Math 8 - Midterm 1 Solutions

October 19, 2007

1. (12 pts) Consider the proposition R

   "If I go surfing or take a nap, then I will not go surfing or I will not take a nap."

   (a) (2 pts) Express this proposition symbolically in terms of propositional variables P and Q and logical connectives. Be sure to say what P and Q represent.

   Solution. $(P \vee Q) \Rightarrow (\sim P \vee \sim Q)$, where P is "I will go surfing." and Q is "I will take a nap."

   (b) (6 pts) Make a truth table for your answer to (a).

   Solution.

P	Q	$P \vee Q$	$\sim P$	$\sim Q$	$\sim P \vee \sim Q$	$(P \vee Q) \Rightarrow (\sim P \vee \sim Q)$
T	T	T	F	F	F	F
T	F	T	F	T	T	T
F	T	T	T	F	T	T
F	F	F	T	T	T	T

   (c) (4 pts) State the converse and contrapositive of R as English sentences (without using phrases like "it is not the case that", etc.). It may help to first write them in terms of P and Q.

   Solution. Converse: "If I don't go surfing or I don't take a nap, then I will go surfing or take a nap." In fact, this is equivalent to the simpler statement "I will go surfing or take a nap."

   Contrapositive: First notice that the negation of the conclusion $(\sim P \vee \sim Q)$ is $P \wedge Q$ by De Morgan's law. Also by De Morgan's law, the negation of the hypothesis $(P \vee Q)$ is $\sim P \wedge \sim Q$. Thus the contrapositive says "If I go surfing and take a nap, then I will not go surfing and I will not take a nap." Of course, it sounds absurd like this, but is equivalent to saying "I won't go surfing and take a nap."

2. (10 pts) Let P be the proposition

   "The sum of a rational number and an irrational number is irrational."

   (a) (2 pts) Rephrase P as a conditional statement. (You may want to introduce some variables x, y.)

   Solution. "If x is rational and y is irrational, then $x + y$ is irrational."

   (b) (2 pts) Express P in terms of symbols and variables only, without using words.

   Solution. $[(x \in \mathbb{Q}) \wedge (y \notin \mathbb{Q})] \Rightarrow (x + y \notin \mathbb{Q})$.

   (c) (6 pts) Prove P is true.

   Solution. We will prove P by contradiction. Assume that P is not true. This means that we have a rational number x and an irrational number y such that

$x + y$ is rational. By definition of rational, we can write $x = p/q$ and $x + y = r/s$ for integers p, q, r, s with $q, s \neq 0$. Thus

$$y = (x + y) - x = \frac{r}{s} - \frac{p}{q} = \frac{rq - ps}{sq},$$

which is a rational number since $rq - ps$ and sq are integers with $sq \neq 0$. However, this contradicts the assumption that y is irrational.

3. (a) (4 pts) List (or otherwise describe) the elements of the set $S = \{x \in \mathbb{R} \mid 3x \in \mathbb{N}\}$.

   Solution. S is the set of all real numbers x such that $3x$ is a natural number. Thus

$$S = \left\{ \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \dots \right\}$$

   is the set of all positive rational numbers that can be written with a 3 in the denominator.

   (b) (4 pts) List (or otherwise describe) the elements of the set $T = \{1/x \mid x \in S\}$.

   Solution. T is the set of all reciprocals of elements of S. Thus

$$T = \left\{ \frac{3}{1}, \frac{3}{2}, \frac{3}{3}, \frac{3}{4}, \frac{3}{5}, \dots \right\}$$

   is the set of all positive rational numbers that can be written with a 3 in the numerator.

4. (10 pts) Prove that an integer n is the product of two even integers if and only if it is a multiple of 4.

   Solution. ($\Rightarrow$) Assume that n is the product of two even integers x and y. Thus $x = 2k$ and $y = 2l$ for integers k and l. Hence $n = xy = (2k)(2l) = 4kl$ is a multiple of 4.

   ($\Leftarrow$) Assume that n is a multiple of 4. Thus we can write $n = 4k$ for some integer k, and then we have $n = 2(2k)$, which is the product of two even integers, 2 and $2k$.