Thm $\mathbb{R}$ is uncountable.

Pf Cantor's Diagonal Argument.

Suppose, by way of contradiction, that there is a bijection $f: \mathbb{N} \to \mathbb{R}$.

Let's write out the images $f(1)$, $f(2)$, etc. as decimals

for example, we might have

$$\left.\begin{aligned}
f(1) &= 1.2345\cdots \\
f(2) &= 19.619\cdots \\
f(3) &= -0.1000\cdots \\
f(4) &= 2371.1\cdots \\
f(5) &= -6.2123\cdots \\
&\ \ \vdots
\end{aligned}\right\}$$

by assumption, f is onto, meaning that every real number eventually appears in this list

We will obtain a contradiction by producing a real number that cannot be in this list.

Take all the numbers on the main diagonal.

$$\begin{array}{llll}
1.2 & 3 & \cdots & \\
1 & 9 & .6 & \\
-0.1 & & 0 & \\
\vdots & & & 1 \\
\vdots & & & \ \ 3 \\
 & & & \ \ \ \ \ddots
\end{array}$$

and change each digit to 1, or to 2 if it already equals 1, (and leave the decimal after the first digit) to define $x \in \mathbb{R}$.

here,

$\Rightarrow x = 2.1121\cdots$

claim x does not appear in the list.

ie, $\nexists\, n \in \mathbb{N}\ (f(n) = x)$.

notice the n^{th} digit of x is obtained by changing the n^{th} digit of $f(n)$.

Thus $x \neq f(n)$ for any $n \in \mathbb{N}$.

This contradicts the assumption that $f: \mathbb{N} \to \mathbb{R}$ was bijective.

note $\mathbb{N}, \mathbb{Z}, \mathbb{N}\times\mathbb{N}, \mathbb{Z}\times\mathbb{Z}, \mathbb{Q}$ are countably infinite.