Math 8 - Solutions to Midterm Review Problems
Winter 2007

1. Prove the logical equivalence:

$$(P \wedge \sim Q) \wedge (R \Rightarrow Q) \equiv \sim [(P \Rightarrow Q) \vee R].$$

Solution. Starting with the left-hand side and using the identity $A \Rightarrow B \equiv \sim A \vee B$ and then the distributive law, we have

$$\begin{aligned}
(P \wedge \sim Q) \wedge (R \Rightarrow Q) &\equiv (P \wedge \sim Q) \wedge (\sim R \vee Q) \\
&\equiv (P \wedge \sim Q \wedge \sim R) \vee (P \wedge \sim Q \wedge Q) \\
&\equiv (P \wedge \sim Q \wedge \sim R) \vee \mathbf{F} \\
&\equiv P \wedge \sim Q \wedge \sim R \\
&\equiv \sim [\sim (P \wedge \sim Q) \vee R] \\
&\equiv \sim [(\sim P \vee Q) \vee R] \\
&\equiv \sim [(P \Rightarrow Q) \vee R].
\end{aligned}$$

Alternatively, we could check that the left and right hand sides have identical truth tables: each side is False only when P is True and Q and R are both False. Another method would be to check that the proposition obtained by changing the "$\equiv$" to a "$\Leftrightarrow$" is a Tautology, by means of a truth table.

2. Simplify the sentential form $(P \wedge \sim Q) \Rightarrow (P \vee Q)$ as much as possible.

Solution.

$$\begin{aligned}
(P \wedge \sim Q) \Rightarrow (P \vee Q) &\equiv \sim (P \wedge \sim Q) \vee (P \vee Q) \\
&\equiv (\sim P \vee Q) \vee (P \vee Q) \\
&\equiv \sim P \vee P \vee Q \vee Q \\
&\equiv Q.
\end{aligned}$$

3. Write the following propositions symbolically with no words. (You do not have to prove them.)

   (a) "There does not exist a largest real number."
   Solution.
   $$\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ (y > x)$$
   or a more literal version would be $\sim [\exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\ (y \leq x)]$.

   (b) "The interval strictly between any two distinct real numbers contains at least one rational number."
   Solution.
   $$\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ [(x < y) \Rightarrow \exists z \in \mathbb{Q}\ (x < z < y)]$$

(c) "Every nonempty set has at least two distinct subsets."
Solution. (We must assume that some set U of sets is given for the domain of interpretation.)

$$\forall A \; [(A \neq \emptyset) \Rightarrow \exists B \; \exists C \; (B \neq C \wedge B \subseteq A \wedge C \subseteq A)]$$

4. Determine whether the following statements are true or false, where the universe of discourse is the set of all real numbers, and give a brief justification.

   (a) $\forall x \; \exists y \; [(y > 0) \Rightarrow (xy > 0)]$
   Solution. True. Notice that the implication is automatically true whenever $y \leq 0$. So for any x, one such y that makes the implication true is $y = 0$.

   (b) $\forall x \; \exists y \; \forall z \; [(x+y)z^2 \leq 0]$
   Solution. True. Since $z^2 \geq 0$ for any z, the inequality will be satisfied if and only if $x + y \leq 0$. So for any x, we can choose $y = -x$ (or any $y < -x$) to make the inequality true for all z.

   (c) $\exists x \; \forall y \; (xy = 1)$
   Solution. False. This says that there is some x that is equal to $1/y$ for every $y \neq 0$. Clearly that is impossible.

   (d) $\forall y \; \exists x \; (x < y < x + 1)$
   Solution. True. Any y satisfies the inequality $y - 1/2 < y < y + 1/2$, so we can take $x = y - 1/2$.

5. Recall that the *Sheffer stroke* of two propostions P and Q is defined as

$$P \uparrow Q \equiv \sim (P \wedge Q).$$

If $A = \{x \mid P(x)\}$ and $B = \{x \mid Q(x)\}$, let $S = \{x \mid P(x) \uparrow Q(x)\}$. (Assume everything is contained in a fixed domain of interpretation U.)

   (a) Describe the set S in terms of A and B, using the standard set operations (eg. union, intersection, set difference, etc.).
   Solution.

$$\begin{aligned} S = \{x \mid P(x) \uparrow Q(x)\} &= \{x \mid \; \sim (P(x) \wedge Q(x))\} \\ &= \{x \mid P(x) \wedge Q(x)\}' \\ &= (A \cap B)'. \end{aligned}$$

   (b) Illustrate S using a Venn Diagram.
   Solution. Everything should be shaded except for the intersection of A and B.

   (c) If we also know that $A \subseteq B$, what else can we say about S?
   Solution. If $A \subseteq B$, then $A \cap B = A$ (think of a Venn diagram, if you are not sure about this). Thus $S = (A \cap B)' = A'$ is the complement of A.

6. Let A be a finite set, and let B be a subset of A. Prove that $A = B$ if and only if $|A| = |B|$. (Recall, $|A|$ is the cardinality of A, i.e., the number of elements of A.)

   Solution. If $A = B$, then A and B have exactly the same elements, and so they must have equal numbers of elements. Conversely, suppose that $|A| = |B|$, and assume by way of contradiction that $A \neq B$. Since $B \subset A$, there exists an $x \in A - B$, and A has at least one more element than B: $|A| \geq |B| + 1$. This contradicts the fact that A and B have the same cardinality.

7. Let A, B, C be sets. Prove:

   (a) If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.
   Solution. Let x be an element of A. Since $A \subseteq B$, we know that $x \in B$, and since $A \subseteq C$ we know that $x \in C$. Since $x \in B$ and $x \in C$, we know $x \in B \cap C$. This shows that any element of A belongs also to $B \cap C$, and hence $A \subseteq B \cap C$.

   (b) If $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.
   Solution. Let x be an element of $A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$, then $A \subseteq C$ implies that $x \in C$. If, on the other hand, $x \in B$, then $B \subseteq C$ implies that $x \in C$. Thus, we see that any element of $A \cup B$ is also an element of C. In other words, $A \cup B \subseteq C$.

8. Consider the proposition: "Every nonzero rational number is equal to a product of two irrational numbers."

   (a) Write this proposition using only symbols and no words.
   Solution. $\forall x \in \mathbb{Q}\ [(x \neq 0) \Rightarrow \exists y \in \mathbb{R}\ \exists z \in \mathbb{R}[(y \notin \mathbb{Q}) \wedge (z \notin \mathbb{Q}) \wedge (x = yz)]]$

   (b) Prove this proposition.
   Solution. Let $x \neq 0$ be a rational number, and let y be any nonzero irrational number (for example, let $y = \sqrt{2}$). Then $x = y(x/y)$. We claim that x/y is also an irrational number. We prove this fact indirectly. Assume, by way of contradiction, that x/y is rational. This means that there are integers $a \neq 0$ and $b \neq 0$ such that $x/y = a/b$. Since x is rational and nonzero, there are integers $c \neq 0$ and $d \neq 0$ such that $x = c/d$. Solving for y we get $y = xb/a = cb/ad \in \mathbb{Q}$. This contradicts the fact that y is irrational. Hence $x = y(x/y)$ is a product of two irrational numbers.

9. Consider the family $\{A_n\}_{n \in \mathbb{N}}$ of subsets

$$A_n = \{x \in \mathbb{R} \mid nx \in \mathbb{Z}\ \}$$

   of $\mathbb{R}$, indexed by the set $\mathbb{N}$ of natural numbers. Prove:

   (a) $\bigcup_{n \in \mathbb{N}} A_n = \mathbb{Q}$.

Solution. We must show two set inclusions $\bigcup_{n \in \mathbb{N}} A_n \subseteq \mathbb{Q}$ and $\mathbb{Q} \subseteq \bigcup_{n \in \mathbb{N}} A_n$ to establish the equality of these two sets. For the first inclusion, it suffices to show that each A_n is a subset of $\mathbb{Q}$. To see this, let $x \in A_n$. Thus $nx = m \in \mathbb{Z}$ and $x = m/n \in \mathbb{Q}$ since $m, n \in \mathbb{Z}$. Thus $A_n \subseteq \mathbb{Q}$ for all n, and it follows (by essentially the same argument as in 7b) that the union of the A_n's is a subset of $\mathbb{Q}$. To prove the reverse inclusion, let $x \in \mathbb{Q}$. Then x can be written as a fraction $x = a/b$ with $a \in \mathbb{Z}$ and $b \in \mathbb{N}$. Thus $bx = a \in \mathbb{Z}$ and it follows that $x \in A_b$ for the natural number b. Hence, x also belongs to the union of all the A_n's. This completes the proof.

(b) $\bigcap_{n \in \mathbb{N}} A_n = \mathbb{Z}$.

Solution. As above, in order to prove that these two sets are equal, we must prove the two inclusions: $\bigcap_{n \in \mathbb{N}} A_n \subseteq \mathbb{Z}$ and $\mathbb{Z} \subseteq \bigcap_{n \in \mathbb{N}} A_n$. To prove the first, suppose that x belongs to A_n for every $n \in \mathbb{N}$ (that is, x belongs to the intersection). Then, in particular, letting $n = 1$ we have $x \in A_1 = \{y \mid 1y \in \mathbb{Z}\} = \mathbb{Z}$. To prove the reverse inclusion, let $x \in \mathbb{Z}$. Then $nx \in \mathbb{Z}$ for any $n \in \mathbb{N}$. Thus $x \in A_n$ for every $n \in \mathbb{N}$, and this is exactly the same as saying that x belongs to the intersection of all the A_n's, as required.