Math 8 - Homework #7 Solutions

1. **Exercises 3.1 p. 143-4:** 4)a; 8)b, i, j; 11) a–e.

   Solutions. 4a. We want to find sets A, B, C, D such that

   $$(A \times B) \cup (C \times D) \neq (A \cup C) \times (B \cup D).$$

   Notice that the set on the right contains $C \times B$, whereas the set on the left typically won't–so an example shouldn't be too hard to find. For instance choose $A = \{a\}, B = \{b\}, C = \{c\}, D = \{d\}$. Then $(A \times B) \cup (C \times D) = \{(a, b), (c, d)\}$, while $(A \cup C) \times (B \cup D) = \{(a, b), (a, d), (c, b), (c, d)\}$.

   Question: Is it true that the set on the left is always a subset of the set on the right?

   8. (b): $R_2 = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = -5x + 2\}$. To find R_2^{-1}, start by replacing (x, y) with (y, x), and then switch x and y and solve for y on the right to get the desired form.

   $$\begin{aligned} R_2^{-1} &= \{(y, x) \in \mathbb{R} \times \mathbb{R} \mid y = -5x + 2\} \\ &= \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x = -5y + 2\} \\ &= \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = (2 - x)/5\}. \end{aligned}$$

   (i):

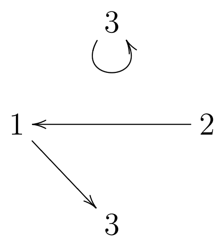
   $$\begin{aligned} R_9 &= \{(y, x) \in P \times P \mid y \text{ is the father of } x\} \\ &= \{(x, y) \in P \times P \mid x \text{ is the father of } y\} \\ &= \{(x, y) \in P \times P \mid y \text{ is the child of } x\}. \end{aligned}$$

   (j):

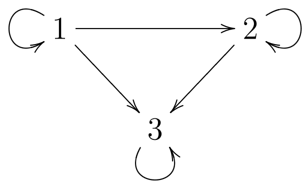
   $$\begin{aligned} R_{10} &= \{(y, x) \in P \times P \mid y \text{ is a sibling of } x\} \\ &= \{(x, y) \in P \times P \mid x \text{ is a sibling of } y\} \\ &= \{(x, y) \in P \times P \mid y \text{ is a sibling of } x\} = R_{10}. \end{aligned}$$

   11. (a) =

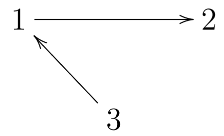
   (b):



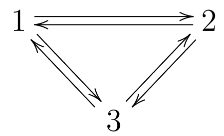
(c): $\leq$



(d): Reverse the arrows in (b):



(e): $\neq$



2. **Exercises 3.2 p. 150-2:** 1)i, j; 2)b, c; 4)a, b, c; 8

Solutions. 1. (i) Symmetric: two lines being perpendicular does not depend on their order. Not reflexive: no line is perpendicular to itself. Not transitive: transitive and symmetric would force it to be reflexive (why?).

(j) Not reflexive: $(2,1)$ is not related to itself since $2+2 > 1+1$. Symmetric: $(x,y)R(z,w) \Leftrightarrow x+z \leq y+w \Leftrightarrow z+x \leq w+y \Leftrightarrow (z,w)R(x,y)$. Not transitive: $(2,1)R(1,2)$ and $(1,2)R(2,1)$ but $(2,1)$ is not related to $(2,1)$.

2. (b) Reflexive, Not Symmetric, Not Transitive: $R = \{(1,1),(2,2),(3,3),(1,2),(2,3)\}$. The first 3 are needed to make it reflexive. It is not symmetric since we have $(1,2)$ but not $(2,1)$. It is not transitive since we have $(1,2)$ and $(2,3)$, but not $(1,3)$.

(c) Symmetric, Not Reflexive, Not Transitive: $R = \{(1,2),(2,1)\}$. It is symmetric since for each ordered pair $(a,b) \in R$, we have $(b,a) \in R$. It is not reflexive since $(1,1) \notin R$. It is not transitive since we have $(1,2)$ and $(2,1)$ but not $(1,1)$.

4. For each relation, we must check that it is Reflexive, Symmetric and Transitive.

(a) Reflexive: By definition $xRx \Leftrightarrow x^2 = x^2$ which is true for all $x \in \mathbb{Z}$.

Symmetric: $xRy \Leftrightarrow x^2 = y^2 \Leftrightarrow y^2 = x^2 \Leftrightarrow yRx$ for all $x, y \in \mathbb{Z}$.

Transitive: Suppose xRy and yRz. By definition, $x^2 = y^2$ and $y^2 = z^2$. Thus $x^2 = z^2$ and hence xRz.

Equivalence classes: $0/R = \{x \in \mathbb{Z} \mid xR0\} = \{x \in \mathbb{Z} \mid x^2 = 0^2\} = \{0\}$.

$4/R = \{x \in \mathbb{Z} \mid xR4\} = \{x \in \mathbb{Z} \mid x^2 = 4^2\} = \{4, -4\}$.

$-72/R = \{x \in \mathbb{Z} \mid xR-72\} = \{x \in \mathbb{Z} \mid x^2 = (-72)^2\} = \{-72, 72\}$.

(b) Reflexive: By definition mRm if and only if m and m have the same digit in the tens' places. Clearly they do.

Symmetric: Clearly, if mRn then m and n have the same tens' digit, and thus so do n and m. Hence nRm.

Transitive: If mRn and nRp then m and n have the same tens' digit, as do n and p. Hence m and p have the same tens' digit, and mRp.

Equivalence classes: $5 = 05 \in 106/R$ since both have tens' digit 0. Between 150 and 300, any n between 200 and 209 has tens' digit 0 and is thus in $106/R$. Greater than 1000, we have $1001 \in 106/R$. For $635/R$, we have $35, 230, 1031 \in 635/R$.

(c) Reflexive: xVx is true for all $x \in \mathbb{R}$ by definition.

Symmetric: $xVy \Leftrightarrow x = y \vee xy = 1 \Leftrightarrow y = x \vee yx = 1 \Leftrightarrow yVx$.

Transitive: Suppose xVy and yVz. Then $x = y$ or $xy = 1$ and $y = z$ or $yz = 1$, and there are 4 different ways these equations can be satisfied. First, if $x = y$ and $y = z$, then $x = z$ so xVz. Next, if $x = y$ and $yz = 1$ then $xz = 1$, so xVz. The case where $xy = 1$ and $y = z$ is similar. Finally, if $xy = 1$ and $yz = 1$, then $x = z$ so we again get xVz.

Equivalence Classes: $3/V = \{x \in \mathbb{R} \mid x = 3 \vee 3x = 1\} = \{3, 1/3\}$.

$-\frac{2}{3}/V = \{x \in \mathbb{R} \mid x = -\frac{2}{3} \vee -\frac{2}{3}x = 1\} = \{-\frac{2}{3}, -\frac{3}{2}\}$.

$0/V = \{x \in \mathbb{R} \mid x = 0 \vee 0x = 1\} = \{0\}$.

8. (a) We check that R is reflexive, symmetric and transitive. If $n \in \mathbb{N}$, then $2|n + n$, and hence nRn. Thus R is reflexive. Now suppose that nRm for $n, m \in \mathbb{N}$. Thus $2|n+m$, and since $n+m = m+n$, we also have $2|m+n$, meaning that mRn. Hence R is symmetric. Now suppose that mRn and nRp. This means that $2|m+n$ and $2|n+p$. Hence $m + p = (m + n) + (n + p) - 2n$ is a sum of multiples of two, and is thus even as well. This shows that mRp, so R is transitive.

(b) S cannot be an equivalence relation since it is not reflexive. For instance, $1+1 = 2$ is not divisible by 3 and thus $(1, 1) \notin S$.

3. **Exercises 3.4 p. 166:** 8.

   Solution. 8. We must check that R is reflexive, antisymmetric and transitive. For $(a, b) \in \mathbb{R}^2$, we have $(a, b)R(a, b)$ since $a \leq a$ and $b \leq b$. Thus R is reflexive. Now suppose $(a, b)R(x, y)$ and $(x, y)R(a, b)$. This means $a \leq x$ and $b \leq y$ and $x \leq a$ and $y \leq b$. Hence we must have $(a, b) = (x, y)$, showing that R is antisymmetric. Now suppose $(a, b)R(c, d)$ and $(c, d)R(e, f)$. This means that $a \leq c$ and $c \leq e$ – so $a \leq e$ – and $b \leq d$ and $d \leq f$ – so $b \leq f$. Thus $(a, b)R(e, f)$ and R is transitive.