathcal{C}$ is the minimum of the projective dimensions attained on the modules in $\mathcal{C}$. In most interesting cases, the projective dimension fails to be constant on all of $\mathcal{C}$, however. (Think, e.g., of the path algebra $\Delta$ of the quiver $1 \to 2$, and let $\mathcal{C}$ be the irreducible component of $\mathbf{Mod}_2(\Delta)$, whose points correspond to the modules with composition factors $S_1, S_2$; here the generic projective dimension is 0, while $\mathrm{p\,dim}(S_1 \oplus S_2) = 1$.) This raises the question of how the following generic variant of the finitistic dimension relates to the classical little finitistic dimension of $\Delta$.

**Definition 3.1.** The *generic left finitistic dimension* of a finite dimensional algebra $\Delta$ is the supremum of the finite numbers gen-$\mathrm{p\,dim}(\mathcal{C})$, where $\mathcal{C}$ traces the irreducible components of the varieties $\mathbf{Mod}_d(\Delta)$; here gen-$\mathrm{p\,dim}(\mathcal{C})$ is the generic value of the function $\mathrm{p\,dim}$, restricted to the modules in $\mathcal{C}$.

Clearly, the (left) generic finitistic dimension of an algebra $\Delta$ is always bounded above by $\mathrm{fin\,dim}\,\Delta$. When are the two dimensions equal? Given an irreducible component $\mathcal{C} \subseteq \mathbf{Mod}_d(\Delta)$, what is the spectrum of values attained by the projective dimension on $\mathcal{C}$?

The completeness with which these questions can be answered in the case of a truncated path algebra $\Lambda$ came as a surprise to us. The resulting picture underscores the pivotal role played by tree modules and supplements the fact that, in the truncated scenario, the irreducible components are fairly well understood. They are in one-to-one correspondence with certain sequences of semisimple modules, as follows:

Recall that, given a finitely generated left $\Lambda$-module $M$, its *radical layering* is $\mathbb{S}(M) = (J^l M/J^{l+1}M)_{0\le l\le L}$. We will identify isomorphic semisimple modules so that the radical layerings of isomorphic $\Lambda$-modules become identical. That the $K$-dimension of $M$ be $d$ evidently translates into the equality $\sum_{0\le l\le L} \dim_K J^l M/J^{l+1}M = d$. For each sequence $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ of semisimple modules $\mathbb{S}_l$ with total dimension $d$, let $\mathbf{Mod}(\mathbb{S})$ be the subset of $\mathbf{Mod}_d(\Lambda)$ consisting of those points which correspond to the modules with radical layering $\mathbb{S}$. Then the locally closed subvariety $\mathbf{Mod}(\mathbb{S})$ of $\mathbf{Mod}_d(\Lambda)$ is irreducible by [1, Theorem 5.3], whence so is its closure in $\mathbf{Mod}_d(\Lambda)$. The maximal candidates among the closures $\overline{\mathbf{Mod}(\mathbb{S})}$, where $\mathbb{S}$ traces the sequences $\mathbb{S}$ of total dimension $d$, are therefore the irreducible components of $\mathbf{Mod}_d(\Lambda)$; indeed, there are only finitely many such sequences. It is, moreover, easy to recognize whether a given sequence $\mathbb{S}$ of semisimple modules as above arises as the radical layering of a $\Lambda$-module, that is, whether $\mathbf{Mod}(\mathbb{S}) \neq \varnothing$ (see [1]). Namely, suppose that $\mathbb{S}_l = \bigoplus_{0\le l\le L} S_i^{s(i,l)}$ and let $P$ be the projective cover of $\mathbb{S}_0$. Then $\mathbf{Mod}(\mathbb{S}) \neq \varnothing$ if and only if there exists a set $\sigma$ of paths in $P$, which is closed under initial subpaths, such that $\sigma$ is *compatible with* $\mathbb{S}$ in the following sense: For each $i \in \{1, \dots, n\}$ and each $l \in \{0, 1, \dots, L\}$, the set $\sigma$ contains precisely $s(i,l)$ paths of length $l$ which end in the vertex $e_i$. Observe that, whenever $M$ is a module with radical layering $\mathbb{S}(M) = \mathbb{S}$, any skeleton of $M$ is compatible with $\mathbb{S}$. Consequently, the requirement that $\mathbf{Mod}(\mathbb{S}) \neq \varnothing$ implies that the $l$-th layer $\mathbb{S}_l$ of $\mathbb{S}$ be a direct summand of the $l$-th layer $J^l P/J^{l+1}P$ in the radical layering of $P$.

**Theorem 3.2.** *Let $\mathbb{S} = (\mathbb{S}_0, \mathbb{S}_1, \dots, \mathbb{S}_L)$ be a sequence of semisimple $\Lambda$-modules such that $\mathbf{Mod}(\mathbb{S}) \neq \varnothing$, and let $P$ a projective cover of $\mathbb{S}_0$. Moreover, suppose*

$$J^l P/J^{l+1}P = \Big(\bigoplus_{1\le i\le n} S_i^{s(i,l)}\Big) \oplus \Big(\bigoplus_{1\le i\le n} S_i^{r(i,l)}\Big)$$

*for suitable nonnegative integers $r(i,l)$; here $s(i,l)$ is the multiplicity of $S_i$ in $\mathbb{S}_l$ as above.*

(1) *The projective dimension of a module $M$ depends only on its radical layering $\mathbb{S}(M)$. In other words, the projective dimension is constant on each of the varieties $\mathbf{Mod}(\mathbb{S})$. This constant value, denoted* $\operatorname{p\,dim}\mathbb{S}$, *is the generic projective dimension of the irreducible subvariety $\overline{\mathbf{Mod}(\mathbb{S})}$ of $\mathbf{Mod}_d(\Lambda)$.*

(2) *If* $\operatorname{p\,dim}\mathbb{S} > 0$, *then*

$$\operatorname{p\,dim}\mathbb{S} = 1 + \sup\{l\text{-deg}(c(e_i)) \mid i \le n,\ l \le L \text{ with } r(i,l) \neq 0\}.$$

(*We adopt the standard convention "$1 + \infty = \infty$".*) *In particular,* $\operatorname{p\,dim}\mathbb{S}$ *is finite if and only if $r(i,l) = 0$ for all cyclebound vertices $e_i$, that is, if and only if every simple module of infinite projective dimension has the same composition multiplicity in $P$ as in $\bigoplus_{0\le l\le L} \mathbb{S}_l$.*

(3) *The generic finitistic dimension of $\Lambda$ coincides with* $\operatorname{fin\,dim}\Lambda$. *It is the projective dimension of a tree module $\mathcal{T}$ – of dimension $d$ say – whose orbit closure is an irreducible component of $\mathbf{Mod}_d(\Lambda)$.*

Computing $\operatorname{p\,dim} \mathbb{S}$ in concrete examples amounts to performing at most $n$ counts: Indeed, if $r(i,l) \neq 0$ for some $l$, then $l$-deg$(c(e_i)) \le l_i$-deg$(c(e_i))$, where $l_i$ is maximal with $r(i,l_i) \neq 0$. Observe moreover that the event $\operatorname{p\,dim} \mathbb{S} = 0$ is readily recognized: It occurs if and only if $\mathbb{S} = \mathbb{S}(P)$; in this case, $\mathbf{Mod}(\mathbb{S})$ consists of the $\mathrm{GL}_d$-orbit of $P$ only.

We smooth the road towards a proof of Theorem 3.2 with two preliminary observations.

**Observation 3.3.** *Given any finitely generated $\Lambda$-module with skeleton $\sigma$, there exists a direct sum of tree modules with the same skeleton.*

*In particular, the syzygy of any finitely generated $\Lambda$-module is isomorphic to the syzygy of a direct sum of tree modules, and all projective dimensions in the set $\{0, 1, \dots, \operatorname{fin\,dim} \Lambda\}$ are attained on tree modules.*

*Proof.* Let $M$ be any finitely generated left $\Lambda$-module, $P = \bigoplus_{1 \le r \le t} \Lambda z_r$ a projective cover of $M$ with $z_r = e(r) \in \{e_1, \dots, e_n\}$, and $\sigma \subseteq P$ a skeleton of $M$. For fixed $r \le t$, let $\sigma^{(r)}$ be the subset of $\sigma$ consisting of all paths in $\sigma$ of the form $pz_r$. Then $\mathcal{T}^{(r)} := \Lambda z_r / (\sum_{q\ \sigma^{(r)}\text{-critical}} \Lambda q)$ is a tree module whose branches are precisely the maximal paths in $\sigma^{(r)}$ relative to the "initial subpath order". Hence, $\bigoplus_{1 \le r \le t} \mathcal{T}^{(r)}$ is a direct sum of tree modules, again having skeleton $\sigma$. Since, by Theorem 2.2, any skeleton of a module determines its syzygy up to isomorphism, the remaining claims follow. $\square$

The next observation singles out candidates for the tree module postulated in Theorem 3.2(3). Let $\epsilon$ be the sum of all non-cyclebound primitive idempotents in the full set $e_1, \dots, e_n$. (In Example 2.4, we have $\epsilon = e_8 + \cdots + e_{15}$.) Clearly, the left ideal $\Lambda\epsilon \subseteq \Lambda$ of finite projective dimension equals $\epsilon\Lambda\epsilon$. In particular, given any left $\Lambda$-module $M$, the subspace $\epsilon M$ is a sub*module* of $M$.

**Observation 3.4.** *Let $e_i$ be any vertex of $Q$. Then $\operatorname{p\,dim} \epsilon J e_i < \infty$, and*

$$\operatorname{p\,dim} \epsilon J e_i \ge \operatorname{p\,dim} \Lambda q,$$

*for every nonzero path $q$ of positive length in $\Lambda$ with starting vertex $e_i$ such that $\operatorname{p\,dim} \Lambda q < \infty$.*

*Moreover: The factor module $\mathcal{T}_i = \Lambda e_i / \epsilon J e_i$ is a tree module. If $\dim_K \mathcal{T}_i = d_i$, and $\mathbb{S}(\mathcal{T}_i) = \mathbb{S}^{(i)}$ is the radical layering of $\mathcal{T}_i$, then the subvariety $\mathbf{Mod}\big(\mathbb{S}^{(i)}\big)$ of $\mathbf{Mod}_{d_i}(\Lambda)$ coincides with the $\mathrm{GL}_{d_i}$-orbit of $\mathcal{T}_i$ and is open in $\mathbf{Mod}_{d_i}(\Lambda)$.*

*Proof.* We first address the second set of claims. Let $p_{ij}$, $1 \le j \le t_i$, be the different paths of positive length in $\Lambda$ which start in $e_i$, end in a non-cyclebound vertex, and are minimal with these properties in the "initial subpath order"; that is, every proper intial subpath of positive length of one of the $p_{ij}$ ends in a cyclebound vertex. Clearly, $\epsilon J e_i = \bigoplus_{1 \le j \le t_i} \Lambda p_{ij}$, which shows in particular that $\mathcal{T}_i$ is a tree module. Moreover, any module $M$ sharing the radical layering of $\mathcal{T}_i$ also has projective cover $\Lambda e_i$, and a comparison of composition factors shows that every epimorphism $\Lambda e_i \to M$ has kernel $\epsilon J e_i$. Thus $M \cong \mathcal{T}_i$, which shows $\mathbf{Mod}\big(\mathbb{S}^{(i)}\big)$ to equal the $\mathrm{GL}_{d_i}$-orbit of $\mathcal{T}_i$. Moreover, it is readily checked that $\mathrm{Ext}^1_\Lambda(\mathcal{T}_i, \mathcal{T}_i) = 0$, whence the orbit $\mathbf{Mod}(\mathbb{S}^{(i)})$ of $\mathcal{T}_i$ is open in $\mathbf{Mod}_{d_i}$ (see [5, Corollary 3]), and the proof of the final assertions is complete.

For the first claim, let $q = qe_i$ be a nonzero path of positive length in $\Lambda$ with $\mathrm{p\,dim}\,\Lambda q < \infty$. Then $q$ ends in a non-cyclebound vertex by Lemma 2.9 – call it $e$ – and hence $q$ has an initial subpath $q'$ among the paths $p_{ij}$; let $e'$ be the (non-cyclebound) terminal vertex of $q'$. If $l$ and $l'$ are the lengths of $q$ and $q'$, respectively, $c(e') - c(e) \geq l - l' \geq 0$, and hence $c(e') + l' \geq c(e) + l$. This shows

$$\mathrm{p\,dim}\,\Lambda q' = l'\text{-deg}(c(e')) \geq l\text{-deg}(c(e)) = \mathrm{p\,dim}\,\Lambda q,$$

which yields the desired inequality. ∎

*Proof of Theorem 3.2.* (1) Let $M$ be a module with radical layering $\mathbb{S}$ and $\sigma$ any skeleton of $M$. By [1, Theorem 5.3], the points in $\mathbf{Mod}_d(\Lambda)$ parametrizing the modules that share this skeleton constitute a dense open subset of $\mathbf{Mod}(\mathbb{S})$. All modules represented by this open subvariety have the same projective dimension as $M$, because any skeleton of a module pins down its syzygy up to isomorphism. Therefore, $\mathrm{p\,dim}\,M$ is the generic value of the function $\mathrm{p\,dim}$ on the irreducible subvariety $\overline{\mathbf{Mod}(\mathbb{S})}$ of $\mathbf{Mod}_d(\Lambda)$.

(2) Suppose that $\mathrm{p\,dim}\,\mathbb{S} > 0$, which means $r(i,l) > 0$ for some pair $(i,l)$. Let $M$ be any module with $\mathbb{S}(M) = \mathbb{S}$. By part (1), $\mathrm{p\,dim}\,\mathbb{S} = \mathrm{p\,dim}\,M$. To scrutinize the projective dimension of $M$, let $\widehat{\sigma}$ be a skeleton of $P$ and $\sigma \subset \widehat{\sigma}$ a skeleton of $M$. We have $\mathbf{\Omega}^1(M) \cong \bigoplus_{q\ \sigma\text{-critical}} \Lambda q$ by Theorem 2.2. Since $r(i,l) > 0$ whenever $q$ is a $\sigma$-critical path of length $l$ ending in $e_i$, we glean that $\mathrm{p\,dim}\,M$ is bounded above by the supremum displayed in part (2) of Theorem 3.2. For the reverse inequality, choose any pair $(i,l)$ with $r(i,l) > 0$. This inequality amounts to the existence of a path $pz_r$ of length $l$ in $\widehat{\sigma} \setminus \sigma$ which ends in $e_i$. Denote by $p'z_r$ the maximal initial subpath of $pz_r$ which belongs to $\sigma$. Since $pz_r \notin \sigma$, there is a unique arrow $\alpha$ such that $\alpha p' z_r$ is in turn an initial subpath of $pz_r$. In particular, if $q = \alpha p'$, then $qz_r$ is a $\sigma$-critical path ending in some vertex $e_j$. Invoking once again the above decomposition of $\mathbf{\Omega}^1(M)$, we deduce that the cyclic left ideal $\Lambda q$ is isomorphic to a direct summand of $\mathbf{\Omega}^1(M)$. By Theorem 2.6, it therefore suffices to show that the length$(q)$-degree of $c(e_j)$ is larger than or equal to $l$-deg$(c(e_i))$. For that purpose, we write $pz_r = q'qz_r$ for a suitable path $q'$ in $\Lambda$. Since $c(e_j) \geq c(e_i) + \text{length}(q')$, we obtain $c(e_j) \geq c(e_i)$, and consequently $c(e_j) + \text{length}(q) \geq c(e_i) + l$. We conclude

$$\left[\frac{c(e_j)}{L+1}\right] + \left[\frac{c(e_j) + \text{length}(q)}{L+1}\right] \geq \left[\frac{c(e_i)}{L+1}\right] + \left[\frac{c(e_i) + l}{L+1}\right] = l\text{-deg}(c(e_i)).$$

Thus $\mathrm{p\,dim}\,M - 1 \geq l$-deg$(c(e_i))$ as required. The final equivalence under (2) is an immediate consequence.

(3) By construction, the tree modules $\mathcal{T}_i$ of Observation 3.4 all have finite projective dimension. Combining the first part of this observation with the final statement of Theorem 2.2, we moreover see that $\mathrm{fin\,dim}\,\Lambda$ equals the maximum of these dimensions. The final statement of Observation 3.4 now completes the proof of (3). ∎

Let $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ again be a sequence of semisimple modules of total dimension $d$ such that $\mathbf{Mod}(\mathbb{S}) \neq \varnothing$. As we saw, the projective dimension $\operatorname{p\,dim} \mathbb{S}$ holds some information about path lengths in $KQ$; namely on the lengths of paths starting in vertices that belong to the support of $\mathbf{\Omega}^1(M)$, where $M$ is any module in $\mathbf{Mod}(\mathbb{S})$. To obtain a tighter correlation between $Q$ and the homology of $\Lambda$, we will next explore the full spectrum of values of the function $\operatorname{p\,dim}$ attained on the closure $\overline{\mathbf{Mod}(\mathbb{S})}$. While those ranges of values are better gauges of how the vertices corresponding to the simples in the various layers $\mathbb{S}_l$ of $\mathbb{S}$ are placed in the quiver $Q$, the refined homological data still do not account for the intricacy of the embedding of $\overline{\mathbf{Mod}(\mathbb{S})}$ into $\mathbf{Mod}_d(\Lambda)$ in general. (See the comments following the next theorem.) On the other hand, for $\operatorname{p\,dim} \mathbb{S} < \infty$ and small $L$, far more of this picture is preserved in the homology than in the hereditary case.

We first recall from [1, Section 2.B] that, for any $M$ in $\overline{\mathbf{Mod}(\mathbb{S})}$, the sequence $\mathbb{S}(M)$ is larger than or equal to $\mathbb{S}$ in the following partial order: Suppose that $\mathbb{S}$ and $\mathbb{S}'$ are semisimple modules with $\bigoplus_{0 \le l \le L} \mathbb{S}_l = \bigoplus_{0 \le l \le L} \mathbb{S}'_l$. Then "$\mathbb{S}' \geq \mathbb{S}$" means that $\bigoplus_{l \le r} \mathbb{S}_l$ is a direct summand of $\bigoplus_{l \le r} \mathbb{S}'_l$, for all $r \geq 0$. In intuitive terms this says that, in the passage from $\mathbb{S}$ to $\mathbb{S}'$, the simple summands of the $\mathbb{S}_l$ are only upwardly mobile relative to the layering.

**Lemma 3.5.** *If* $\mathbb{S}' \geq \mathbb{S}$ *and* $\mathbf{Mod}(\mathbb{S}') \neq \varnothing$, *then* $\operatorname{p\,dim} \mathbb{S}' \geq \operatorname{p\,dim} \mathbb{S}$.

*Proof.* Let $P$ be a projective cover of $\mathbb{S}_0$ as before and $P'$ a projective cover of $\mathbb{S}'_0$. Decompose the radical layers of $P'$ in analogy with the decomposition given for $P$ above:

$$J^l P'/J^{l+1} P' = \bigoplus_{1 \le i \le n} S_i^{s'(i,l)} \oplus \bigoplus_{1 \le i \le n} S_i^{r'(i,l)},$$

where $\mathbb{S}'_l = \bigoplus_{1 \le i \le n} S_i^{s'(i,l)}$. It follows immediately from the definition of the partial order on sequences of semisimples that, whenever $r(i,l) > 0$, there exists $l' \geq l$ with $r'(i,l') > 0$. In light of Theorem 3.2, this proves the lemma. ◻

We give two descriptions of the range of values of $\operatorname{p\,dim}$ on the closure $\overline{\mathbf{Mod}(\mathbb{S})}$. For a combinatorial version, we keep the notation of Theorem 3.2 and the proof of Lemma 3.5: Namely, $\mathbb{S}_l = \bigoplus_{1 \le i \le n} S_i^{s(i,l)}$, and $P$ is a projective cover of $\mathbb{S}_0$. From $\mathbf{Mod}(\mathbb{S}) \neq \varnothing$, one then obtains $J^l P/J^{l+1} P = \mathbb{S}_l \oplus \bigoplus_{1 \le i \le n} S_i^{r(i,l)}$. In our graph-based description of the values $\operatorname{p\,dim} M > \operatorname{p\,dim} \mathbb{S}$, where $M$ traces $\overline{\mathbf{Mod}(\mathbb{S})}$, the exponents $s(i,l)$ take over the role played by the $r(i,l)$ relative to the generic projective dimension, $\operatorname{p\,dim} \mathbb{S}$: Recall from Theorem 3.2 that, whenever $\operatorname{p\,dim} \mathbb{S}$ is nonzero, it is the maximum of the values $1 + l\text{-deg}(c(e_i)) \in \mathbb{N} \cup \{0, \infty\}$ which accompany the pairs $(i,l)$ with $r(i,l) > 0$. (Note: In view of $\mathbb{S}_0 = P/JP$, the inequality $r(i,l) > 0$ entails $l \geq 1$.)

Now, we consider the different candidates $n_1, \dots, n_v$ among those elements in $\mathbb{N} \cup \{0, \infty\}$ which have the form

$$1 + l\text{-deg}(c(e_j)),\ l \geq 1,\ S_j \subseteq \mathbb{S}_l$$

and are *strictly larger* than $\operatorname{p\,dim} \mathbb{S}$. In other words,

$$\{n_1, \dots, n_v\} = [\operatorname{p\,dim} \mathbb{S} + 1\,,\ \infty] \cap \{1 + l\text{-deg}(c(e_j)) \mid l \geq 1,\ s(j,l) > 0\}.$$

**Theorem 3.6.** *Let $\mathbb{S}$ be a semisimple sequence of total dimension $d$ with $\mathbf{Mod}(\mathbb{S}) \neq \varnothing$. The range of values,*

$$\{\operatorname{p\,dim} M \mid M \text{ in } \overline{\mathbf{Mod}(\mathbb{S})}\},$$

*of the function* $\operatorname{p\,dim}$ *on the closure of* $\mathbf{Mod}(\mathbb{S})$ *in* $\mathbf{Mod}_d$*, is equal to the following coinciding sets:*

$$\{\operatorname{p\,dim} \mathbb{S}' \mid \mathbb{S}' \geq \mathbb{S},\, \mathbf{Mod}(\mathbb{S}') \neq \varnothing\} = \{\operatorname{p\,dim} \mathbb{S}\} \cup \{n_1, \dots, n_v\}.$$

In general, describing the closure of $\mathbf{Mod}(\mathbb{S})$ in $\mathbf{Mod}_d(\Lambda)$ is an intricate representation-theoretic task, a fact not reflected by the homology. For instance: • When $\mathbb{S}'$ is a sequence of semisimple modules such that $\mathbb{S}' \geq \mathbb{S}$ and $\mathbf{Mod}(\mathbb{S}') \neq \varnothing$, the intersection $\overline{\mathbf{Mod}(\mathbb{S})} \cap \mathbf{Mod}(\mathbb{S}')$ may still be empty. • The condition $\overline{\mathbf{Mod}(\mathbb{S})} \cap \mathbf{Mod}(\mathbb{S}') \neq \varnothing$ does not imply $\mathbf{Mod}(\mathbb{S}') \subseteq \overline{\mathbf{Mod}(\mathbb{S})}$. See the final discussion of our example for illustration.

*Proof.* Set $\mathcal{P} = \{\operatorname{p\,dim} M \mid M \text{ in } \overline{\mathbf{Mod}(\mathbb{S})}\}$. We already know that

$$\mathcal{P} \subseteq \{\operatorname{p\,dim} \mathbb{S}' \mid \mathbb{S}' \geq \mathbb{S}\};$$

indeed, this is immediate from Lemma 3.5 and the remarks preceding it.

Suppose that $\mathbb{S}'$ is a sequence of semisimple modules with $\mathbb{S}' \geq \mathbb{S}$ and $\mathbf{Mod}(\mathbb{S}') \neq \varnothing$. Assume $\operatorname{p\,dim} \mathbb{S}' > \operatorname{p\,dim} \mathbb{S}$, which, in particular, implies $\operatorname{p\,dim} \mathbb{S}' > 0$. To show that $\operatorname{p\,dim} \mathbb{S}'$ equals one of the $n_k$, we adopt the notation used in the proof of Lemma 3.5. By Theorem 3.2, $\operatorname{p\,dim} \mathbb{S}' = 1 + a\text{-deg}(c(e_i))$ for some pair $(i,a)$ with $r'(i,a) > 0$. Again invoking Theorem 3.2, we moreover infer that $r(i,a) = 0$ from $\operatorname{p\,dim} \mathbb{S} < \operatorname{p\,dim} \mathbb{S}'$. If $s(i,a) > 0$, we are done, since necessarily $a \geq 1$. So let us suppose that also $s(i,a) = 0$, meaning that $S_i$ fails to be a summand of the $a$-th layer $J^a P / J^{a+1} P$ of $P$. In light of $S_i \subseteq J^a P' / J^{a+1} P'$, this entails the existence of a simple $S_j \subseteq \mathbb{S}'_0 / \mathbb{S}_0$ with the property that $S_i \subseteq J^a e_j / J^{a+1} e_j$. Consequently, $c(e_j) \geq c(e_i) + a$. On the other hand, $S_j \subseteq \bigoplus_{l \geq 1} \mathbb{S}_l$, because the total multiplicities of the simple summands of $\mathbb{S}$ and $\mathbb{S}'$ coincide. This means $s(j,k) > 0$ for some $k \geq 1$. In light of $\mathbb{S}_0 \oplus S_j \subseteq \mathbb{S}'_0$ and $\bigoplus_{0 \leq l \leq L} \mathbb{S}_l = \bigoplus_{0 \leq l \leq L} \mathbb{S}'_l$, we deduce that $r'(j,b) > 0$ for some pair $(j,b)$ with $b \geq 1$ and $s(j,b) > 0$. Another application of Theorem 3.2 thus yields

$$\begin{aligned} \operatorname{p\,dim} \mathbb{S}' - 1 \geq b\text{-deg}(c(e_j)) &= \left[\frac{c(e_j)}{L+1}\right] + \left[\frac{c(e_j) + b}{L+1}\right] \\ &\geq \left[\frac{c(e_i) + a}{L+1}\right] + \left[\frac{c(e_i) + a + b}{L+1}\right] \geq a\text{-deg}(c(e_i)) = \operatorname{p\,dim} \mathbb{S}' - 1. \end{aligned}$$

We conclude that all inequalities along this string are actually equalities, that is,

$$b\text{-deg}(c(e_j)) = a\text{-deg}(c(e_i)).$$

This shows that $\operatorname{p\,dim} \mathbb{S}' = 1 + b\text{-deg}(c(e_j))$ for a pair $(j, b)$ with $s(j, b) > 0$ as required.

Finally, we verify that each of the numbers $n_k$ belongs to $\mathcal{P}$. By definition, $n_k$ is of the form $1 + l\text{-deg}(c(e_i))$ for some pair $(i, l)$ with $l \geq 1$ and $s(i, l) > 0$. Let $D$ be any direct sum of tree modules with $\mathbb{S}(D) = \mathbb{S}$; in light of $\mathbf{Mod}(\mathbb{S}) \neq \varnothing$, such a module $D$ exists by Observation 3.3. Then there is a tree direct summand $\mathcal{T}$ of $D$ with a branch that contains an initial subpath $q$ of length $l$ ending in the vertex $e_i$. The direct sum of tree modules $D' = (\mathcal{T}/\Lambda q) \oplus \Lambda q \oplus D/\mathcal{T}$ belongs to $\overline{\mathbf{Mod}(\mathbb{S})}$. In fact, $D'$ is well known to belong to the closure of the orbit of $D$ in $\mathbf{Mod}_d(\Lambda)$; see, e.g., [3, Section 3, Lemma 2]. Therefore $\operatorname{p\,dim} D' \in \mathcal{P}$. As for the value of this projective dimension: Up to isomorphism, $\Lambda q$ is a direct summand of the syzygy of the tree module $\mathcal{T}/\Lambda q$: indeed, $q$ is $\sigma$-critical relative to the obvious skeleton $\sigma$ of $\mathcal{T}/\Lambda q$ consisting of all initial subpaths of the branches (see Theorem 2.2 and the comments accompanying Definition 2.7). Theorems 2.6 and 3.2 moreover yield

$$\operatorname{p\,dim} D' - 1 \geq \operatorname{p\,dim} \Lambda q = l\text{-deg}(c(e_i)) = n_k - 1 > \operatorname{p\,dim} \mathbb{S} - 1 = \operatorname{p\,dim} D - 1,$$

whence $\operatorname{p\,dim}(\mathcal{T}/\Lambda q) = n_k = \operatorname{p\,dim} D'$. This shows $n_k$ to belong to $\mathcal{P}$ and completes the argument. ∎

**A final visit to Example 2.4.** (a) First, let $\mathbb{S} = \mathbb{S}(\Lambda e_1)$ be the radical layering of the projective tree module $\mathcal{T} = \Lambda e_1$. By Theorems 3.2 and 3.6, the values of $\operatorname{p\,dim}$ on $\overline{\mathbf{Mod}(\mathbb{S})}$ are $\operatorname{p\,dim} \mathbb{S} = 0, 2, 3, 4, \infty$. For instance, $\operatorname{p\,dim}\big((\mathcal{T}/\Lambda\beta_3\alpha_2\beta_1) \oplus (\Lambda\beta_3\alpha_2\beta_1)\big)$ equals 2. Note that the value 1, on the other hand, is not attained.

(b) Next we justify the comments following the statement of Theorem 3.6. Let $\mathbb{S}$ and $\mathbb{S}'$ be the radical layerings of the modules $M$ and $M'$ with the following graphs, respectively:

$$\begin{array}{ccccc} & & 9 & & \\ & & | & & \\ & & 10 & & \\ & / & & \backslash & \\ 11 & & & & 13 \\ | & & & & \\ 12 & & & & \end{array} \qquad\qquad \begin{array}{ccc} 9 & & 12 \\ | & & | \\ 10 & \oplus & 13 \\ | & & \\ 11 & & \end{array}$$

Then $\mathbb{S}' \geq \mathbb{S}$, while $\overline{\mathbf{Mod}(\mathbb{S})} \cap \mathbf{Mod}(\mathbb{S}') = \varnothing$.

On the other hand, if $\mathbb{S} := \mathbb{S}(N)$, $\mathbb{S}' := \mathbb{S}(N')$, and $\mathbb{S}'' := \mathbb{S}(N'')$ where $N$, $N'$, and $N''$ are given by the graphs

$$\begin{array}{ccc} 10 & & \\ | & \backslash & \\ 11 & & 13 \\ | & & \\ 12 & & \end{array} \qquad\qquad \begin{array}{ccccc} 10 & & & & 12 \\ | & \backslash & & \oplus & \\ 11 & & 13 & & \end{array} \qquad\qquad \begin{array}{ccc} 10 & & 12 \\ | & \oplus & | \\ 11 & & 13 \end{array}$$

then $\mathbb{S}' = \mathbb{S}'' \geq \mathbb{S}$, and the intersection $\overline{\mathbf{Mod}(\mathbb{S})} \cap \mathbf{Mod}(\mathbb{S}')$ contains $N'$, but not $N''$.

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