1

Math 8 Solutions to Final Review Problems

1a) $\sim$ is not an equivalence relation on $\mathcal{P}(\mathbb{N})$, because it is not transitive or reflexive (it is symmetric). $\{1\} \not\sim \{1\}$ since $\{1\} \cup \{1\} = \{1\} \neq \mathbb{N}$.

b) $\approx$ is an equivalence relation on $\mathcal{P}(\mathbb{N})$:

reflexive: $A \approx A \iff \min A = \min A$ = true. ✓

symmetric: $A \approx B \iff \min A = \min B$
$\iff \min B = \min A$
$\iff B \approx A$. ✓

transitive: $A \approx B \wedge B \approx C \Rightarrow \min A = \min B \wedge \min B = \min C$
$\Rightarrow \min A = \min C$
$\Rightarrow A \approx C$.

There is 1 possible equivalence class for each $n \in \mathbb{N}$, consisting of all the sets $A \subseteq \mathbb{N}$ whose smallest element equals n.

2.a) Basis Step: $n=1$

$P(1) \equiv (x-1)(x+1) = x^2 - 1$ is true ✓

Inductive Step: Assume $P(k)$:

$$(x-1)(x^k + x^{k-1} + \cdots + x + 1) = x^{k+1} - 1 \qquad (k \geq 1)$$

$P(k+1)$ states $(x-1)(x^{k+1} + x^k + \cdots + x + 1) = x^{k+2} - 1$.

to prove this,

$$(x-1)(x^{k+1} + x^k + \cdots + x + 1) = (x-1)x^{k+1} + \underbrace{(x-1)(x^k + x^{k-1} + \cdots + x + 1)}$$
$$= x^{k+2} - x^{k+1} + \underbrace{x^{k+1} - 1}_{\text{by Induction Hypothesis}}$$
$$= x^{k+2} - 1$$

so $P(k+1)$ is true.

Thus $P(n)$ is true $\forall n \in \mathbb{N}$ by Induction.

2b) $P(n) \equiv 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$.

Basis Step $n=1$: $P(1) \equiv 1 \leq 2 - 1/1 = 1 \equiv$ True ✓.

Inductive Step Assume $P(k)$ is true for some $k \geq 1$.
$P(k) \equiv 1 + \frac{1}{2^2} + \cdots + \frac{1}{k^2} \leq 2 - 1/k$:

Then $P(k+1)$ states $1 + \frac{1}{2^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}$.
To prove this,

$$1 + \frac{1}{2^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \quad \text{by Ind. Hyp.}$$

$$= 2 - \frac{(k+1)^2 - k}{k(k+1)^2} = 2 - \frac{k^2+k+1}{k(k+1)^2}$$

$$\leq 2 - \frac{k^2+k}{k(k+1)^2}$$

$$= 2 - \frac{1}{k+1}.$$

Thus $P(k+1)$ is true, & by induction $P(n)$ is true $\forall n \in \mathbb{N}$.

3) a) Let $1 \leq k \leq n$, $\frac{n}{k}\binom{n-1}{k-1} = \frac{n}{k} \frac{(n-1)!}{(k-1)!\,(n-1-(k-1))!}$

$$= \frac{n(n-1)!}{k(k-1)!\,(n-k)!}$$

$$= \frac{n!}{k!\,(n-k)!}$$

$$= \binom{n}{k}.$$

b) [The problem should read "for any integer $n \geq 1$".
Although it is true for $n=0$, since
$\sum_{k=1}^{0} k\binom{0}{k}$ is the empty sum, which equals 0 by ~~default~~ definition.]

method 1: $\sum_{k=1}^{n} k\binom{n}{k} = \sum_{k=1}^{n} n\binom{n-1}{k-1} = n\sum_{\ell=0}^{n-1}\binom{n-1}{\ell} = n \cdot 2^{n-1}$.

letting $\ell = k-1$.

method 3: By the binomial theorem:

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k .$$

Differentiating both sides yields:

$$n(x+1)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}x^{k-1} = \sum_{k=1}^{n} k\binom{n}{k}x^{k-1}.$$

now let $x=1$:

$$n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}.$$

method 4 If we choose the committee chairman first, there are n ways to do this, and then there are 2^{n-1} subsets of the remaining $n-1$ people. Since any of these can be chosen for the remaining committee members, the total # of ways to pick a committee & chairman is $n\cdot 2^{n-1}$. On the other hand, if the committee has k people there are $\binom{n}{k}$ ways of choosing them & then k ways of electing a chairman among them. Since the committee may be of any size k $(1\le k\le n)$ the total # of possible ways to select it & a chairman is $\sum_{k=1}^{n} k\binom{n}{k}$. Thus $n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$.

4. a) $f:\mathbb{N}\to\mathbb{N}$ given by $f(x)=x+1$ is injective but not surjective, since $1\notin f(\mathbb{N})$.

b) Impossible: Suppose $f:\mathbb{N}_n\to\mathbb{N}_n$ is surjective. & that $f(a)=f(b)$. Assume $a\neq b$.
Let $B=\mathbb{N}_n-\{a,b\}$ so $|B|=n-2$.
& let $A=\mathbb{N}_n-\{f(a)\}$ so $|A|=n-1$.
Define $g:A\to B$ by $g(x)=$ some $y\in B$ s.t. $f(y)=x$.
makes sense since f is surjective.
& since $x\neq f(a)=f(b) \Rightarrow y\neq a,b$.
g is easily seen to be injective, but this is a contradiction

c) Impossible: If $f: \mathbb{R} \to \mathbb{N}$ were injective, then the standard injection $g: \mathbb{N} \to \mathbb{R}$ given by $g(x) = x$, & the Schroeder-Bernstein Thm would imply that $\mathbb{N} \approx \mathbb{R}$. But we proved in class that $\mathbb{N} \not\approx \mathbb{R}$.

d) $f: \mathbb{N} \to [0,1]$ given by $f(n) = 1/n$ is injective

e) If C has only one element, any map $g: B \to C$ has to be surjective, and no maps $h: A \to C$ can be injective when $|A| > 1$. So let $A = \{1,2\}$, $C = \{1\}$. and $B = \{1,2\}$. Let $f: A \to B$ be the identity function, which is clearly injective. & let $g: B \to C$ be the only possible function (ie, $g(1) = g(2) = 1$) which is clearly surjective. However, $g \circ f$ is not injective: $g \circ f(1) = g \circ f(2) = 1$.

f) True: Suppose $f: A \to B$ is injective. Then every

should be stated in problem. → (If A is not empty! False if A is the empty set)

element of B in $f(A)$ has a unique preimage in A. So we can define $g: B \to A$ by

$$g(b) = \begin{cases} a, & \text{if } f(a) = b. \\ a_0, & \text{if } b \notin f(A). \end{cases}$$ where $a_0 \in A$ is some fixed element.

Since any $a \in A$ has an image in B, g is surjective: given $a \in A$, $a = g(f(a))$.

g) False Let $A = \{1,2\}$ & $B = \{1\}$ & $f: \{1,2\} \to \{1\}$ the only possible function: $f(1) = f(2) = 1$. If $C = \{1\}$ & $D = \{2\}$, then $C \cup D = A$ & $f|_C$ & $f|_D$ are bijections. But, clearly f is not injective.

h) (i) Typo! It should say "$X \subseteq f^{-1}(f(X))$".

Pf let $x \in X$. Then $f(x) \in f(X)$ by def. of $f(X)$

But $f^{-1}(f(X)) = \{a \in A \mid f(a) \in f(X)\}$.

$\therefore$ $f(x) \in f(X) \Rightarrow x \in f^{-1}(f(X))$.

(ii) Assume f is injective. We need to show $f^{-1}(f(X)) \subseteq X$: Let $a \in f^{-1}(f(X))$.

By definition of $f^{-1}(-)$, this means $f(a) \in f(X)$.
By definition of $f(X)$, this means $\exists x \in X (f(a) = f(x))$
Since f is injective, $f(a) = f(x) \Rightarrow a = x$.
$\therefore$ $a = x \in X$, as we needed to show.

(iii) by (ii) our example $f: A \to B$ cannot be injective, so take the simplest example of a non-injective function $f: \{1,2\} \to \{1\}$, $f(1) = f(2) = 1$.
Let $X = \{1\} \subseteq \{1,2\}$. Then $f(X) = \{f(1)\} = \{1\}$.
& $f^{-1}(f(X)) = f^{-1}(\{1\}) = \{1,2\} \neq X$.

5.a) (Typo: It should say "finite, nonempty sets")

Pf $\Rightarrow$: Assume $|A| = n \geq m = |B| > 0$.
Then $\exists$ bijections $f: A \to \mathbb{N}_n$ & $g: B \to \mathbb{N}_m$.
and there is a surjection $h: \mathbb{N}_n \to \mathbb{N}_m$ given by $h(x) = \begin{cases} x, & \text{if } 1 \leq x \leq m \\ m, & \text{if } x > m \end{cases}$. ~~Thus~~

$$\begin{tikzcd} A \arrow[r, dashed] \arrow[d, "f"] & B \arrow[d, "g"] \\ \mathbb{N}_n \arrow[r, "h"'] & \mathbb{N}_m \end{tikzcd}$$

Thus $g^{-1} \circ h \circ f: A \to B$ is surjective.

$\Leftarrow$: Assume there is a surjection $h: A \to B$. Similar to (4f) we can define an injection $f: B \to A$, by $f(b) =$ any element a s.t. $h(a) = b$.
If $f(b_1) = f(b_2) = a$, then $h(a) = b_1 = b_2$ so f is injective. By Theorem from class, $|B| \leq |A|$.

⑥

b) 4.2 #4) a) $h:[0,1]\to(0,1)$ by $h(x)=\begin{cases} 1/2, & \text{if } x=0 \\ 1/n+2, & \text{if } x=1/n, n\in\mathbb{N}. \\ x, & \text{otherwise}. \end{cases}$

Injectivity: Suppose $h(x)=h(y)$ for $x,y\in[0,1]$.
- There are 6 cases to check depending on which of the hypotheses are satisfied by x & y:

case 1 $x=y=0$ ✓

case 2 $x=1/n$, $y=1/m$ for $n,m\in\mathbb{N}$.
$h(x)=h(y) \Rightarrow 1/n+2 = 1/m+2 \Rightarrow n=m \Rightarrow x=y$ ✓

case 3 $x,y\neq 0, 1/n$. $h(x)=h(y) \Rightarrow x=y$ ✓

case 4 $x=0$, $y=1/n$ for $n\in\mathbb{N}$.
$h(x)=1/2=h(y)=1/n+2 \Rightarrow n=0$ contradiction.

case 5 $x=0$, $y\neq 0, 1/n$.
$h(x)=1/2=h(y)=y \Rightarrow y=1/2 \Rightarrow h(y)=1/4$, contradiction

case 6 $x=1/n$, $y\neq 0, 1/n$.
$h(x)=1/n+2=h(y)=y \Rightarrow h(y)=1/n+4$. contradiction.

Surjectivity: Let $y\in(0,1)$. If $y=1/2$, $y=h(0)$.
If $y=1/n$ for $n\geq 3$, $y=h\left(\frac{1}{n-2}\right)$
Otherwise $y=h(y)$.

4.2 #5 a) the Identity function $1_A: A\to A$ restricts to an injection $1_A|_{A-B}: A-B\to A$.
Since $A-B\subseteq A$. $\therefore |A-B|\leq|A|$.

b) $|A|\leq|B| \Rightarrow \exists f: A\to B$ injective.
Define $g: A\times C\to B\times C$ by $g(a,c)=(f(a),c)$.

g is injective since if $g(a_1,c_1)=g(a_2,c_2)$, then
$(f(a_1),c_1)=(f(a_2),c_2) \Rightarrow f(a_1)=f(a_2) \wedge c_1=c_2$
$\Rightarrow a_1=a_2 \wedge c_1=c_2$
$\Rightarrow (a_1,c_1)=(a_2,c_2)$.

Thus $|A\times C|\leq|B\times C|$.

c) False. If $A=\{1,2\}$ & $B=\{3\}$, $A-B=\{1,2\}$,
& $|A-B|=2>|B|=1$.

7

4.2 6) If $|A| < |B|$, then there is an injection $f: A \to B$. We use f to define an injection $g: \mathcal{P}(A) \to \mathcal{P}(B)$. by $g(X) = \{f(x) \in B \mid x \in X\}$ for any $X \subseteq A$.

(In our other notation, $g(X)$ was written as $f(X)$ - it is just the image of the set X under f, which is a subset of B)

If $g(X_1) = g(X_2)$, then $\{f(x) \mid x \in X_1\} = \{f(x) \mid x \in X_2\}$
we must show $X_1 = X_2$.
let $x \in X_1$. Then $f(x) \in \{f(x) \mid x \in X_1\} = \{f(x) \mid x \in X_2\}$
$\Rightarrow \exists y \in X_2$ s.t. $f(x) = f(y) \Rightarrow x = y$ since f = injective.
$\Rightarrow x \in X_2$. $\therefore X_1 \subseteq X_2$
The same argument shows $X_2 \subseteq X_1$, so we can conclude $X_1 = X_2$. Thus g is injective & $|\mathcal{P}(A)| \le |\mathcal{P}(B)|$.

4.3 4) True. Assume A is countable & $B \subseteq A$ is finite.
Then 1) $A - B$ is infinite.
Pf If not, $A - B$ = finite, $\Rightarrow A = B \cup (A - B)$ = finite, contradiction.

2) $A - B \approx \mathbb{N}$.
Pf A is countable so $A \approx \mathbb{N}$.
$A - B$ is infinite so $|\mathbb{N}| \le |A - B|$
(There is an injection $\mathbb{N} \xrightarrow{f} A - B$ by a Thm from class)
There is an injection $A - B \xrightarrow{g} A$, namely $g = 1_A|_{A-B}$, $g(a) = a$. So $|A - B| \le |A| = |\mathbb{N}|$.
By the Schroeder Bernstein Theorem,
$|A - B| = |A| = |\mathbb{N}|$.