Math 8 - Final Exam Review Problems
Winter 2007

While the final exam will focus on topics covered since the first midterm (Ch. 2.9-2.10, 3.1-3.3, 4.1-4.3, 5.8), you will still be expected to know all the material from chapters 1 and 2 as well. In particular, you should be able to:

- Understand and Prove conditional and biconditional statements (using direct and/or indirect proofs). (Ch. 1.3-1.4)
- Understand and use quantifier notation (Ch. 2.3)
- Understand definitions of sets, and related notation (eg. $\in, \subseteq$, Ch. 2.1, 2.4)
- Prove that one set is a subset of another, or that two sets are equal.(Ch. 2.4) (One method is using logical equivalence, Ch. 1.5)
- Know the definitions of Unions, Intersections, and Complements. (Ch. 2.5)
- Know the definitions of the Power Set (Ch. 2.7) and Cartesian Product (Ch. 2.8)

New Topics

1. **(Equivalence) Relations.**

   (a) Consider the relation $\sim$ on the power set $\mathcal{P}(\mathbb{N})$ of the set $\mathbb{N}$ of natural numbers, defined by
   $$A \sim B \Leftrightarrow A \cup B = \mathbb{N},$$
   for subsets $A, B \subseteq \mathbb{N}$. Is $\sim$ an equivalence relation? Justify your answer. If so, describe the equivalence classes.

   (b) Consider the relation $\approx$ on the power set $\mathcal{P}(\mathbb{N})$ of the set $\mathbb{N}$ of natural numbers, defined by
   $$A \approx B \Leftrightarrow \min A = \min B,$$
   for subsets $A, B \subseteq \mathbb{N}$. Here, $\min A$ denotes the smallest element of A. Is $\approx$ an equivalence relation? Justify your answer. If so, describe the equivalence classes.

   (c) Text: 2.9, Ex. 20.

2. **Induction.**

   (a) Prove that for any integer $n \geq 1$,
   $$(x-1)(x^n + x^{n-1} + \cdots + x + 1) = x^{n+1} - 1.$$

   (b) Prove that for any integer $n \geq 1$,
   $$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

3. **Binomial Coefficients.**

   (a) Prove that for any integers k, n with $1 \leq k \leq n$, $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$.

   (b) Prove that for any integer $n \geq 0$,

$$\sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}.$$

   (4 Hints: 1) use the previous problem; or 2) use induction on n; or 3) use the binomial theorem and some calculus; or 4) show that both sides count the number of ways to choose a committee and chairman from n people.)

4. **Functions.** (a)-(e) Give examples of the following, or explain why no example exists.

   (a) An injection $f : \mathbb{N} \to \mathbb{N}$ that is not surjective.

   (b) A surjection $f : \mathbb{N}_n \to \mathbb{N}_n$ that is not injective.

   (c) An injection $f : \mathbb{R} \to \mathbb{N}$.

   (d) An injection $f : \mathbb{N} \to [0, 1]$.

   (e) An injection $f : A \to B$ and a surjection $g : B \to C$ such that $g \circ f$ is not injective.

   (f) True or False: Let A and B be sets, and suppose $f : A \to B$ is an injection. Then there exists a surjection $g : B \to A$. Give a proof or counterexample.

   (g) True or False: If $f : A \to B$ is a function such that there are two subsets $C, D \subseteq A$ with $f|_C$ and $f|_D$ injective and $C \cup D = A$, then f is injective. Give a proof or counterexample.

   (h) If $f : A \to B$, and $X \subseteq A$, and $Y \subseteq B$, we write

$$f^{-1}(Y) = \{a \in A \mid f(a) \in Y\} \subseteq A$$

   and

$$f(X) = \{f(x) \mid x \in X\} \subseteq B.$$

   (i) Prove that $f^{-1}(f(X)) \subseteq X$.
   (ii) If f is injective, show that $f^{-1}(f(X)) = X$.
   (iii) Give an example where $f^{-1}(f(X)) \neq X$.

5. **Cardinality**

   (a) Prove that for finite sets A and B, $|A| \geq |B|$ if and only if there exists a surjection $g : A \to B$.

   (b) Text: Ch. 4.2, Ex. 4, 5, 6.

   (c) Text: Ch. 4.3, Ex. 1, 4, 5, 11.