HW 8 Solutions

p417–418

Determine where the series converge:

(1C)

$$\sum_{n=1}^{\infty} \frac{1}{nx^{2n}}$$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{1}{(n+1)x^{2n+2}} \cdot \frac{nx^{2n}}{1} \right| = \lim_{n\to\infty} \left| \frac{n\,x^2}{n+1} \right| = |x|^2 < 1$$

So radius of convergence is 1

Check endpoints:

$\underline{x=-1)}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(-1)^{2n}} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$\underline{x=1)}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(1)^{2n}} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

So $\sum \frac{1}{nx^{2n}}$ converges for $-1 < x < 1$;

(1E) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{(1-x)^n}$

Root Test:

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{x^n}{(1-x)^n}\right|} = \left|\frac{x}{1-x}\right| < 1$$

So $|x| < |1-x| \Rightarrow x < \frac{1}{2}$

Now check the endpoint $x = \frac{1}{2}$:

$$\sum_{n=1}^{\infty} \frac{(\frac{1}{2})^n}{(1-\frac{1}{2})^n} = \sum_{n=1}^{\infty} 1^n = \sum_{n=1}^{\infty} 1 \quad \text{diverges}$$

So $\sum \frac{x^n}{(1-x)^n}$ converges for $x < \frac{1}{2}$

(1F) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n \sin^n x}{n^2}$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{2^{n+1} \sin^{n+1} x}{(n+1)^2} \cdot \frac{n^2}{2^n \sin^n x} \right| = \lim_{n\to\infty} \left| \frac{(2\sin x)\, n^2}{(n+1)^2} \right|$$

$$= |2\sin x| < 1$$

$$\Rightarrow \quad -\tfrac{1}{2} < \sin x < \tfrac{1}{2}$$

$$\Rightarrow \quad \pi(2n - \tfrac{1}{6}) < x < \pi(2n + \tfrac{1}{6})$$

or $\pi(2n + \frac{5}{6}) < x < \pi(2n + \frac{7}{6})$ for any $n \in \mathbb{Z}$.

Check endpoints:

At any endpoint x, $\sin x = \frac{1}{2}$ and the series becomes $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges So we have convergence for

$$\pi(2n - \tfrac{1}{6}) \le x \le \pi(2n + \tfrac{1}{6})$$

and $\pi(2n + \frac{5}{6}) \le x \le \pi(2n + \frac{7}{6})$ for any $n \in \mathbb{Z}$, Prove uniform convergence:

2A

$$\sum_{n=1}^{\infty} \frac{x^n}{n^3} \qquad -1 \le x \le 1$$

If $-1 \le x \le 1$, then $\left|\frac{x^n}{n^3}\right| \le \frac{1}{n^3}$, and since

$\sum 1/n^3$ converges, then by the Weierstrass M-Test $\sum_{n=1}^{\infty} \frac{x^n}{n^3}$ converges uniformly on $-1 \le x \le 1$.

2C $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2+1}$ for all x

Since $|\sin nx| \le 1$, then $\left|\frac{\sin nx}{n^2+1}\right| \le \frac{1}{n^2+1}$.

Furthermore $\sum \frac{1}{n^2+1}$ converges, so by the Weierstrass M-Test $\sum \frac{\sin nx}{n^2+1}$ converges uniformly for all x.

2D $\sum_{n=1}^{\infty} \frac{e^{nx}}{2^n} \qquad x \le \log 3/2$

If $x \le \log 3/2$, then $e^{nx} \le e^{n \log 3/2} = \left(\frac{3}{2}\right)^n$.

So $\left|\frac{e^{nx}}{2^n}\right| \le \frac{(3/2)^n}{2^n} = \left(\frac{3}{4}\right)^n$ and since $\sum \left(\frac{3}{4}\right)^n$ converges, then the Weierstrass M-Test implies uniform convergence of $\sum \frac{e^{nx}}{2^n}$ for $x \le \log 3/2$.

2F $\sum_{n=1}^{\infty} nx^n \qquad -\frac{1}{2} \leq x \leq \frac{1}{2}$

If $-\frac{1}{2} \leq x \leq \frac{1}{2}$, then

$$|nx^n| \leq n\left(\tfrac{1}{2}\right)^n = \frac{n}{2^n}$$

So since $\sum \frac{n}{2^n}$ converges (by the ratio test) then the Weierstrass M-Test gives us that $\sum_{n=1}^{\infty} nx^n$ converges uniformly for $-\frac{1}{2} \leq x \leq \frac{1}{2}$.