# Math 8 - Midterm Solutions

February 8, 2007

1. (8 points) Use a truth table (or other methods) to prove the logical equivalence

$$(P \Rightarrow Q) \Rightarrow (Q \Rightarrow P) \equiv (Q \Rightarrow P).$$

**Solution.**

| $P$ | $Q$ | $P \Rightarrow Q$ | $Q \Rightarrow P$ | $(P \Rightarrow Q) \Rightarrow (Q \Rightarrow P)$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | T |
| F | T | T | F | F |
| F | F | T | T | T |

Since the truth values of $(P \Rightarrow Q) \Rightarrow (Q \Rightarrow P)$ and $Q \Rightarrow P$ are the same for all possible truth values of $P$ and $Q$, these two sentential forms are logically equivalent.

The logical equivalence could also be proved symbolically, as follows:

$$\begin{aligned}
(P \Rightarrow Q) \Rightarrow (Q \Rightarrow P) &\equiv \sim (P \Rightarrow Q) \vee (Q \Rightarrow P) \\
&\equiv \sim (\sim P \vee Q) \vee (\sim Q \vee P) \\
&\equiv (P \wedge \sim Q) \vee \sim Q \vee P \\
&\equiv (P \vee \sim Q \vee P) \wedge (\sim Q \vee \sim Q \vee P) \\
&\equiv (P \vee \sim Q) \wedge (P \vee \sim Q) \\
&\equiv P \vee \sim Q \\
&\equiv Q \Rightarrow P.
\end{aligned}$$

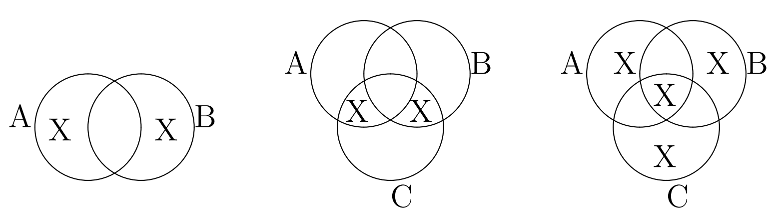
2. (12 points) The **Symmetric Difference** of two sets $A$ and $B$ is defined as

$$A \oplus B = A \cup B - A \cap B.$$

(a) (8 pts) Draw 3 separate Venn diagrams for the sets (i) $A \oplus B$; (ii) $(A \oplus B) \cap C$; and (iii) $(A \oplus B) \oplus C$.

**Solution.** The X's denote the regions that should be shaded.

(i) $A \oplus B$ (ii) $(A \oplus B) \cap C$ (iii) $(A \oplus B) \oplus C$



(b) (4 pts) If $A = \{0, 1\}$ and $B = \{0, 2\}$, what is $\mathcal{P}(A) \oplus \mathcal{P}(B)$?

**Solution.** $\mathcal{P}(A) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$ and $\mathcal{P}(B) = \{\emptyset, \{0\}, \{2\}, \{0, 2\}\}$. So the elements of $\mathcal{P}(A) \oplus \mathcal{P}(B)$ are those sets that belong to $\mathcal{P}(A)$ or to $\mathcal{P}(B)$ but not to both:

$$\mathcal{P}(A) \oplus \mathcal{P}(B) = \{\{1\}, \{2\}, \{0, 1\}, \{0, 2\}\}.$$

3. (12 points) State whether the following propositions are True or False, and give a brief justification for your answer (a single example may suffice). **In (a)-(c) the domain of interpretation is the set $\mathbb{Z}$ of integers.**

   (a) $\forall a\ \forall b\ \exists c\ (a \leq c \leq b)$

   **Solution.** False. If $a = 2$ and $b = 1$, there does not exist a $c$ with $2 \leq c \leq 1$.

   (b) $\forall n\ \exists k\ [(n = 2k) \vee (n = 2k + 1)]$

   **Solution.** True. Every integer is even or odd.

   (c) $\exists x\ \forall y\ [xy > 0]$

   **Solution.** False. For any value of $x$, letting $y = 0$ makes $xy = 0$.

   (d) If $A$ is any set, then $A \cap \mathcal{P}(A) = \emptyset$.

   **Solution.** False. If $A = \{a, \{a\}\}$ then $\mathcal{P}(A) = \{\emptyset, \{a\}, \{\{a\}\}, \{a, \{a\}\}\}$. So $\{a\} \in A \cap \mathcal{P}(A)$ and thus the intersection is not the empty set.

4. (8 points) Let $A$ and $B$ be sets. Prove $A \cup B = A \cap B \Leftrightarrow A = B$. (A picture may help, but it is not a proof.)

   **Solution.** First assume that $A = B$. Then $A \cup B = A \cup A = \{x \mid x \in A \vee x \in A\} = A$, and $A \cap B = A \cap A = \{x \mid x \in A \wedge x \in A\} = A$. Thus $A \cup B = A = A \cap B$.

   To prove the converse, assume that $A \cap B = A \cup B$. Then $A \subseteq A \cup B$ and $A \cap B \subseteq B$ imply that $A \subseteq A \cup B = A \cap B \subseteq B$, so $A \subseteq B$. Similarly, we have $B \subseteq A \cup B = A \cap B \subseteq A$, so $B \subseteq A$. Finally, we know that $A \subseteq B$ and $B \subseteq A$ implies that $A = B$.

5. (10 points) Consider the proposition: "For any two real numbers $x$ and $y$, if the product $xy$ is irrational, then either $x$ or $y$ is irrational."

   (a) (3 pts) Write this proposition using only symbols and no words.

   **Solution.** $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ [xy \notin \mathbb{Q} \Rightarrow (x \notin \mathbb{Q} \vee y \notin \mathbb{Q})]$

   (b) (5 pts) Prove this proposition.

   **Solution.** Let $x$ and $y$ be any real numbers. In order to prove the implication $xy \notin \mathbb{Q} \Rightarrow (x \notin \mathbb{Q} \vee y \notin \mathbb{Q})$, it suffices to prove its contrapositive

   $$(x \in \mathbb{Q} \wedge y \in \mathbb{Q}) \Rightarrow xy \in \mathbb{Q}.$$

   If $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$, there exist integers $a, b, c, d$ such that $x = a/b$ and $y = c/d$. Thus $xy = (a/b)(c/d) = ac/bd$ is also a ratio of two integers, and hence it belongs to $\mathbb{Q}$.

(c) (2 pts) Is the converse of this proposition also true (for all real numbers $x, y$)? Explain.

**Solution.** The converse states: "For all real numbers $x$ and $y$, if $x$ or $y$ is irrational, then the product $xy$ is irrational." But clearly this is false, as can be seen by letting $x$ be any irrational number and $y = 0$ (so $xy = 0$ is rational), or by letting $x$ be any irrational number and $y = 1/x$ (so $xy = 1$ is rational), or by letting $x = y = \sqrt{2}$ (so $xy = 2$ is rational), etc.