Math 8 - Solutions to Final Exam Review Problems
Spring 2009

1. **Induction (Ch. 2.4-2.5).**

   (a) Prove that for any integer $n \geq 1$,

   $$(x-1)(x^n + x^{n-1} + \cdots + x + 1) = x^{n+1} - 1.$$

   Solution. Basis Step: Let $n = 1$. We must check that $(x-1)(x+1) = x^2 - 1$, but this is obvious.
   Inductive Step: Assume that for some $n \geq 1$ we have $(x-1)(x^n + \cdots + x + 1) = x^{n+1} - 1$. We must prove that $(x-1)(x^{n+1} + x^n + \cdots + x + 1) = x^{n+2} - 1$. To prove this we start with the left hand side, use the induction hypothesis and expand:

   $$\begin{aligned} (x-1)(x^{n+1} + x^n + \cdots + x + 1) &= (x-1)x^{n+1} + (x-1)(x^n + \cdots + x + 1) \\ &= x^{n+2} - x^{n+1} + x^{n+1} - 1 \\ &= x^{n+2} - 1 \end{aligned}$$

   Thus, by induction, $(x-1)(x^n + \cdots + x + 1) = x^{n+1} - 1$ is true for all $n \in \mathbb{N}$.

   (b) Prove that for any integer $n \geq 1$,

   $$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

   Solution. Basis Step: Let $n = 1$. We must check that $\frac{1}{1^2} \leq 2 - \frac{1}{1}$. Simplifying, we have $1 \leq 1$, which is true.
   Inductive Step: Assume that for some $n \geq 1$ we have $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$. We must check that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1}$. So we start with the left hand side and simplify it using the induction hypothesis.

   $$\begin{aligned} \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} &= \left(\frac{1}{1^2} + \cdots + \frac{1}{n^2}\right) + \frac{1}{(n+1)^2} \\ &\leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \\ &= 2 - \frac{(n+1)^2 - n}{n(n+1)^2} \\ &= 2 - \frac{n^2 + n + 1}{n(n+1)^2} \\ &\leq 2 - \frac{n^2 + n}{n(n+1)^2} \\ &= 2 - \frac{1}{n+1}. \end{aligned}$$

   Thus, by induction, the desired inequality is true for all $n \in \mathbb{N}$.

2. **Relations (Ch. 3).**

(a) Consider the relation $\sim$ on the power set $\mathcal{P}(\mathbb{N})$ of the set $\mathbb{N}$ of natural numbers, defined by

$$A \sim B \Leftrightarrow A \cup B = \mathbb{N},$$

for subsets $A, B \subseteq \mathbb{N}$. Is $\sim$ an equivalence relation? Justify your answer. If so, describe the equivalence classes.

Solution. No, it is not an equivalence relation because it is not reflexive (you may also want to check that it is symmetric, but not transitive). To see this consider $A = \{1\} \subseteq \mathbb{N}$. Clearly $A \cup A = A \neq \mathbb{N}$, and thus $A \not\sim A$.

(b) Consider the relation $\approx$ on the power set $\mathcal{P}(\mathbb{N})$ of the set $\mathbb{N}$ of natural numbers, defined by

$$A \approx B \Leftrightarrow \min A = \min B,$$

for subsets $A, B \subseteq \mathbb{N}$. Here, $\min A$ denotes the smallest element of A, or $\min A = 0$ if $A = \emptyset$. Is $\approx$ an equivalence relation? Justify your answer. If so, describe the equivalence classes.

Solution. We show that $\approx$ is an equivalence relation by checking that it is reflexive, symmetric and transitive. (Note that this is a different meaning for $\approx$ than how we have recently used it in class for equipotency/equivalency of two sets.)

Reflexive: For any set $A \subseteq \mathbb{N}$, we clearly have $\min A = \min A$. Thus $A \approx A$.

Symmetric: If $A \approx B$, then $\min A = \min B$, and hence $\min B = \min A$. Hence $B \approx A$.

Transitive: If $A \approx B$ and $B \approx C$, then $\min A = \min B = \min C$. Since $\min A = \min C$, we have $A \approx C$.

Equivalence Classes: Two sets are equivalent under $\approx$ if and only if they have the same minimum element. So the equivalence class of a set A with $\min A = n$ would be

$$A/\approx = \{B \subseteq \mathbb{N} \mid \min B = n\} = \{\{n\} \cup C \mid C \subseteq \{n+1, n+2, \ldots\}\}.$$

(c) Define a relation R on $\mathbb{N} \times \mathbb{N}$ by

$$(a, b)R(c, d) \Leftrightarrow a + b \leq c + d.$$

Is R a partial order? Which of the properties Reflexive, Symmetric, Antisymmetric, Transitive hold for R?

Solution. R is reflexive since $a + b \leq a + b$ implies that $(a, b)R(a, b)$ for any $a, b \in \mathbb{N}$. R is not symmetric since, $a + b \leq c + d$ does not imply $c + d \leq a + b$: for instance, $(1, 1) \sim_R (2, 2)$ but $(2, 2) \not\sim_R (1, 1)$. R is not antisymmetric either since $(a, b)R(c, d)$ and $(c, d)R(a, b)$ both hold wheneve $a+b = c+d$, and this does not imply that $(a,b)=(c,d)$. For instance, we have $(1,3)R(2,2)$ and $(2,2)R(1,3)$ but $(2,2)\neq(1,3)$. R is transitive since if $(a,b)R(c,d)$ and $(c,d)R(e,f)$ then $a+b\leq c+d$ and $c+d\leq e+f$, which implies that $a+b\leq e+f$. Hence $(a,b)R(e,f)$. However, R is not a partial order since it is not antisymmetric.

3. **Functions (Ch. 4).** Give examples of the following, or explain why no example exists.

   (a) A one-to-one function $f:\mathbb{N}\to\mathbb{N}$ that is not onto.

   Solution. Let $f(n)=n+1$. This is injective since for any $a,b\in\mathbb{N}$, $f(a)=f(b)$ implies $a+1=b+1$, which implies $a=b$. This is not surjective since for all $a\in\mathbb{N}$, $f(a)\neq 1$.

   (b) An onto function $f:\mathbb{R}\to\mathbb{Z}$.

   Solution. Define $f:\mathbb{R}\to\mathbb{Z}$ by rounding off a real number x to the nearest integer (with the convention that $n.5$ is always rounded up to $n+1$.) Then $f(n)=n$ for any $n\in\mathbb{Z}$, so f is definitely onto.

   (c) A one-to-one function $f:\mathbb{N}\to[0,1]$.

   Solution. Let $f(n)=1/n$. This is injective since for any $a,b\in\mathbb{N}$, $f(a)=f(b)$ implies $1/a=1/b$, which implies $a=b$.

   (d) A one-to-one function $f:A\to B$ and an onto function $g:B\to C$ such that $g\circ f$ is not one-to-one.

   Solution. Let $A=\{0,1\}$, $B=\{2,3,4\}$ and $C=\{5\}$. Define f by the set of ordered pairs $\{(0,2),(1,3)\}$ (ie., $f(0)=2$ and $f(1)=3$), and define g by the set of ordered pairs $\{(2,5),(3,5),(4,5)\}$ (this is the only function from B to C here). Clearly f is injective since $f(0)\neq f(1)$, g is surjective since $g(2)=5$, but $g\circ f$ is not injective since $g(f(0))=5=g(f(1))$.

   (e) True or False: Let A and B be sets, and suppose $f:A\to B$ is one-to-one. Then there exists an onto function $g:B\to A$. Give a proof or counterexample.

   Solution. (Trick Question!) FALSE! This is not true if $A=\emptyset$. Any function $f:\emptyset\to B$ (in fact there is only one) is automatically one-to-one, since in order not to be one-to-one there must be two elements of $\emptyset$ that produce the same output. However, there are NO functions $g:B\to\emptyset$ whenever $B\neq\emptyset$, since there are no possible outputs in $\emptyset$ for the elements of B.

   However, if A is assumed to be nonempty, it is TRUE. A surjection g can be constructed as follows. If $b=f(a)$ for some $a\in A$ then this a is unique (since f is one-to-one) and we may define $g(b)=a$. Otherwise, define $g(b)=a_0$ where a_0 is some fixed element of A. If $a\in A$, then $a=g(f(a))$ by definition of g, so g is surjective.

   Are the following functions bijective? Give a proof or explain why not.

(f) $f : P \to P$, where P is the set of people and $f(p)$ is defined to be p's mother.

Solution. If f were a bijection, that would mean that every person q is the mother of exactly one (input) person p. This fails in two ways: 1) Not every person is somebody's mother (i.e., f is not onto); and 2) One person can be the mother of multiple (input) people (i.e., f is not one-to-one).

(g) $f : (0,1) \to (-2,2)$ defined by $f(x) = 4x - 2$.

Solution. One-to-one: Suppose $f(x) = f(y)$ for $x, y \in (0,1)$. This means that $4x - 2 = 4y - 2$, which clearly implies that $x = y$. Hence f is one-to-one.

Onto: Let $y \in (-2,2)$, then for $f(x) = y$, we need $4x - 2 = y$, which forces $x = (y+2)/4$. Since $-2 < y < 2$ we have $0 < x = (y+2)/4 < 1$. So any $y \in (-2,2)$ equals $f(x)$ for some x in the domain of f, showing that f is onto.

Since f is one-to-one and onto it is bijective.

(h) $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ defined by $f(x) = (x,x)$.

Solution. f is not onto since there is no $x \in \mathbb{N}$ with $f(x) = (1,2) \in \mathbb{N}^2$. ($f$ is one-to-one since $(x,x) = (y,y)$ implies $x = y$.)

4. **Cardinality (Ch. 5).**

(a) Suppose $A \approx C$ and $B \approx D$. Prove that $A \times B \approx C \times D$.

Solution. Assume $A \approx C$ and $B \approx D$. This means that we have bijections $f : A \to C$ and $g : B \to D$. Define $h : A \times B \to C \times D$ by $h(a,b) = (f(a), g(b))$ for all $a \in A$ and $b \in B$. We check that h is one-to-one and onto.

One-to-one: Suppose $h(a,b) = h(a',b')$. This means that $(f(a), g(b)) = (f(a'), g(b'))$, which implies that $f(a) = f(a')$ and $g(b) = g(b')$. Since f and g are one-to-one, we can conclude that $a = a'$ and $b = b'$. Hence $(a,b) = (a',b')$.

Onto: Let $(c,d) \in C \times D$. Since f and g are onto, there exist $a \in A$ and $b \in B$ such that $f(a) = c$ and $g(b) = d$. Thus $h(a,b) = (f(a), g(b)) = (c,d)$.

(b) Show that $\mathbb{N} \approx \mathbb{N} \times \{0,1\}$.

Solution. $\mathbb{N} \times \{0,1\} = \{(1,0), (1,1), (2,0), (2,1), (3,0), (3,1), \ldots\}$. Thus we can find a bijection between $\mathbb{N} \times \{0,1\}$ and $\mathbb{N}$ by matching the ordered pairs $(n,0)$ with the odd natural numbers and the pairs $(n,1)$ with the even natural numbers. More precisely, we define a function $f : \mathbb{N} \times \{0,1\} \to \mathbb{N}$ by

$$f(n,a) = 2n + a - 1, \quad \text{for } n \in \mathbb{N},\ a \in \{0,1\}.$$

We check that f is one-to-one and onto. First suppose that $f(n,a) = f(m,b)$, so that $2n + a - 1 = 2m + b - 1$. Since a and b are each either 1 or 0, $2n + a = 2m + b$ implies that $a = b$ and then it follows easily that $n = m$. Hence f is injective. Now suppose $m \in \mathbb{N}$. If m is even, $m = 2n = 2n + 1 - 1 = f(n,1)$. If m is odd, $m = 2n - 1 = 2n + 0 - 1 = f(n,0)$. This shows that f is onto. Hence f is a bijection. (Can you find a simple formula for f^{-1}?)