# A NOTE ON STABLE EQUIVALENCES OF MORITA TYPE

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Abstract. We investigate when an exact functor $F \cong - \otimes_\Lambda M_\Gamma : \text{mod-}\Lambda \to \text{mod-}\Gamma$ which induces a stable equivalence is part of a stable equivalence of Morita type. If $\Lambda$ and $\Gamma$ are finite dimensional algebras over a field $k$ whose semisimple quotients are separable, we give a necessary and sufficient condition for this to be the case. This generalizes a result of Rickard's for self-injective algebras. As a corollary, we see that the two functors given by tensoring with the bimodules in a stable equivalence of Morita type are right and left adjoints of one another, provided that these bimodules are indecomposable. This fact has many interesting consequences for stable equivalences of Morita type. In particular, we show that a stable equivalence of Morita type induces another stable equivalence of Morita type between certain self-injective algebras associated to the original algebras. We further show that when there exists a stable equivalence of Morita type between $\Lambda$ and $\Gamma$, it is possible to replace $\Lambda$ by a Morita equivalent $k$-algebra $\Delta$ such that $\Gamma$ is a subring of $\Delta$ and the induction and restriction functors induce inverse stable equivalences.

## 1. Introduction

Given a stable equivalence $\alpha$ between two finite dimensional $k$-algebras $\Lambda$ and $\Gamma$, it is natural to ask whether $\alpha$ is induced by a functor between the entire module categories of the two algebras. That is, given an equivalence $\alpha$ does there exist a functor $F$ making the following diagram commute up to isomorphism?

$$\begin{array}{ccc} \text{mod } \Lambda & \overset{F}{\dashrightarrow} & \text{mod } \Gamma \\ \downarrow & & \downarrow \\ \underline{\text{mod}}\ \Lambda & \xrightarrow{\alpha} & \underline{\text{mod}}\ \Gamma \end{array}$$

Clearly such an $F$ must take projective $\Lambda$-modules to projective $\Gamma$-modules. In order to make this problem somewhat more tractable, we shall consider only the case when $F$ can be chosen to be exact. In particular, as is well-known, the right-exactness of $F$ implies that $F \cong - \otimes_\Lambda M_\Gamma$, for some bimodule ${}_\Lambda M_\Gamma$. Recall that $M$ is nothing more than $F(\Lambda_\Lambda)$, which has a natural left $\Lambda$-module structure that makes it a bimodule. Now, $F$ is exact if and only if ${}_\Lambda M$ is projective, while $F(\text{proj-}\Lambda) \subseteq \text{proj-}\Gamma$ if and only if $F(\Lambda) \cong M_\Gamma$ is projective. Hence, $\alpha$ is induced by an exact functor between module categories if and only if it is induced by a functor of the form $- \otimes_\Lambda M_\Gamma$ for a bimodule $M$ which is projective as a right $\Gamma$-module and as a left $\Lambda$-module. Pogorzaly has already studied bimodules with this property in connection with stable equivalence in [16], and we shall continue to call such bimodules **left-right projective**. Clearly, every projective bimodule is left-right projective, but not conversely. The following definition, due to Broué, includes sufficient conditions for such a bimodule to induce a stable equivalence.

**Definition 1** ([6]). *A pair of left-right projective bimodules ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$ is said to induce a stable equivalence of **Morita type** between $\Lambda$ and $\Gamma$ if we have the following isomorphisms* *of bimodules:*

$$_{\Lambda}M \otimes_{\Gamma} N_{\Lambda} \cong {_{\Lambda}\Lambda_{\Lambda}} \oplus {_{\Lambda}P_{\Lambda}} \text{ and } {_{\Gamma}N} \otimes_{\Lambda} M_{\Gamma} \cong {_{\Gamma}\Gamma_{\Gamma}} \oplus {_{\Gamma}Q_{\Gamma}},$$

*where* $_{\Lambda}P_{\Lambda}$ *and* $_{\Gamma}Q_{\Gamma}$ *are projective bimodules.*

It is immediate from the above definition that the functors $-\otimes_{\Lambda} M$ and $-\otimes_{\Gamma} N$ induce inverse stable equivalences between $\Lambda$ and $\Gamma$. Hence we shall also say that a stable equivalence $\alpha : \underline{\text{mod}}\text{-}\Lambda \to \underline{\text{mod}}\text{-}\Gamma$ is of Morita type if there exist a pair of bimodules $M$ and $N$ as above such that $-\otimes_{\Lambda} M$ induces $\alpha$ between the stable categories. In some instances in the literature, $\alpha$ is said to be of Morita type simply if it and a suitable inverse equivalence are induced by tensoring with left-right projective bimodules $M$ and $N$ respectively, without requiring the bimodule isomorphisms in the above definition. A theorem of Rickard's [17] partially resolves this discrepancy by proving that any stable equivalence between self-injective algebras that lifts to an exact functor is of Morita type, provided the algebras split over a separable extension of the ground field $k$. In this article we present a partial generaliziation of this theorem to arbitrary finite dimensional algebras subject to the same separability hypothesis. While it does not appear to be the case that these two types of stable equivalences necessarily coincide in general, we give necessary and sufficient conditions on a left-right projective bimodule $M$ that induces a stable equivalence to be one of a pair of bimodules inducing a stable equivalence of Morita type. As a result, we will see that many facts about stable equivalences of Morita type for self-injective algebras remain true in general. Furthermore, we use these facts to show that in the context of [15] a stable equivalence of Morita type between two algebras induces another stable equivalence of Morita type between their associated self-injective algebras. Finally, we derive an interesting corollary, which essentially states that all stable equivalences of Morita type can be realized by induction and restriction functors.

Throughout this article we shall assume that the algebras $\Lambda$ and $\Gamma$ are finite dimensional over a field $k$ and have no semisimple blocks. Furthermore, as mentioned above, we shall need their semisimple quotients $\Lambda/\text{rad}\,\Lambda$ and $\Gamma/\text{rad}\,\Gamma$ to be *separable*, meaning that they remain semisimple in any extension of scalars to a field $K$ containing $k$ (see [7] for more details). This is equivalent to the algebras $\Lambda$ and $\Gamma$ having splitting fields which are separable extensions of $k$, and this is always the case, for instance, if $k$ is a perfect field. This assumption is necessary to ensure that $\Lambda/\text{rad}\,\Lambda \otimes_k \Gamma^{op}/\text{rad}\,\Gamma^{op}$ is a semisimple ring, and thus that $\Lambda \otimes_k \Gamma^{op}/\text{rad}\,(\Lambda \otimes_k \Gamma^{op}) \cong \Lambda/\text{rad}\,\Lambda \otimes_k \Gamma/\text{rad}\,\Gamma$ as left $(\Lambda \otimes_k \Gamma^{op})$-modules. Except where noted otherwise, we identify $(\Lambda,\Gamma)$-bimodules with *right* modules over the algebra $\Lambda^{op} \otimes_k \Gamma$ in the natural way. We also note that we shall identify pairs of isomorphic functors. In particular, we shall say that a functor $\alpha$, between stable module categories, is induced by or lifts to a functor $F$, between module categories, as long as $\alpha$ is isomorphic to the functor induced by $F$.

## 2. Exact functors inducing stable equivalences

Our goal in this section is to compare stable equivalences induced by exact functors with stable equivalences of Morita type. Unlike in the self-injective case, these two types of stable equivalences do not appear to coincide in general. However, we will see that a left-right projective bimodule $_{\Lambda}M_{\Gamma}$, for which $-\otimes_{\Lambda} M_{\Gamma}$ induces a stable equivalence, is part of a stable equivalence of Morita type if and only if $\text{Hom}_{\Lambda}(M,\Lambda)$ is a projective left $\Gamma$-module. The basic strategy, as in Rickard's proof, is to apply the theory of adjoint functors to $-\otimes_{\Lambda} M_{\Gamma}$ and its left adjoint. In fact we will see that, just as in the self-injective case, the exact functors given by tensoring with the two indecomposable bimodules that induce a stable equivalence of Morita type are left and right adjoints of one another.

We start out with a simple preliminary result regarding the indecomposabilty of bimodules inducing a stable equivalence of Morita type. The following lemma, whose proof is due to Rouquier, appears for self-injective algebras in [10].

**Lemma 2.1.** *Suppose that* ${}_\Lambda M_\Gamma$ *and* ${}_\Gamma N_\Lambda$ *are left-right projective bimodules that give a stable equivalence of Morita type between* $\Lambda$ *and* $\Gamma$*, and assume that either* $\Lambda$ *or* $\Gamma$ *is indecomposable as an algebra. Then* $M$ *and* $N$ *each have a unique (up to isomorphism) indecomposable, nonprojective bimodule summand. If we denote these summands as* $M'$ *and* $N'$ *respectively, then* $M'$ *and* $N'$ *also induce a stable equivalence of Morita type.*

*Proof.* Without loss of generality, suppose $\Lambda$ is indecomposable. Thus ${}_\Lambda\Lambda_\Lambda$ is an indecomposable bimodule. Let $M\otimes_\Gamma N \cong \Lambda\oplus U$ and $N\otimes_\Lambda M \cong \Gamma\oplus V$, where $U$ and $V$ are projective bimodules. If $M = M'\oplus M''$ as bimodules, then $\Lambda\oplus U \cong M'\otimes_\Gamma N\oplus M''\otimes_\Gamma N$. Since $\Lambda$ is indecomposable, either $M'\otimes_\Gamma N$ or $M''\otimes_\Gamma N$ is a projective bimodule. If, without loss of generality, $M''\otimes_\Gamma N$ is projective, so is $M''$ as it is a summand of $M''\otimes_\Gamma N\otimes_\Lambda M$, which is clearly projective.

Similarly, if $N = N'\oplus N''$, we have $\Lambda\oplus U \cong M\otimes_\Gamma N'\oplus M\otimes_\Gamma N''$, and this implies that, without loss of generality, $M\otimes_\Gamma N''$ is projective. As above, we now tensor on the left with $N$ to conclude that $N''$ must be projective. For the final statement, we have $\Lambda\oplus U = (M'\oplus M'')\otimes_\Gamma (N'\oplus N'')$ where $M''$ and $N''$ are projective bimodules. Clearly, once we expand the right hand side, the only nonprojective term is $M'\otimes_\Gamma N'$, and it follows that this is isomorphic to $\Lambda\oplus U'$ for some summand $U'$ of $U$. Likewise, we also obtain $N\otimes_\Lambda M\cong \Gamma\oplus V'$ for a summand $V'$ of $V$. □

In the sequel our arguments will require that the bimodules inducing a stable equivalence of Morita type be indecomposable. The above result shows that there is no real loss in generality assuming this, provided that at least one of the algebras is indecomposable.

Next, we prove a similar result for arbitrary left-right projective bimodules inducing stable equivalences. In general we cannot conclude that such a bimodule has a unique *nonprojective* indecomposable summand, unless we assume that either $\Lambda/\text{rad }\Lambda$ or $\Gamma/\text{rad }\Gamma$ is separable (see the remark after Theorem 2.8). The proof of this result is completely different from the above proof, and in fact extends Linckelmann's original methods for the self-injective case (cf. [9], Proposition 2.4).

**Proposition 2.2.** *Suppose* ${}_\Lambda M_\Gamma$ *is a left-right projective bimodule such that* $-\otimes_\Lambda M_\Gamma$ *induces a stable equivalence. If* $\Lambda$ *is indecomposable as an algebra, then* ${}_\Lambda M_\Gamma$ *has an indecomposable nonprojective* $(\Lambda,\Gamma)$*-bimodule summand* $M'$ *such that* $-\otimes_\Lambda M'_\Gamma \cong -\otimes_\Lambda M_\Gamma$ *as functors from* $\underline{\text{mod}}$-$\Lambda$ *to* $\underline{\text{mod}}$-$\Gamma$.

*Proof.* Suppose ${}_\Lambda M_\Gamma \cong {}_\Lambda A_\Gamma\oplus {}_\Lambda B_\Gamma$. We shall show that either $X\otimes_\Lambda A_\Gamma$ is projective for every $X_\Lambda$ or $X\otimes_\Lambda B_\Gamma$ is projective for every $X_\Lambda$. It will then follow that $-\otimes_\Lambda M_\Gamma$ is isomorphic to $-\otimes_\Lambda B_\Gamma$ or to $-\otimes_\Lambda A_\Gamma$, respectively, as functors between stable categories. Clearly, we can then complete the argument by induction on the number of indecomposable bimodule summands of $M$.

Notice that since $M$ induces a stable equivalence, for each indecomposable nonprojective module $X_\Lambda$, there is a unique indecomposable nonprojective summand of $X\otimes_\Lambda M_\Gamma$. It follows that exactly one of $X\otimes_\Lambda A_\Gamma$ and $X\otimes_\Lambda B_\Gamma$ is projective. On the other hand, $X_\Lambda$ is projective if and only if both $X\otimes_\Lambda A_\Gamma$ and $X\otimes_\Lambda B_\Gamma$ are projective. Furthermore, note that if $X_\Lambda$ and $Y_\Lambda$ are nonisomorphic indecomposable modules then the nonprojective summands of $X \otimes_\Lambda M_\Gamma$ and $Y \otimes_\Lambda M_\Gamma$ must be nonisomorphic.

Now let $S_\Lambda$ and $T_\Lambda$ be nonisomorphic simple $\Lambda$-modules such that $0 \to T \longrightarrow U \longrightarrow S \to 0$ is a nonsplit short exact sequence. Without loss of generality, suppose $S \otimes_\Lambda A_\Gamma$ is not projective. Hence,

$$0 \to T \otimes_\Lambda B_\Gamma \longrightarrow U \otimes_\Lambda B_\Gamma \longrightarrow S \otimes_\Lambda B_\Gamma \to 0$$

splits. Since the nonprojective part of $T \otimes_\Lambda M_\Gamma$ cannot be a summand of $U \otimes_\Lambda M_\Gamma$, it follows that $T \otimes_\Lambda B_\Gamma$ and $U \otimes_\Lambda B_\Gamma$ are both projective. Therefore, we have shown that if $\mathrm{Ext}^1_\Lambda(S, T) \neq 0$ and $S \otimes_\Lambda B_\Gamma$ is projective, then $T \otimes_\Lambda B_\Gamma$ is also projective.

Now suppose in addition that $\mathrm{Ext}^1_\Lambda(S', T) \neq 0$ for another simple module $S'_\Lambda$. We shall show that $S' \otimes_\Lambda B_\Gamma$ is projective. If not, we must have $S' \otimes_\Lambda A_\Gamma$ projective, which as above implies that $T \otimes_\Lambda A_\Gamma$ is projective. Hence $T_\Lambda$ is a simple projective, and we have a nonsplit short exact sequence $0 \to U \longrightarrow V \longrightarrow S' \to 0$, which can be obtained as the pushout of a nonsplit sequence $0 \to T \longrightarrow W \longrightarrow S' \to 0$ with respect to the inclusion $T \to U$ above.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & T & \longrightarrow & W & \longrightarrow & S' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & S' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & S & = & S & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Notice that $V$ is not projective, for we have a short exact sequence $0 \to T \longrightarrow U \oplus W \longrightarrow V \to 0$ which does not split since $\mathrm{Hom}_\Lambda(U \oplus W, T) = 0$. Now, in the above diagram the bottom sequence must split upon tensoring with $A$, yielding $V \otimes_\Lambda A_\Gamma \cong S' \otimes_\Lambda A_\Gamma \oplus U \otimes_\Lambda A_\Gamma$. On the other hand, the vertical sequence $0 \to W \longrightarrow V \longrightarrow S \to 0$ must split upon tensoring with $B$, yielding $V \otimes_\Lambda B_\Gamma \cong S \otimes_\Lambda B_\Gamma \oplus W \otimes_\Lambda B_\Gamma$. However, this implies that the unique indecomposable nonprojective summand of $V \otimes_\Lambda M_\Gamma$ is isomorphic to the unique indecomposable nonprojective summand of either $U \otimes_\Lambda M_\Gamma$ or $W \otimes_\Lambda M_\Gamma$, which is a contradiction as $U, W$ and $V$ are all indecomposable and nonisomorphic.

Finally, since $\Lambda$ is indecomposable, its Ext quiver is connected and the above argument shows that $S \otimes_\Lambda B_\Gamma$ is projective for every simple module $S_\Lambda$. It now follows by induction on the length of $X_\Lambda$ that $X \otimes_\Lambda B_\Gamma$ is projective for every $X_\Lambda$. $\square$

We remark that the assumption that $\Lambda$ is indecomposable appears to be necessary, although we do not require any similar assumption about $\Gamma$. Still, between algebras that are not self-injective, even assuming that they have no semisimple blocks, there exist stable equivalences that do not preserve the decomposabilty of the algebras. This occurs, for example, with constructions of nodes as described in [14]. However, we point out here that a construction of nodes can never lift to an exact functor. This is most easily seen by noting that any exact functor that induces a stable equivalence must preserve projective dimensions, whereas a construction of nodes does not: if $S_\Lambda$ is a simple projective that corresponds to a node $T_\Gamma$, we have pdim $TrDT =$ pdim $\alpha(TrDS) \neq$ pdim $TrDS = 1$. Still, it would be interesting to know if there are other examples of stable equivalences that do not preserve decomposability of the algebras but are nevertheless induced by exact functors.

It is not hard to see that if a pair of bimodules ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$ induce a stable equivalence of Morita type, then $M$ and $N$ are projective generators as left or right modules [11]. We now proceed to show that the same holds for any left-right projective bimodule ${}_\Lambda M_\Gamma$ that induces a stable equivalence. At the same time, we will obtain some interesting information about the inverse stable equivalence.

**Proposition 2.3.** *Suppose ${}_\Lambda M_\Gamma$ is a bimodule such that $M_\Gamma$ is projective and $-\otimes_\Lambda M_\Gamma$ induces a stable equivalence. Then $M_\Gamma$ is a projective generator.*

*Proof.* Let $\beta : \underline{\text{mod}}\text{-}\Gamma \to \underline{\text{mod}}\text{-}\Lambda$ be an equivalence inverse to the one induced by $-\otimes_\Lambda M_\Gamma$. If $S_\Gamma$ is simple and nonprojective, take a projective cover $\pi : P \to \beta(S)$ in mod-$\Lambda$. Then $\pi \otimes 1_M$ is an epimorphism from $P \otimes_\Lambda M_\Gamma$ to $\beta(S) \otimes_\Lambda M_\Gamma \cong S \oplus Q$ for some projective right $\Gamma$-module $Q$. But $P \otimes_\Lambda M_\Gamma \in \text{add}(M_\Gamma)$, and hence the projective cover of $S$ belongs to $\text{add}(M_\Gamma)$.

Next suppose that $S_\Gamma$ is a simple projective module. Since $S$ cannot be injective, as $\Gamma$ has no semisimple blocks, there exists an almost split sequence $0 \to S \longrightarrow P \longrightarrow TrDS \to 0$ with $P_\Gamma$ projective. It follows from Proposition 2.6 in [3] that we have an almost split sequence $0 \to U \xrightarrow{f} Q \xrightarrow{g} \beta(TrDS) \to 0$ in mod-$\Lambda$ with $Q_\Lambda$ projective. Furthermore, Theorem V.3.3 of [5] implies that the maps $P \to TrDS$ and $Q \to \beta(TrDS)$ are projective covers. If we tensor the latter almost split sequence with $M$ we obtain the exact sequence

$$U \otimes_\Lambda M_\Gamma \longrightarrow Q \otimes_\Lambda M_\Gamma \xrightarrow{g\otimes 1} TrDS \oplus P' \to 0,$$

where $P'_\Gamma$ is projective. Thus $U \otimes_\Lambda M_\Gamma$ admits an epimorphism onto $\ker(g \otimes 1)$ which is isomorphic to the direct sum of $S$ with some projective $\Gamma$-module, as $Q \otimes_\Lambda M_\Gamma$ is projective. Finally, if $R_\Lambda$ is the projective cover of $U_\Lambda$, we obtain an epimorphism $R \otimes_\Lambda M_\Gamma \to U \otimes_\Lambda M_\Gamma \to \ker(g \otimes 1) \to S_\Gamma$. Again, since $R \otimes_\Lambda M_\Gamma \in \text{add}(M_\Gamma)$, so is $S_\Gamma$. $\square$

Our next lemma is a special case of Proposition 1.1 in [1].

**Lemma 2.4.** *Let $F : \text{mod-}\Lambda \to \text{mod-}\Gamma$ be left adjoint to $G : \text{mod-}\Gamma \to \text{mod-}\Lambda$, and assume both functors take projectives to projectives. Then $F$ and $G$ induce a pair of adjoint functors $\underline{F}$ and $\underline{G}$ between $\underline{\text{mod}}\text{-}\Lambda$ and $\underline{\text{mod}}\text{-}\Gamma$.*

In this context, one easily checks that the adjunction $\eta : \text{Hom}_\Gamma(F-,-) \to \text{Hom}_\Lambda(-,G-)$ induces the adjunction between $\underline{F}$ and $\underline{G}$. Hence, if $\epsilon_X : X \to GFX$ is the unit of $\eta$, then $\underline{\epsilon_X} : \underline{X} \to \underline{GFX}$ gives the unit of the induced adjunction between $\underline{F}$ and $\underline{G}$, and similarly for the counit. In particular, if we know that $\underline{G}$ is an equivalence, then $\underline{F}$ must be an inverse equivalence, and the unit and counit of the adjunction $\eta$ induce isomorphisms in the stable categories.

We are interested in applying this lemma when $G = -\otimes_\Lambda M_\Gamma$, for a left-right projective bimodule $M$, induces a stable equivalence $\alpha$. In this case, letting ${}_\Gamma N_\Lambda = {}_\Gamma M^*_\Lambda = \text{Hom}_\Lambda({}_\Lambda M_\Gamma, {}_\Lambda \Lambda_\Lambda)$, we have that $-\otimes_\Gamma N_\Lambda$ is left adjoint to $-\otimes_\Lambda M_\Gamma$. Since $N_\Lambda$ is projective, the lemma implies that $-\otimes_\Gamma N_\Lambda$ induces a stable equivalence $\beta$, which is inverse to $\alpha$. Furthermore, Propostion 2.3 now tells us that $N_\Lambda$ is a projective generator. Since $N_\Lambda$ is the $\Lambda$-dual of the projective module ${}_\Lambda M$, we see that ${}_\Lambda M$ is also a projective generator. Hence we have proven the following.

**Proposition 2.5.** *If ${}_\Lambda M_\Gamma$ is a left-right projective bimodule such that $- \otimes_\Lambda M_\Gamma$ induces a stable equivalence, then both ${}_\Lambda M$ and $M_\Gamma$ are projective generators. In particular, $M$ is faithful on either side.*

Now returning to the above context, we analyze the unit $\varepsilon_X : X_\Gamma \to X \otimes_\Gamma N \otimes_\Lambda M_\Gamma$ of the adjunction. According to Lemma 2.4 and the remarks thereafter, $\varepsilon_X$ induces an isomorphism in the stable category for any $X_\Gamma$. Notice that, since ${}_\Lambda M$ is projective, we have a $(\Gamma,\Gamma)$-bimodule isomorphism $N \otimes_\Lambda M \cong \mathrm{End}_\Lambda(M)$. With this identification, it is easy to see that the map $h : X \to X \otimes_\Gamma \mathrm{End}_\Lambda(M)$ given by $h(x) = x \otimes 1_M$ corresponds to $1_{X\otimes N}$ via the sequence of isomorphisms

$$\begin{aligned} \mathrm{Hom}_\Gamma(X, X \otimes_\Gamma \mathrm{End}_\Lambda(M)) &\cong \mathrm{Hom}_\Gamma(X, X \otimes_\Gamma N \otimes_\Lambda M) \\ &\cong \mathrm{Hom}_\Gamma(X, \mathrm{Hom}_\Lambda(N, X \otimes_\Gamma N)) \\ &\cong \mathrm{End}_\Lambda(X \otimes_\Gamma N). \end{aligned}$$

It follows that $\varepsilon_X$ is given by $h$. In particular, $\varepsilon_\Gamma : \Gamma \to \Gamma \otimes_\Gamma N \otimes_\Lambda M \cong \mathrm{End}_\Lambda(M)$ is given by $\varepsilon_\Gamma(\gamma) = \gamma \cdot 1_M$, and this is clearly a $(\Gamma,\Gamma)$-bimodule map with $\varepsilon_X = 1_X \otimes \varepsilon_\Gamma$ for any $X_\Gamma$. Furthermore, $\varepsilon_\Gamma$ is injective since $M_\Gamma$ is faithful. This yields the following short exact sequence of bimodules

$$\eta : 0 \to \Gamma \xrightarrow{\varepsilon_\Gamma} N \otimes_\Lambda M \longrightarrow P \to 0.$$

Tensoring this sequence with any nonprojective indecomposable $X_\Gamma$ on the left, we obtain a right exact sequence

$$X \xrightarrow{\varepsilon_X} X \otimes_\Gamma N \otimes_\Lambda M \longrightarrow X \otimes_\Gamma P \to 0.$$

Since $\varepsilon_X$ induces an isomorphism in $\underline{\mathrm{mod}}$-$\Gamma$, it is not contained in the radical of mod-$\Gamma$, and hence must be a split monomorphism with a projective cokernel. Thus, for each indecomposable nonprojective $X_\Gamma$, $X \otimes_\Gamma \eta$ is exact and $X \otimes_\Gamma P_\Gamma$ is projective. In addition, we shall need the following.

**Lemma 2.6.** *If ${}_\Gamma P_\Gamma = \mathrm{coker}\ \varepsilon_\Gamma$, then $P_\Gamma$ is projective.*

*Proof.* It suffices to check that $R \otimes_\Gamma P_\Gamma$ is projective for each indecomposable projective $\Gamma$-module $R$. Consider a minimal projective presentation $P_1 \to P_0 \to X_\Gamma \to 0$ with $X$ indecomposable and nonprojective. Tensoring this sequence with $\eta$ we obtain a commutative diagram in mod-$\Gamma$ with exact rows and columns.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \xrightarrow{\varepsilon_{P_1}} & P_1 \otimes_\Gamma N \otimes_\Lambda M & \longrightarrow & P_1 \otimes_\Gamma P & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle g_1} & & \downarrow & & \\
0 & \longrightarrow & P_0 & \xrightarrow{\varepsilon_{P_0}} & P_0 \otimes_\Gamma N \otimes_\Lambda M & \longrightarrow & P_0 \otimes_\Gamma P & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle g_0} & & \downarrow & & \\
0 & \longrightarrow & X & \xrightarrow{\varepsilon_X} & X \otimes_\Gamma N \otimes_\Lambda M & \longrightarrow & X \otimes_\Gamma P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The bottom sequence splits since $X \otimes_\Gamma P_\Gamma$ is projective. Let $h$ be a splitting for $\varepsilon_X$, and let $h_0 : P_0 \otimes_\Gamma N \otimes_\Lambda M \to P_0$ and $h_1 : P_1 \otimes_\Gamma N \otimes_\Lambda M \to P_1$ be lifts of $h$. We have $f_0 h_0 \varepsilon_{P_0} = h g_0 \varepsilon_{P_0} = h \varepsilon_X f_0 = f_0$. Since $f_0$ is a projective cover, $h_0 \varepsilon_{P_0}$ is an isomorphism and thus the middle sequence splits. It follows that $P_0 \otimes_\Gamma P_\Gamma$ is projective. Similarly,

$f_1 h_1 \varepsilon_{P_1} = h_0 g_1 \varepsilon_{P_1} = h_0 \varepsilon_{P_0} f_1$ which is also a projective cover of $\Omega X$ as $h_0 \varepsilon_{P_0}$ restricts to an isomorphism on $\Omega X \subset P_0$. Thus $h_1 \varepsilon_{P_1}$ must be an isomorphism and the top sequence splits as well, yielding that $P_1 \otimes_\Gamma P_\Gamma$ is also projective.

Finally, if $R_\Gamma$ is a projective indecomposable and not simple, $R$ is the projective cover of its top and the above implies that $R \otimes_\Gamma P_\Gamma$ is projective. On the other hand, if $S_\Gamma$ is a simple projective, we have an almost split sequence $0 \to S \longrightarrow R \longrightarrow TrDS \to 0$ which is in fact a minimal projective presentation of $TrDS$. Hence, the above argument also shows that $S \otimes_\Gamma P_\Gamma$ is projective. □

We now know that $X \otimes_\Gamma P_\Gamma$ is projective for any $X_\Gamma$. Henceforth, we shall call any bimodule with this property **strongly right projective**. For example, any bimodule of the form ${}_\Lambda X \otimes_k Q_\Gamma$, where $X$ is any left $\Lambda$-module and $Q_\Gamma$ is projective, is strongly right projective. Of course, any direct summand or direct sum of strongly right projective bimodules is also strongly right projective. With these simple observations we can now prove one of our key results.

**Theorem 2.7.** *If ${}_\Lambda M_\Gamma$ and ${}_\Gamma L_\Lambda$ are indecomposable bimodules inducing a stable equivalence of Morita type between $\Lambda$ and $\Gamma$, then $L \cong N = \mathrm{Hom}_\Lambda(M, \Lambda)$ as $(\Gamma, \Lambda)$-bimodules.*

*Proof.* Let $M \otimes_\Gamma L \cong \Lambda \oplus U$ and $L \otimes_\Lambda M \cong \Gamma \oplus V$ for projective bimodules $U$ and $V$. Since $M_\Gamma$ is projective we have an exact sequence of $(\Lambda, \Gamma)$-bimodules

$$M \otimes_\Gamma \eta : 0 \to M \xrightarrow{1_M \otimes \varepsilon_\Gamma} M \otimes_\Gamma N \otimes_\Lambda M \longrightarrow M \otimes_\Gamma P \to 0.$$

Identifying $N \otimes_\Lambda M$ with $\mathrm{End}_\Lambda(M)$, the map $1_M \otimes \varepsilon_\Gamma$ corresponds to the map sending $m \in M$ to $m \otimes 1_M \in M \otimes_\Gamma \mathrm{End}_\Lambda(M)$. It is clear that this is a $(\Lambda, \Gamma)$-bimodule homomorphism, and it is straightforward to check that it is split by the map sending $m \otimes f \in M \otimes_\Gamma \mathrm{End}_\Lambda(M)$ to $(m)f \in M$, which is also a $(\Lambda, \Gamma)$-bimodule homomorphism. Hence we have $M \otimes_\Gamma N \otimes_\Lambda M \cong M \oplus M \otimes_\Gamma P$ as bimodules. We now tensor this identity on the right with $L$ to obtain

$$M \otimes_\Gamma N \otimes_\Lambda M \otimes_\Gamma L \cong M \otimes_\Gamma L \oplus M \otimes_\Gamma P \otimes_\Gamma L \cong \Lambda \oplus U \oplus M \otimes_\Gamma P \otimes_\Gamma L.$$

On the other hand, we have

$$M \otimes_\Gamma N \otimes_\Lambda M \otimes_\Gamma L \cong M \otimes_\Gamma N \otimes_\Lambda (\Lambda \oplus U) \cong M \otimes_\Gamma N \oplus M \otimes_\Gamma N \otimes_\Lambda U.$$

But observe that the bimodules $U$, $M \otimes_\Gamma P \otimes_\Gamma L$ and $M \otimes_\Gamma N \otimes_\Lambda U$ are all strongly right projective, as $U$ is projective, $P$ is strongly right projective and $L_\Lambda$ is projective. Thus, the first direct sum decomposition shows us that all bimodule summands of $M \otimes_\Gamma N \otimes_\Lambda M \otimes_\Gamma L$ that are not strongly right projective must occur as a summand of $\Lambda$. Moreover, since $\Lambda$ is assumed to have no semisimple blocks, $\Lambda$ can have no strongly right projective bimodule summands. If we now compare the two decompositions, we see that $M \otimes_\Gamma N \cong \Lambda \oplus Q$ as bimodules, where $Q$ is strongly right projective.

Next, we tensor this identity on the left with $L$ to obtain the following pair of decompositions:

$$L \otimes_\Lambda M \otimes_\Gamma N \cong L \oplus L \otimes_\Lambda Q \text{ and } L \otimes_\Lambda M \otimes_\Gamma N \cong (\Gamma \oplus V) \otimes_\Gamma N \cong N \oplus V \otimes_\Gamma N.$$

Clearly, $L \otimes_\Lambda Q$ and $V \otimes_\Gamma N$ are strongly right projective. Since $L$ and $N$ are not strongly right projective, and we have assumed $L$ is indecomposable, $L$ must be isomorphic to a direct summand of $N$. Hence it remains only to check that $N$ is indecomposable. To verify this, notice that the isomorphism between ${}_\Lambda M$ and its double dual with respect to $\Lambda$ is in fact an isomorphism of $(\Lambda, \Gamma)$-bimodules, that is, $M \cong \mathrm{Hom}_\Lambda(N, \Lambda)$ as bimodules. Therefore, if $N = A \oplus B$ for nonzero bimodules $A$ and $B$, we would have $M \cong \mathrm{Hom}_\Lambda(A, \Lambda) \oplus \mathrm{Hom}_\Lambda(B, \Lambda)$

as bimodules, and neither summand vanishes as $A_\Lambda$ and $B_\Lambda$ must be projective since $N_\Lambda$ is. But this contradicts the indecomposability of $M$. □

In the following sections we shall investigate the implications this theorem has for stable equivalences of Morita type. First, however, we prove a result in the other direction. Notice that one consequence of the preceding theorem is that ${}_\Gamma N$ is projective. In fact, under a mild separability hypothesis on the algebras $\Lambda$ and $\Gamma$, this is a sufficient condition for $M$ and $N$ to give a stable equivalence of Morita type. The following result is due to Auslander and Reiten, although we state a slightly more general version since the proof is identical. We point out that this is the only part of our argument that requires the semisimple quotients of $\Lambda$ and $\Gamma$ to be separable.

**Theorem 2.8** ([4]). *If $\Lambda$ and $\Gamma$ are finite dimensional $k$-algebras such that either $\Lambda/\mathrm{rad}\ \Lambda$ or $\Gamma/\mathrm{rad}\ \Gamma$ is separable, then the following are equivalent for a bimodule ${}_\Lambda P_\Gamma$ with ${}_\Lambda P$ projective.*

1. *The bimodule ${}_\Lambda P_\Gamma$ is projective.*
2. *For every $X_\Lambda$, the tensor product $X \otimes_\Lambda P_\Gamma$ is a projective right $\Gamma$-module.*
3. *If $J = \mathrm{rad}\ \Lambda$, then $\Lambda/J \otimes_\Lambda P_\Gamma$ is a projective right $\Gamma$-module.*

*Remarks.* (1) We note that the separability hypothesis in the above theorem is essential, as is illustrated by the following example found in [8]. Suppose that $k$ is a nonperfect field of characteristic $p$ with $a \in k \setminus k^p$, and let $K = k(\alpha)$ where $\alpha^p = a$. Then $K^{op} \otimes_k K$ is isomorphic to $K[t]/(t-\alpha)^p$, which is not semisimple. Hence there exists a nonprojective $(K, K)$-bimodule $P$. However, since $K$ is a field, ${}_K P$ and $X \otimes_K P_K$ are clearly projective over $K$ for any $X_K$.

(2) Notice that in the context of the proof of Proposition 2.2 above, the bimodule $B$ satisfies condition (2) of the theorem. Hence, if the algebras satisfy the separability hypothesis of the theorem, the conclusion of Proposition 2.2 can be strengthened to say that $M$ contains, up to isomorphism, a unique indecomposable nonprojective bimodule summand.

**Theorem 2.9.** *Let $\Lambda$ and $\Gamma$ be finite dimensional $k$-algebras whose semisimple quotients are separable, and suppose that ${}_\Lambda M_\Gamma$ is a left-right projective bimodule such that $- \otimes_\Lambda M_\Gamma$ induces a stable equivalence $\underline{\mathrm{mod}}\text{-}\Lambda \to \underline{\mathrm{mod}}\text{-}\Gamma$. If ${}_\Gamma N_\Lambda = \mathrm{Hom}_\Lambda(M, \Lambda)$ is projective over $\Gamma$, then $M$ and $N$ induce a stable equivalence of Morita type between $\Lambda$ and $\Gamma$.*

*Proof.* Recall that for any $X_\Gamma$, the sequence $X \otimes_\Gamma \eta$ remains exact. It follows that $\eta$ is pure exact as a sequence of left $\Gamma$-modules. Since ${}_\Gamma P$ is finitely presented, $\eta$ must split on the left. Therefore ${}_\Gamma P$ is a direct summand of ${}_\Gamma N \otimes_\Lambda M$ which is projective since ${}_\Lambda M$ and ${}_\Gamma N$ are. Since $P$ is strongly right projective, Theorem 2.8 implies that it is a projective bimodule and thus $\eta$ splits as a sequence of bimodules, yielding ${}_\Gamma N \otimes_\Lambda M_\Gamma \cong {}_\Gamma \Gamma_\Gamma \oplus {}_\Gamma P_\Gamma$.

In order to obtain the second required isomorphism, we now consider the counit $\delta$ of the adjunction between $- \otimes_\Gamma N_\Lambda$ and $- \otimes_\Lambda M_\Gamma$. For any $Y_\Lambda$, the morphism $\delta_Y : Y \otimes_\Lambda M \otimes_\Gamma N \to Y$ corresponds to $1_{Y \otimes M}$ via the sequence of isomorphisms

$$\mathrm{Hom}_\Lambda(Y \otimes_\Lambda M \otimes_\Gamma N, Y) \cong \mathrm{Hom}_\Gamma(Y \otimes_\Lambda M, \mathrm{Hom}_\Lambda(N, Y)) \cong \mathrm{Hom}_\Gamma(Y \otimes_\Lambda M, Y \otimes_\Lambda M).$$

Thus, for $y \in Y$, $m \in M$, and $u \in N = \mathrm{Hom}_\Lambda(M, \Lambda)$, we have $\delta_Y(y \otimes m \otimes u) = y \cdot (m)u$. In particular, $\delta_\Lambda : \Lambda \otimes_\Lambda M \otimes_\Gamma N \to \Lambda$ can be identified with the evaluation morphism taking $m \otimes u \in M \otimes_\Gamma N$ to $(m)u$. It is not hard to see that this is in fact a $(\Lambda, \Lambda)$-bimodule map. Furthermore, if we identify $Y \otimes_\Lambda \Lambda_\Lambda$ with $Y_\Lambda$, the map $1_Y \otimes \delta_\Lambda$ corresponds to $\delta_Y$.

Observe next that $\delta_\Lambda$ is surjective. Clearly, the image of $\delta_\Lambda$ equals the trace ideal $\operatorname{tr}({}_\Lambda M) = \Lambda$, since ${}_\Lambda M$ is a projective generator according to Proposition 2.5. It follows that we have a short exact sequence of $(\Lambda, \Lambda)$-bimodules

$$\mu : 0 \to Q \xrightarrow{i} M \otimes_\Gamma N \xrightarrow{\delta_\Lambda} \Lambda \to 0.$$

We first check that $Q$ is a projective bimodule. Since $\mu$ must split as a sequence of left $\Lambda$-modules, it remains exact upon tensoring (over $\Lambda$) with any $Y_\Lambda$ on the left. Therefore, if $Y_\Lambda$ is indecomposable and nonprojective, $\ker \delta_Y \cong Y \otimes_\Lambda Q_\Lambda$ must be projective since $\underline{\delta}_Y$ is an isomorphism in $\underline{\text{mod}}$-$\Lambda$. We also know that $M \otimes_\Gamma N$ is left-right projective, and since $\mu$ also splits as a sequence of right $\Lambda$-modules, $Q$ is left-right projective. Thus $Q$ is a projective bimodule by Theorem 2.8.

Finally, we are ready to show that $\mu$ splits as a sequence of bimodules. Since $\Lambda$ is clearly finitely presented as a right module over the enveloping algebra $\Lambda^e = \Lambda^{op} \otimes_k \Lambda$, it suffices to show that $\mu$ is pure exact over $\Lambda^e$. First notice that for any $Y_\Lambda$, since $\delta_Y$ induces an isomorphism in $\underline{\text{mod}}$-$\Lambda$ when $Y$ is not projective, $Y \otimes_\Lambda \mu$ is split exact in mod-$\Lambda$. Hence for any $\Lambda$-modules $Y_\Lambda$ and ${}_\Lambda X$, the sequence $\mu \otimes_{\Lambda^e} (Y \otimes_k X) \cong Y \otimes_\Lambda \mu \otimes_\Lambda X$ is exact. In particular, noting that the indecomposable injective left $\Lambda^e$-modules have the form

$$D(\Lambda^e(e_i \otimes e_j))^* \cong D((e_i \otimes e_j)\Lambda^e) \cong \operatorname{Hom}_k(\Lambda e_i \otimes_k e_j\Lambda, k) \cong D(\Lambda e_i) \otimes_k D(e_j\Lambda),$$

where $e_i$ and $e_j$ belong to a complete set of pairwise orthogonal primitive idempotents for $\Lambda$, we have that $\mu \otimes_{\Lambda^e} I$ is exact for any injective left $\Lambda^e$-module $I$. Now, if $A$ is an arbitrary left $\Lambda^e$-module, let $f : A \hookrightarrow I$ be an injective envelope and tensor $f$ on the left with $\mu$ to obtain the following commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & Q \otimes_{\Lambda^e} A & \xrightarrow{i\otimes 1} & (M \otimes_\Gamma N) \otimes_{\Lambda^e} A & \longrightarrow & \Lambda \otimes_{\Lambda^e} A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1\otimes f} & & \downarrow{\scriptstyle 1\otimes f} & & \downarrow & & \\
0 & \longrightarrow & Q \otimes_{\Lambda^e} I & \xrightarrow{i\otimes 1} & (M \otimes_\Gamma N) \otimes_{\Lambda^e} I & \longrightarrow & \Lambda \otimes_{\Lambda^e} I & \longrightarrow & 0,
\end{array}$$

where $1_Q \otimes f$ is injective since $Q$ is a projective right $\Lambda^e$-module. A simple diagram chase now shows that $i \otimes 1_A$ is injective, and hence $\mu \otimes_{\Lambda^e} A$ is exact. $\square$

Unfortunately, examples of indecomposable left-right projective bimodules ${}_\Lambda M_\Gamma$ that induce a stable equivalence are hard to come by, and we are unaware of any such $M$ for which $N = \operatorname{Hom}_\Lambda(M, \Lambda)$ is not projective over $\Gamma$. However, by Theorem 2.7, such an example would show that Rickard's theorem for self-injective algebras does not extend to arbitrary algebras. It would even be interesting to determine whether such an $M$ must induce a stable equivalence of Morita type, under the additional assumption that there is a left-right projective bimodule $L$ that induces the inverse stable equivalence.

## 3. Consequences for stable equivalences of Morita type

In this section we shall use Theorem 2.7 to derive some interesting results on stable equivalences of Morita type, which generalize known results for self-injective algebras.

**Corollary 3.1.** *Assume ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$ are indecomposable bimodules that induce a stable equivalence of Morita type between $\Lambda$ and $\Gamma$. Then the following are true.*

(1) *We have bimodule isomorphisms* $N \cong \mathrm{Hom}_\Lambda(M,\Lambda) \cong \mathrm{Hom}_\Gamma(M,\Gamma)$ *and* $M \cong \mathrm{Hom}_\Lambda(N,\Lambda) \cong \mathrm{Hom}_\Gamma(N,\Gamma)$.
(2) *The functor* $-\otimes_\Lambda M_\Gamma$ *is right and left adjoint to* $-\otimes_\Gamma N_\Lambda$

*Proof.* The first isomorphism in (1) is the conclusion of Theorem 2.7. Applying the theorem with $N$ in place of $M$ we get $M \cong \mathrm{Hom}_\Gamma(N,\Gamma)$ as bimodules. Since tensoring with $M$ and $N$ on the left also induces an equivalence between the stable left module categories $\Lambda$-$\underline{\mathrm{mod}}$ and $\Gamma$-$\underline{\mathrm{mod}}$, all of our above arguments can be carried out in a similar manner for left modules. Thus we obtain dual isomorphisms $N \cong \mathrm{Hom}_\Gamma(M,\Gamma)$ and $M \cong \mathrm{Hom}_\Lambda(N,\Lambda)$.

For (2), we have already noted that $-\otimes_\Gamma N_\Lambda$ is left adjoint to $-\otimes_\Lambda M_\Gamma$. But the latter also has a right adjoint given by $-\otimes_\Gamma \mathrm{Hom}_\Gamma(M,\Gamma) \cong -\otimes_\Gamma N_\Lambda$, since for any $X_\Lambda$ and $Y_\Gamma$ we have natural isomorphisms

$$\mathrm{Hom}_\Gamma(X\otimes_\Lambda M_\Gamma, Y) \cong \mathrm{Hom}_\Lambda(X, \mathrm{Hom}_\Gamma(M,Y)) \cong \mathrm{Hom}_\Lambda(X, Y\otimes_\Gamma \mathrm{Hom}_\Gamma(M,\Gamma)).\square$$

We now present two simple consequences that will be essential in the next section. We note that neither of them appears to follow directly from the definition of a stable equivalence of Morita type. For a one-sided module $A$ over a ring $R$ we shall write $A^*$ to denote its dual $\mathrm{Hom}_R(A,R)$ with respect to the ring $R$, where the ring $R$ will be made clear by context. Notice that for the bimodules $M$ and $N$ as above, the preceding corollary shows that there is no ambiguity in the notation $M^*$ or $N^*$. We shall also denote by $\nu_R$ the Nakayama functor $D\mathrm{Hom}_R(-,R)$, where $D$ is the standard duality with respect to the ground field $k$.

**Lemma 3.2.** *Suppose* ${}_\Lambda M_\Gamma$ *is an indecomposable bimodule inducing a stable equivalence of Morita type. Then, if* $I_\Lambda$ *is injective, so is* $I\otimes_\Lambda M_\Gamma$.

*Proof.* Recall that $-\otimes_\Gamma N_\Lambda$ is exact and left adjoint to $-\otimes_\Lambda M_\Gamma$ where $N = \mathrm{Hom}_\Lambda(M,\Lambda)$. Hence, we have an isomorphism of functors $\mathrm{Hom}_\Gamma(-, I\otimes_\Lambda M_\Gamma) \cong \mathrm{Hom}_\Lambda(-\otimes_\Gamma N_\Lambda, I_\Lambda)$, and the latter is exact as it is the composite of $-\otimes_\Gamma N_\Lambda$ and $\mathrm{Hom}_\Lambda(-,I)$. □

**Lemma 3.3.** *Suppose* ${}_\Lambda M_\Gamma$ *and* ${}_\Gamma N_\Lambda$ *are indecomposable bimodules that induce a stable equivalence of Morita type. Then, for every* $X_\Lambda$ *there exist natural isomorphisms* $(X\otimes_\Lambda M_\Gamma)^* \cong {}_\Gamma N\otimes_\Lambda X^*$ *and* $\nu_\Gamma(X\otimes_\Lambda M_\Gamma) \cong \nu_\Lambda X\otimes_\Lambda M_\Gamma$.

*Proof.* We have

$$\begin{aligned}(X\otimes_\Lambda M_\Gamma)^* &= \mathrm{Hom}_\Gamma(X\otimes_\Lambda M_\Gamma, \Gamma)\\ &\cong \mathrm{Hom}_\Lambda(X, \mathrm{Hom}_\Gamma(M,\Gamma))\\ &\cong \mathrm{Hom}_\Lambda(X,N)\\ &\cong {}_\Gamma N\otimes_\Lambda X^*\end{aligned}$$

since $N_\Lambda$ is projective. If we now apply the duality $D$ to each side, we obtain

$$\begin{aligned}\nu_\Gamma(X\otimes_\Lambda M_\Gamma) &\cong \mathrm{Hom}_k(N\otimes_\Lambda X^*, k)\\ &\cong \mathrm{Hom}_\Lambda(N, \mathrm{Hom}_k(X^*,k))\\ &\cong DX^*\otimes_\Lambda \mathrm{Hom}_\Lambda(N,\Lambda)\\ &\cong \nu_\Lambda X\otimes_\Lambda M_\Gamma. \square\end{aligned}$$

In fact, if $X$ is a $(\Gamma,\Lambda)$-bimodule, then the first sequence of isomorphisms in the above proof consists of $(\Gamma,\Gamma)$-bimodule isomorphisms. As a consequence, we obtain the following.

**Proposition 3.4.** *Suppose* ${}_\Lambda M_\Gamma$ *and* ${}_\Gamma N_\Lambda$ *are indecomposable bimodules that induce a stable equivalence of Morita type, and write* $M\otimes_\Gamma N \cong \Lambda\oplus U$ *and* $N\otimes_\Gamma M \cong \Gamma\oplus V$ *for projective* *bimodules $U$ and $V$. Then $U \cong U^* = \mathrm{Hom}_\Lambda(U_\Lambda, \Lambda_\Lambda)$ and $V \cong V^* = \mathrm{Hom}_\Gamma(V_\Gamma, \Gamma_\Gamma)$ as bimodules.*

*Proof.* By the previous lemma and the remark afterwards, we have $(\Gamma, \Gamma)$-bimodule isomorphisms

$$\begin{aligned} \Gamma^* \oplus V^* &\cong \mathrm{Hom}_\Gamma(\Gamma_\Gamma \oplus V_\Gamma, \Gamma_\Gamma) \\ &\cong \mathrm{Hom}_\Gamma(N \otimes_\Lambda M_\Gamma, \Gamma_\Gamma) \\ &\cong N \otimes_\Lambda N^* \\ &\cong N \otimes_\Lambda M \\ &\cong \Gamma \oplus V. \end{aligned}$$

Since $\Gamma \cong \Gamma^*$ as bimodules, $V$ must be isomorphic to its dual as well. Analogously, one proves $U \cong U^*$. □.

It is well-known that if two algebras are stably equivalent, then the injectively stable module categories are equivalent as well. This follows from the fact that $DTr : \underline{\mathrm{mod}}\text{-}\Lambda \to \overline{\mathrm{mod}}\text{-}\Lambda$ is an equivalence [5]. However, in the case of stable equivalences of Morita type, we actually have the same functor inducing both equivalences.

**Corollary 3.5.** *If ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$ are indecomposable bimodules that induce a stable equivalence of Morita type, then the functors $-\otimes_\Lambda M_\Gamma$ and $-\otimes_\Gamma N_\Lambda$ also induce inverse equivalences between the injectively stable categories $\overline{\mathrm{mod}}\text{-}\Lambda$ and $\overline{\mathrm{mod}}\text{-}\Gamma$.*

*Proof.* We have already seen that tensoring with $M$ or $N$ takes injectives to injectives, and hence induces a functor between the injectively stable module categories. To complete the proof it suffices to show that $X \otimes_\Lambda U_\Lambda$ and $Y \otimes_\Gamma V_\Gamma$ are injective for all $X_\Lambda$ and all $Y_\Gamma$. We will establish the former by showing that the functor $\mathrm{Hom}_\Lambda(-, X \otimes_\Lambda U_\Lambda)$ is exact. To see this, notice that it is isomorphic to

$$\mathrm{Hom}_\Lambda(-, \mathrm{Hom}_\Lambda(U^*, X)) \cong \mathrm{Hom}_\Lambda(- \otimes_\Lambda U^*, X) \cong \mathrm{Hom}_\Lambda(- \otimes_\Lambda U, X).$$

Since $U$ is a projective bimodule, $- \otimes_\Lambda U$ takes exact sequences to split exact sequences, and hence $\mathrm{Hom}_\Lambda(- \otimes_\Lambda U, X)$ is exact for any $X_\Lambda$. □

## 4. Associated self-injective algebras

Recently, Liu and Xi ([12], [13]) have obtained various methods for constructing new stable equivalences of Morita type between non self-injective algebras out of such stable equivalences between pairs of self-injective algebras. In this section, we shall apply the above ideas to prove a result in the other direction. That is, a stable equivalence of Morita type between non self-injective algebras naturally induces a stable equivalence of Morita type between certain associated self-injective algebras. In fact these associated self-injective algebras are the same as those studied by the second author in [15], and we now review their construction.

Express the algebra $\Lambda$ as a direct sum of indecomposable projective right $\Lambda$-modules $\Lambda_\Lambda = \oplus_{i=1}^n e_i\Lambda$, and let $\mathcal{P} = \{i \mid \nu^j(e_i\Lambda) \text{ is projective injective for all } j \geq 0\}$ where $\nu = D\mathrm{Hom}_\Lambda(-, \Lambda)$ is the Nakayama functor. Now define $e = \sum_{i\in\mathcal{P}} e_i$ and $P_\Lambda = e\Lambda$, and recall from [15] that $\Delta = \mathrm{End}_\Lambda(P) \cong e\Lambda e$ is self-injective. Furthermore, the functor $- \otimes_\Delta P_\Lambda$ : mod-$\Delta \to$ mod-$\Lambda$ induces an equivalence between mod-$\Delta$ and the full subcategory of mod-$\Lambda$ whose objects are the modules $X_\Lambda$ having a projective presentation $P_1 \to P_0 \to X \to 0$

with $P_1, P_0 \in \mathrm{add}(P)$. An inverse equivalence is provided by the functor $\mathrm{Hom}_\Lambda(P, -) : \text{mod-}\Lambda \to \text{mod-}\Delta$. Note that this equivalence also induces a full embedding of stable categories $\underline{\text{mod}}\text{-}\Delta \hookrightarrow \underline{\text{mod}}\text{-}\Lambda$.

Now suppose that $\Gamma$ is stably equivalent to $\Lambda$ and that $e'$, $P'_\Gamma$ and $\Delta' = \mathrm{End}_\Gamma(P') \cong e'\Gamma e'$ are defined as above. In [15] it is shown that any stable equivalence between $\Lambda$ and $\Gamma$ restricts to a stable equivalence between the associated self-injective algebras $\Delta$ and $\Delta'$, where we identify $\underline{\text{mod}}\text{-}\Delta$ and $\underline{\text{mod}}\text{-}\Delta'$ with the equivalent subcategories of $\underline{\text{mod}}\text{-}\Lambda$ and $\underline{\text{mod}}\text{-}\Gamma$, respectively. We shall prove that if the original stable equivalence between $\Lambda$ and $\Gamma$ is of Morita type, then so is the induced stable equivalence between $\Delta$ and $\Delta'$.

From Lemma 3.2 we know that a stable equivalence of Morita type takes injective projectives to injective projectives, and combining this with Lemma 3.3 we can show that it must take $\mathrm{add}(P_\Lambda)$ to $\mathrm{add}(P'_\Gamma)$.

**Lemma 4.1.** *Let $P_\Lambda$, $P'_\Gamma$ and ${}_\Lambda M_\Gamma$ be as above. Then $P \otimes_\Lambda M_\Gamma \in \mathrm{add}(P'_\Gamma)$.*

*Proof.* By Lemma 3.3, for each $j \geq 0$ we have $\nu^j(P \otimes_\Lambda M_\Gamma) \cong \nu^j P \otimes_\Lambda M_\Gamma$, which is projective-injective by Lemma 3.2 since $\nu^j P$ is. □

**Theorem 4.2.** *Suppose that ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$ are indecomposable bimodules inducing a stable equivalence of Morita type between $\Lambda$ and $\Gamma$, and let $\Delta = \mathrm{End}_\Lambda(P)$ and $\Delta' = \mathrm{End}_\Gamma(P')$ be the associated self-injective algebras as defined above. Then, provided the semisimple quotients of $\Delta$ and $\Delta'$ are separable, the induced stable equivalence between $\Delta$ and $\Delta'$ is of Morita type. In fact, it is induced by the bimodules ${}_\Delta eMe'_{\Delta'}$ and ${}_{\Delta'}e'Ne_\Delta$.*

*Proof.* Let $F$ and $G$ denote the restrictions of the functors $-\otimes_\Lambda M_\Gamma$ and $-\otimes_\Gamma N_\Lambda$ to mod-$\Delta$ and mod-$\Delta'$ respectively. Since $F$ is exact and takes $P_\Lambda$ to a projective module in $\mathrm{add}(P'_\Gamma)$, $F$ sends mod-$\Delta$ to mod-$\Delta'$, and similarly $G$ must send mod-$\Delta'$ to mod-$\Delta$. It is clear that, between stable categories, $F$ and $G$ induce the restrictions of the stable equivalences induced by $-\otimes_\Lambda M_\Gamma$ and $-\otimes_\Gamma N_\Lambda$ respectively, and hence $F$ and $G$ induce inverse stable equivalences. Notice that $F$ is right exact since the embedding of mod-$\Delta$ in mod-$\Lambda$ is given by $-\otimes_\Delta P_\Lambda$ which is right exact. Moreover, $G$ is left adjoint to $F$, as $F$ and $G$ are the restrictions of a pair of adjoint functors to full subcategories. But since $F$ has a left adjoint, it preserves kernels, and hence $F$ must be exact. By the theorem of Rickard's that we have mentioned earlier (Theorem 3.2 in [17]), $F$ and $G$ give a stable equivalence of Morita type. Taking into account the equivalence given above between mod-$\Delta$ and the corresponding full subcategory of mod-$\Lambda$, we have $F \cong \mathrm{Hom}_\Gamma(P', -\otimes_\Delta P \otimes_\Lambda M_\Gamma) \cong -\otimes_\Delta P \otimes_\Lambda M \otimes_\Gamma (P')^*_{\Delta'}$, which simplifies to

$$-\otimes_\Delta e\Lambda \otimes_\Lambda M \otimes_\Gamma \Gamma e'_{\Delta'} \cong -\otimes_\Delta eMe'_{\Delta'},$$

and similarly $G \cong -\otimes_{\Delta'} e'Ne_\Delta$. □

We remark that it would be interesting to know whether the converse of this result is also true. That is, given two stably equivalent algebras, does a stable equivalence of Morita type between their associated self-injective algebras necessarily extend to a stable equivalence of Morita type between the given algebras? As a special case, we might consider when the associated self-injective algebras happen to be zero, that is the *self-injective free* case, using the terminology of [15]. Here we can show that any stable equivalence of Morita type between such algebras is in fact a Morita equivalence. This is actually a weaker version of a question raised in [15] as to whether there exist nonisomorphic stably equivalent basic, self-injective free algebras without nodes or semisimple blocks.

**Theorem 4.3.** *If ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$ are indecomposable bimodules inducing a stable equivalence of Morita type between two self-injective free algebras $\Lambda$ and $\Gamma$, then $M$ and $N$ induce a Morita equivalence.*

*Proof.* We write $M \otimes_\Gamma N \cong \Lambda \oplus U$ and $N \otimes_\Lambda M \cong \Gamma \oplus V$ for projective bimodules $U$ and $V$. It suffices to show that $U$ and $V$ must be zero. Since $U$ is projective, it must be isomorphic to a direct sum of bimodules of the form $\Lambda e_i \otimes_k e_j \Lambda$ where $e_i$ and $e_j$ belong to a complete set of primitive orthogonal idempotents for $\Lambda$. But notice that we have bimodule isomorphisms

$$\begin{aligned} (\Lambda e_i \otimes_k e_j\Lambda)^* &= \mathrm{Hom}_\Lambda(\Lambda e_i \otimes_k e_j\Lambda, \Lambda_\Lambda) \\ &\cong \mathrm{Hom}_k(\Lambda e_i, \mathrm{Hom}_\Lambda(e_j\Lambda, \Lambda)) \\ &\cong \mathrm{Hom}_k(\Lambda e_i, \Lambda e_j) \\ &\cong \Lambda e_j \otimes_k D(\Lambda e_i). \end{aligned}$$

By Proposition 3.4, we know that $U \cong U^*$, so if $\Lambda e_i \otimes_k e_j\Lambda$ is a summand of $U$, so is $\Lambda e_j \otimes_k D(\Lambda e_i)$ as well as its $\Lambda$ dual, which is isomorphic to $(D(\Lambda e_i))^* \otimes_k \nu_\Lambda(e_j\Lambda)$ as above. In particular, since $U_\Lambda$ is projective, $\nu_\Lambda(e_j\Lambda)$ must be projective. Repeating this argument, we see that $\nu^i_\Lambda(e_j\Lambda)$ is projective for all $i \geq 0$, or that $e_j\Lambda \in \mathrm{add}(P_\Lambda)$ to use the notation introduced above. However, by definition $P_\Lambda = 0$ since $\Lambda$ is self-injective free. Thus $U$, and similarly $V$, must vanish as well. □

*Remark.* We take a moment to look at the above results in the context of [12], Section 5. There, Liu and Xi define a full, extension-closed subcategory $\mathcal{C}$ of mod-$\Lambda$, whose objects are the $\Lambda$-modules $X_\Lambda$ such that $X \otimes_\Lambda U_\Lambda = 0$. Our analysis in this section shows that $\mathcal{C}$ always contains all the simple modules $S \notin \mathrm{add}(P/PJ)$, where $P_\Lambda$ is as defined at the beginning of this section. To see this, notice that by the proof of the last theorem, the projective bimodule $U$ is a direct sum of bimodules of the form $\Lambda e_i \otimes_k e_j\Lambda$ with $e_i\Lambda \in \mathrm{add}(P)$. However, if $S$ is as above, $S \otimes_\Lambda \Lambda e_i \otimes_k e_j\Lambda \cong Se_i \otimes_k e_j\Lambda = 0$, since $Se_i \cong \mathrm{Hom}_\Lambda(e_i\Lambda, S) = 0$.

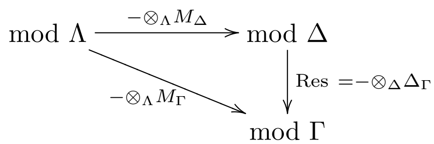
Furthermore, when $\Lambda$ and $\Gamma$ are self-injective, Liu and Xi show that if the idempotent $e \in \Lambda$ is chosen so that the objects in $\mathcal{C}$ are precisely those $X$ with $Xe = 0$, and $f \in \Gamma$ is chosen similarly, then the stable equivalence of Morita type between $\Lambda$ and $\Gamma$ induces a stable equivalence of Morita type between the corner rings $e\Lambda e$ and $f\Gamma f$ (see Theorem 5.7 in [12] and the appendix in [13]). Furthermore, their proof in the appendix of [13] only makes use of the self-injective hypothesis to obtain the adjoint relations between $-\otimes_\Lambda M_\Gamma$ and $-\otimes_\Gamma N_\Lambda$, which we have generalized in Corollary 3.1. The rest of their proof, therefore, carries over without difficulty to this more general situation. For the reader's convenience, we summarize this below as a corollary. Note, in particular, how Liu and Xi's approach avoids any separability assumptions on the semisimple quotients of $\Lambda$ and $\Gamma$, or their corner rings.

**Corollary 4.4.** *Let $\Lambda$ and $\Gamma$ be arbitrary $k$-algebras without semisimple blocks, and assume that ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$ are indecomposable bimodules that induce a stable equivalence of Morita type between $\Lambda$ and $\Gamma$. If $e$ and $f$ are the idempotents defined above, then there exists a stable equivalence of Morita type between the algebras $e\Lambda e$ and $f\Gamma f$, given by the pair of bimodules $eMf$ and $fNe$. Furthermore, $e\Lambda e$ and $f\Gamma f$ are always corner rings inside the associated self-injective algebras of $\Lambda$ and $\Gamma$.*

## 5. Change of Rings

In this section we present an interesting corollary that shows stable equivalences of Morita type can be realized by restriction and induction functors. We start by supposing that ${}_\Lambda M_\Gamma$

is a left-right projective bimodule such that $- \otimes_\Lambda M_\Gamma$ induces a stable equivalence between $\Lambda$ and $\Gamma$. By Proposition 2.5, ${}_\Lambda M$ is a projective generator, and hence $\Delta := \mathrm{End}_\Lambda(M)$ is Morita equivalent to $\Lambda$, and $- \otimes_\Lambda M_\Delta$ gives an equivalence of categories. By the same theorem, $M_\Gamma$ is faithful, and hence the ring homomorphism $\Gamma \to \Delta$, which makes $M$ a right $\Gamma$-module, is injective. Moreover, the following diagram of functors commutes.




It follows immediately that $\mathrm{Res}_\Gamma^\Delta$ takes projectives to projectives, that is $\Delta_\Gamma$ is projective, and induces a stable equivalence between $\Delta$ and $\Gamma$. By Lemma 2.4, the induction functor $\mathrm{Ind}_\Gamma^\Delta = - \otimes_\Gamma \Delta_\Delta$, being left adjoint to $\mathrm{Res}_\Gamma^\Delta$, induces an inverse stable equivalence. In case $M$ is part of a stable equivalence of Morita type we can say more.

**Corollary 5.1.** *Let $\Lambda$ and $\Gamma$ be finite dimensional $k$-algebras whose semisimple quotients are separable. If at least one of them is indecomposable, then the following are equivalent.*

(1) *There exists a stable equivalence of Morita type between $\Lambda$ and $\Gamma$.*
(2) *There exists a $k$-algebra $\Delta$, Morita equivalent to $\Lambda$, and an injective ring homomorphism $\Gamma \hookrightarrow \Delta$ such that the restriction and induction functors are exact and induce inverse stable equivalences.*
(3) *There exists a $k$-algebra $\Delta$, Morita equivalent to $\Lambda$, and an injective ring homomorphism $\Gamma \hookrightarrow \Delta$ such that*

$$ {}_\Gamma\Delta_\Gamma = {}_\Gamma\Gamma_\Gamma \oplus {}_\Gamma P_\Gamma \text{ and } {}_\Delta\Delta \otimes_\Gamma \Delta_\Delta \cong {}_\Delta\Delta_\Delta \oplus {}_\Delta Q_\Delta $$

*for projective bimodules ${}_\Gamma P_\Gamma$ and ${}_\Delta Q_\Delta$.*

*Proof.* (1) $\Rightarrow$ (2) : Suppose ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$ are indecomposable bimodules that induce a stable equivalence of Morita type. As above, let $\Delta = \mathrm{End}_\Lambda(M)$. Then $\mathrm{Res}_\Gamma^\Delta$ is exact and induces a stable equivalence, with $\mathrm{Ind}_\Gamma^\Delta$ inducing an inverse stable equivalence. It only remains to show that $\mathrm{Ind}_\Gamma^\Delta$ is exact. But notice that $\mathrm{Hom}_\Delta(M,-) \circ \mathrm{Ind}_\Gamma^\Delta$ is left adjoint to $\mathrm{Res}_\Gamma^\Delta \circ (- \otimes_\Lambda M_\Delta) \cong - \otimes_\Lambda M_\Gamma$, which is right adjoint to $- \otimes_\Gamma N_\Lambda$ by Theorem 2.7, and hence $\mathrm{Hom}_\Delta(M,-) \circ \mathrm{Ind}_\Gamma^\Delta \cong - \otimes_\Gamma N_\Lambda$. Clearly this shows that $\mathrm{Ind}_\Gamma^\Delta$ is exact.

(2) $\Rightarrow$ (3) : Since $\mathrm{Res}_\Gamma^\Delta = - \otimes_\Delta \Delta_\Gamma$ is exact and induces a stable equivalence, we can apply Theorem 2.9 with $M = {}_\Delta\Delta_\Gamma$. We must simply check that $\mathrm{Hom}_\Delta({}_\Delta\Delta_\Gamma, \Delta) \cong {}_\Gamma\Delta_\Delta$ is projective over $\Gamma$, and this is clearly equivalent to $\mathrm{Ind}_\Gamma^\Delta$ being exact.

(3) $\Rightarrow$ (2) : This is immediate since (3) implies that the restriction and induction functors induce a stable equivalence of Morita type.

(2) $\Rightarrow$ (1) : We know that $\Delta = \mathrm{End}_\Lambda(M)$ for some projective generator ${}_\Lambda M$, and the ring homomorphism $\Gamma \to \Delta$ makes $M$ a $(\Lambda, \Gamma)$-bimodule. As we saw earlier, $- \otimes_\Lambda M_\Gamma = \mathrm{Res}_\Gamma^\Delta \circ (- \otimes_\Lambda M_\Delta)$, and hence $- \otimes_\Lambda M_\Gamma$ is an exact functor inducing a stable equivalence. To apply Theorem 2.9, we must check that the left adjoint of $- \otimes_\Lambda M_\Gamma$ is also exact. But, as stated above, such a left adjoint is given by $\mathrm{Hom}_\Delta(M,-) \circ \mathrm{Ind}_\Gamma^\Delta$, which is exact by assumption. □

*Remarks.* (1) Notice that we have equality $\Delta = \Gamma \oplus P$ in (3). This follows from the proof of Theorem 2.9, noting that the inclusion of $\Gamma$ in $\Delta$ can also be obtained from the split exact sequence $0 \to \Gamma \longrightarrow \Delta \otimes_\Delta \Delta \longrightarrow P \to 0$ of $(\Gamma, \Gamma)$-bimodules once we identify the middle term with $\Delta$. Of course, we are identifying $\Gamma$ with its image in $\Delta$.

(2) By Corollary 3.1, in the context of (3), the induction functor is also right adjoint to restriction, and hence naturally isomorphic to the coinduction functor $\mathrm{Hom}_\Gamma({}_\Delta\Delta_\Gamma, -)$.

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