Math 8 - Propositions vs. Sets - Solutions

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1. For each of the following, state whether it is a Proposition, an Open Proposition, a Set, or none of the above. The letters $P, Q, R, \ldots$ represent Propositions, and the letters $A, B, C, \ldots$ represent Sets.

   (a) $x \leq y$. Open Proposition. It is ambiguous as expressed, but will be either a True or False statement once values for x and y are specified.

   (b) $\forall x \in \mathbb{R}\ \exists y \in \mathbb{N}\ (x \leq y)$. Proposition. It is equivalent to the True statement "For any real number, there is a larger natural number." Any True statement is automatically a Proposition, and likewise for False statements.

   (c) $\{x \in \mathbb{R} \mid x \leq 2\}$. Set. This is the set of all reals that are less than or equal to 2.

   (d) $\{x \leq 2\}$. None of the Above. $x \leq 2$ without the curly braces would be an Open Proposition. But the curly braces suggest that we are defining a set. This could be referring to the same set as in (d), but this is poor notation, as it does not specify what Universe x belongs to. For instance, would this be the set $\{x \in \mathbb{Z} \mid x \leq 2\}$ or would it be $\{x \in \mathbb{R} \mid x \leq 2\}$?

   (e) $3 \in \{0, 1, 2\}$. Proposition. It is a False statement, and so automatically a proposition.

   (f) $(P \Leftrightarrow Q) \subseteq \mathbb{R}$. None of the Above. This is nonsense: $P \Leftrightarrow Q$ denotes a biconditional Proposition, and a Proposition cannot be a subset of the Set $\mathbb{R}$. Compare this with the following Proposition, which makes sense since we are saying that one set is a subset of another set: $\{x \mid P(x) \Leftrightarrow Q(x)\} \subseteq \mathbb{R}$.

   (g) $\mathbb{N} \subseteq \mathbb{R}$. Proposition. This is the True statement "The natural numbers are a subset of the real numbers."

   (h) $A = B \Leftrightarrow A \subseteq B$. Open Proposition. It is making a statement about two sets A and B. Since A and B are not specified, it is Open.

   (i) $\mathbb{Q} \cup \{x \in \mathbb{R} \mid x\sqrt{2} \in \mathbb{Q}\}$. Set. We have the union of two sets, so it is a set.

   (j) $A \cap B \neq \emptyset$. Open Proposition. It is stating that A and B have a nonempty intersection, i.e., that they share at least one element. Since A and B are not specified, it is Open.

   (k) $\{x \in \mathbb{R} \mid x \in \mathbb{Z}\} \Leftrightarrow \{x \in \mathbb{R} \mid -x \in \mathbb{Z}\}$. None of the Above. We have two sets with a Biconditional sign ($\Leftrightarrow$) between them. In general, $\Leftrightarrow$ can be place between 2 Propositions, or an Equality sign ($=$) can be placed between two Sets, to get a valid Proposition.

2. Below is a proof of the statement: For any two sets A and B,

$$(A \subseteq B) \Leftrightarrow (A \subseteq A \cap B).$$

Describe what is wrong with the proof. Are any elements of it correct? How could you rewrite it so that it makes sense?

Proof.

$$\begin{aligned} A \subseteq B &= \{x \mid x \in A \Rightarrow x \in B\} \\ &= \{x \mid x \in A \Rightarrow (x \in B \wedge x \in A)\} \\ &= \{x \mid x \in A \Rightarrow x \in A \cap B\} \\ &= A \subseteq A \cap B \end{aligned}$$

Therefore $A \subseteq B \Leftrightarrow (A \subseteq A \cap B)$. Q.E.D.

The first mistake is in the first line: "$A \subseteq B$" is a (Open) Proposition, and so it makes no sense to say that it Equals the Set on the right hand side. The same error occurs on the last line, which states that a Set is equal to the Proposition "$A \subseteq A \cap B$".

The equalities of Sets on the Right Side of the first 3 lines are essentially correct. This is because the propositions used to define these sets are logically equivalent. $x \in A \Rightarrow x \in B \equiv (x \in A \Rightarrow (x \in B \wedge x \in A))$ is a tautology, and this is equivalent to $x \in A \Rightarrow x \in A \cap B$ by the definition of the Intersection $A \cap B$.

To correct it, we can forget about the set notation (curly braces), and just use the definitions and tautologies to obtain a sequence of equivalent Propositions:

$$\begin{aligned} A \subseteq B &\equiv \forall x\ (x \in A \Rightarrow x \in B) && \text{(By definition of } \subseteq.) \\ &\equiv \forall x\ (x \in A \Rightarrow (x \in B \wedge x \in A)) && \text{(By the tautology } P \Rightarrow Q \equiv P \Rightarrow (P \wedge Q).) \\ &\equiv \forall x\ (x \in A \Rightarrow x \in A \cap B) && \text{(By definition of } \cap.) \\ &\equiv A \subseteq A \cap B && \text{(By definition of } \subseteq.) \end{aligned}$$

Therefore $A \subseteq B \Leftrightarrow (A \subseteq A \cap B)$. Q.E.D.