Math 8 - Homework #3 Solutions

October 18, 2007

For exercises 1-3, do the following:

(a) Rewrite the given proposition as a conditional (if-then) statement.

(b) Prove the proposition or give a counterexample.

(c) If you prove it, say whether your proof is direct, indirect or by contradiction.

1. The sum of any two rational numbers is rational.

   Solution. (a) If x and y are rational, then $x + y$ is rational.

   (b) Assume that x and y are rational. This means that $x = a/b$ and $y = c/d$ for integers a, b, c, d with $b, d \neq 0$. Thus

   $$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

   which is rational since $ad + bc$ and bd are integers and $bd \neq 0$.

   (c) This is a direct proof.

2. The product of any two irrational real numbers is irrational.

   Solution. (a) If x and y are irrational real numbers, then xy is irrational.

   (b) This statement is false. A counterexample is given by letting $x = y = \sqrt{2}$, which we know is irrational from class. However, $xy = (\sqrt{2})^2 = 2$ is rational.

3. For every odd prime number p, at least one of the numbers $p + 2,\ p + 4$ is also prime.

   Solution. (a) If p is an odd prime number, then at least one of $p + 2,\ p + 4$ is also prime.

   (b) This statement is false. A counterexample is provided by the prime number $p = 23$ (the primes $31, 47, 53$ and many more also work as counterexamples), since neither 25 nor 27 is prime.

4. Let n be an integer. Prove that if n^2 is even, then n is even.

   Solution. We shall prove the contrapositive, which states that if n is odd, then n^2 is odd. Assume that n is odd. Thus $n = 2k + 1$ for some integer k. Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is clearly odd.

5. Prove that the sum of two integers a and b is even if and only if a and b are both even or both odd.

   Solution. We first prove the backward direction ($\Leftarrow$), namely the proposition "If a and b are both even or both odd, then $a + b$ is even." We have two cases to consider.

First, suppose that a and b are both even. This means that $a = 2k$ and $b = 2l$ for some integers k and l. Thus $a + b = 2k + 2l = 2(k + l)$ is even. For the second case, suppose that a and b are both odd. This means that $a = 2k + 1$ and $b = 2l + 1$ for some integers k and l. Thus $a + b = 2k + 1 + 2l + 1 = 2(k + l + 1)$ is even.

We now prove the forward direction ($\Rightarrow$), namely that "If $a+b$ is even, then a and b are both even or both odd." We will prove this indirectly by proving the contrapositive, which says that "If one of a, b is even and the other is odd, then $a + b$ is odd." Thus, let us assume that one of a, b is even and the other is odd. Without loss of generality, we may assume that a is even and b is odd (the proof in the other case is similar). So $a = 2k$ and $b = 2l+1$ for some integers k and l. Thus $a+b = 2k+2l+1 = 2(k+l)+1$, which is odd.

6. Prove that 5 is a prime number.

   Solution. We prove this by contradiction. Assume that 5 is not prime. This means that 5 has a factorization $5 = ab$ for integers a and b that lie strictly between 1 and 5. Hence, each of a and b is either $2, 3$ or 4. A simple check now shows that the possible values of the product ab are $4, 6, 8, 9, 12, 16$. Since $5 = ab$, 5 must equal one of these numbers, but this is clearly a contradiction.