Name: Solution

Perm No.: ____________

Section Time :

Math 8 - Final Exam

March 22, 2007

Instructions:

- This exam consists of 8 problems totalling 100 points, and one bonus problem worth up to 10 points.
- You must show all your work and fully justify your answers in order to recieve full credit. Partial credit will be given for work that is correct and relevant. Your proofs will be graded for clarity and organization, in addition to correctness.
- No books, notes or calculators are allowed.
- Write your answers on the test itself, in the space alotted. You may attach additional pages if necessary.

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Bonus	
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1. (15 pts) Write the following statements using symbols only, and no words. (You do not have to prove them.)

   (a) "Every rational number is equal to the ratio of two integers."

   $$\forall x \in \mathbb{Q}\ \exists a \in \mathbb{Z}, \exists b \in \mathbb{Z}\ (x = a/b)$$

   (b) "The square of any even integer is even."

   $$\forall n \in \mathbb{Z} \left[\exists k\, (n = 2k) \Rightarrow \exists m\, (n^2 = 2m) \right]$$

   (c) "Any set of natural numbers contains a least element."

   $$\forall A \subseteq \mathbb{N} \left(\exists x \in A \left(\forall y \in A\ (x \leq y) \right) \right)$$

   or $\forall A \in \mathcal{P}(\mathbb{N})$.

2. (15 pts) Are the following propositions True or False? Give brief justifications for your answers. (The domain of interpretation for all variables is $\mathbb{R}$.)

(a) $\forall x\ \exists y\ (-y < x < y)$.

True: choose any $y > |x|$.

(b) $\exists x\ \forall y\ (y > 0 \Rightarrow xy > 1)$.

False: given any x, $y = \frac{1}{x}$ (or $\frac{1}{x+1}$ if $x = 0$) does not satisfy the implication: $xy = 1 \not> 1$.

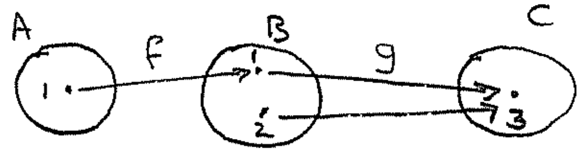
(c) $\forall a\ \exists b\ \exists c\ (a + b = c)$.

True: For any a, we can choose $b = 0$ & $c = a$, so that $a + b = a + 0 = a = c$.

3. (10 pts) Are the following statements true for all functions $f : A \to B$ and $g : B \to C$? Give a proof or a counterexample for each.

(a) If $g \circ f$ is bijective, then f is bijective.

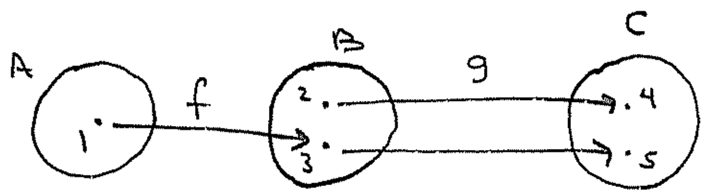
False:



$g \circ f(1) = 3$ is bijective between $A = \{1\}$ & $C = \{3\}$.
But $f : \{1\} \to \{1, 2\}$ is not surjective.

(b) If g is surjective, then $g \circ f$ is surjective.

False:



$g : \{2, 3\} \to \{4, 5\}$ is bijective.
But $g \circ f : \{1\} \to \{4, 5\}$ is not surjective.

4. (10 pts) Give examples of functions with the given properties, or explain why none can exist.

(a) An injective function $f : \mathbb{N}_3 \times \mathbb{N}_3 \to \mathcal{P}(\mathbb{N}_3)$.

Impossible: since $|\mathbb{N}_3 \times \mathbb{N}_3| = 3^2 = 9$

& $|\mathcal{P}(\mathbb{N}_3)| = 2^3 = 8$.

& $9 > 8$.

(b) An injective function $g : \mathbb{R} \to \mathbb{R}$ that is not surjective. (It may help to think of the graph of such a function.)

$g(x) = e^x$. or $g(x) = \begin{cases} x, & x \geq 0 \\ x-1 & x < 0 \end{cases}$

-1 is not in the image of $g(x)$.

$-\frac{1}{2}$ is not in the image of $g(x)$.

5. (15 pts) Consider the relation $\equiv$ on $\mathbb{R}$, defined by

$$a \equiv b \Leftrightarrow a - b \in \mathbb{Z}$$

for any $a, b \in \mathbb{R}$.

(a) Show that $\equiv$ defines an equivalence relation on $\mathbb{R}$.

reflexive: $a \equiv a \iff a - a \in \mathbb{Z} \iff 0 \in \mathbb{Z}$ true ✓

Symmetric: $a \equiv b \iff a - b \in \mathbb{Z} \iff -(a-b) \in \mathbb{Z}$

$\iff b - a \in \mathbb{Z} \iff b \equiv a$.

Transitive: $a \equiv b \wedge b \equiv c \implies a - b \in \mathbb{Z} \wedge b - c \in \mathbb{Z}$

$\implies (a-b) + (b-c) \in \mathbb{Z}$

$\implies a - c \in \mathbb{Z}$

$\implies a \equiv c$

(b) Describe the equivalence classes of $\equiv$.

If $x \in \mathbb{R}$, $[x] = \{y \in \mathbb{R} \mid y \equiv x\}$

$= \{y \in \mathbb{R} \mid y - x \in \mathbb{Z}\}$

$= \{y \in \mathbb{R} \mid \exists n \in \mathbb{Z}\,(y = x + n)\}$.

consists of all the numbers obtained by adding any integer to x.

There is thus exactly one equivalence class for each real number $x \in [0,1)$.

6. (10 pts) Let $x > -1$ be a real number. Prove for any integer $n \geq 1$ that

$$(1+x)^n \geq 1 + nx$$

We use Induction on n.

Basis Step : $n = 1$: $(1+x)^1 \geqslant 1+x$ ✓ True.

Inductive Step : Assume $(1+x)^k \geqslant 1+kx$ for some integer $k \geqslant 1$.

Then $(1+x)^{k+1} = (1+x)(1+x)^k \geqslant (1+x)(1+kx)$ by the Inductive hypothesis and since $x > -1 \Rightarrow 1+x > 0$.

$$(1+x)(1+kx) = 1 + x + kx + kx^2 \geqslant 1 + (k+1)x$$

$$\therefore \ (1+x)^{k+1} \geqslant 1 + (k+1)x$$

By Induction $(1+x)^n \geqslant 1+nx \ \ \forall n \in \mathbb{N}$.

7. (15 pts) For any function $f : A \to B$, we define functions $g : \mathcal{P}(A) \to \mathcal{P}(B)$ and $h : \mathcal{P}(B) \to \mathcal{P}(A)$ as follows.

$$g(X) = \{b \in B \mid \exists x \in X\ [b = f(x)]\} \text{ for any } X \subseteq A,$$

$$h(Y) = \{a \in A \mid f(a) \in Y\} \text{ for any } Y \subseteq B.$$

(a) Prove that $g(h(Y)) \subseteq Y$ for any $Y \subseteq B$.

Let $b \in g(h(Y))$. By Def. of g, $\exists x \in h(Y)\ (b = f(x))$

By Def. of h, $f(x) \in Y$.

$\therefore\ b = f(x) \in Y$.

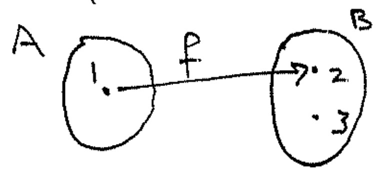
This shows $g(h(Y)) \subseteq Y$.

(b) Prove that if f is surjective, then $g(h(Y)) = Y$ for any $Y \subseteq B$.

Assume f is surjective. By (a) we know $g(h(Y)) \subseteq Y$, so we only need to show $Y \subseteq g(h(Y))$. Let $y \in Y$. Since f is surjective $\exists x \in A$ s.t. $f(x) = y$. Thus, $f(x) = y \in Y \Rightarrow x \in h(Y)$ by Def. of h. Now $\exists x \in h(Y)\ (y = f(x))$, so by Def. of g, $y \in g(h(Y))$. $\therefore Y = g(h(Y))$.

(c) Give an example of a function $f : A \to B$ and a subset $Y \subseteq B$ such that $g(h(Y)) \neq Y$.

By (b) f cannot be surjective. So let f be the function



and let $Y = B = \{2, 3\}$.

$h(Y) = \{a \in A \mid f(a) \in \{2,3\}\} = \{1\}$.

$g(h(Y)) = g(\{1\}) = \{b \in \{2,3\} \mid \exists x \in \{1\} [b = f(x)]\}$

$= \{f(1)\} = \{2\} \neq Y$.

8. (10 pts) Suppose A and B are countable sets. Prove that $A \cup B$ is also countable. (Hint: Can you partition $\mathbb{N}$ into two countable sets?)

Note $\mathbb{N} = \underbrace{\{n \in \mathbb{N} \mid \exists k \ n = 2k\}}_{\text{even \#'s}} \cup \underbrace{\{n \in \mathbb{N} \mid \exists k \ n = 2k+1\}}_{\text{odd \#'s}}$.

Since A & B are countable, there exist bijections $f: A \to \mathbb{N}$ and $g: B \to \mathbb{N}$.

We can use these to define an <u>injection</u> $h: A \cup B \to \mathbb{N}$ as follows:

$$h(x) = \begin{cases} 2f(x), & \text{if } x \in A \\ 2g(x)+1, & \text{if } x \notin A \end{cases}.$$

sending all elements of A to even integers and all elements of $B-A$ to odd integers.

<u>h=injective</u>

<u>pf</u> clearly if $h(x)=h(y)$, x & y are either both in A or both not in A.

If $x, y \in A$, $h(x)=h(y) \Rightarrow 2f(x)=2f(y) \Rightarrow f(x)=f(y)$ $\Rightarrow x=y$ since f is injective.

If $x, y \notin A$, $h(x)=h(y) \Rightarrow 2g(x)+1 = 2g(y)+1 \Rightarrow g(x)=g(y)$ $\Rightarrow x=y$ since g is injective.

. If $i: A \to A \cup B$ is the inclusion function ($i(a)=a$.) it is injective and thus $i \circ f^{-1}: \mathbb{N} \to A \to A \cup B$ is injective.

By the Schroeder-Bernstein Theorem, since $|A \cup B| \le |\mathbb{N}|$ (via h) & $|\mathbb{N}| \le |A \cup B|$ (via $i \circ f^{-1}$), we know $|A \cup B| = |\mathbb{N}|$. That is, $A \cup B$ is countable.

9. (Bonus, 10 pts) A class of students is told that instead of a final exam, they will each be given a blue or red hat (so that each can see the color of everyone else's hat except for his/her own), and be asked to guess the color of their own hat. If every student guesses the correct color of his/her hat, OR if every student guesses the incorrect color, then everyone will get A's. Otherwise (i.e., if some guess right and some guess wrong), everyone will fail. Of course, no talking or other communication is permitted during the "test", but the class can plan a strategy beforehand. Describe a strategy that guarantees they will all pass.

The students should agree on the following strategy:

- each student counts the number of **RED** hats he/she sees.
- If he/she gets an odd number, he/she should guess RED for his/her own hat.
- Otherwise he/she should guess BLUE.

-Think about why this works.

Consider 2 cases: <u>case</u> 1 there are an odd # of Red hats.

<u>Case</u> 2 there are an even # of Red hats.