Homework 6 Solutions

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① (a) $\lim_{n\to\infty} \sin \frac{n^2\pi}{2}$ does not exist so $\sum \sin \frac{n^2\pi}{2}$ diverges

(b) $\lim_{n\to\infty} \frac{2^n}{n^3} = \lim_{n\to\infty} \frac{(\ln 2) 2^n}{3n^2} = \lim_{n\to\infty} \frac{(\ln 2)^2 2^n}{6n} = \lim_{n\to\infty} \frac{(\ln 2)^3 2^n}{6} = \infty$

so $\sum \frac{2^n}{n^3}$ diverges

② (a) $\left|\frac{1}{n^3-1}\right| \leq \left|\frac{2}{n^3}\right|$ for $n \geq 2$ and $\sum \frac{2}{n^3}$ is a convergent p-series, thus $\sum \frac{1}{n^3-1}$ converges

(b) $\left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ is a convergent p-series, thus $\sum \frac{\sin(n)}{n^2}$ converges

③ (a) $\left|\frac{n+5}{n^2-3n-5}\right| \geq \left|\frac{n}{n^2-3n-5}\right| \geq \left|\frac{n}{n^2}\right| = \frac{1}{n}$ for $n \geq 5$

and $\sum \frac{1}{n}$ is a divergent harmonic series (by Integral Test). Thus $\sum \frac{n+5}{n^2-3n-5}$ diverges.

(b) Consider the function $f(x) = \log x - \sqrt{x}$ defined for $x > 0$.

Since $f'(x) = \frac{1}{x} - \frac{1}{2} \cdot \frac{1}{\sqrt{x}} = \frac{2 - \sqrt{x}}{2x} < 0$ whenever $x > 4$

and $f(4) = (\log 4) - 2 = 2\log 2 - 2 = 2\,[(\log 2) - 1] < 0$

this implies that $f(x) < 0$ for all $x > 4$.

Thus $\sqrt{x} > \log x$ for all $x > 4$

So $\left| \frac{1}{\sqrt{n}\log n} \right| \geq \left| \frac{1}{\sqrt{n}\sqrt{n}} \right| = \frac{1}{n}$ for $n > 4$.

Since $\sum \frac{1}{n}$ diverges (by Integral test),

this implies that $\sum \frac{1}{\sqrt{n}\log n}$ diverges.

(4)

(a) $\int_1^\infty \frac{1}{x^2+1}\,dx = \lim_{n\to\infty} \left[\tan^{-1}(n) - \tan^{-1}(1) \right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} < \infty$

Thus $\sum \frac{1}{n^2+1}$ converges

(b) $\int_2^\infty \frac{dx}{x \log^2 x}$: let $u = \log x$ so $du = \frac{1}{x}\,dx$

and we have now

$$\int_2^\infty \frac{dx}{x\log^2 x} = \lim_{b\to\infty} \int_2^b \frac{dx}{x\log^2 x} = \lim_{b\to\infty} \int_{\log 2}^{\log b} \frac{du}{u^2}$$

$$= \lim_{b\to\infty} \left(-\frac{1}{u} \right)_{\log 2}^{\log b} = \lim_{b\to\infty} \left(-\frac{1}{\log b} + \frac{1}{\log 2} \right) = \frac{1}{\log 2} < \infty$$

By the integral test, $\sum \frac{1}{n \log^2 n}$ converges.

(5) (a)

$$\int_1^\infty \frac{n}{n^2+1} \text{ diverges by the Integral Test}$$

$$\int_1^\infty \frac{x\,dx}{x^2+1} \qquad \text{Let } u = x^2+1 \text{, so } du = 2x\,dx$$

$$\int_1^\infty \frac{x\,dx}{x^2+1} = \lim_{b\to\infty} \int_1^b \frac{x\,dx}{x^2+1} = \lim_{b\to\infty} \int_2^{b^2+1} \frac{\frac{1}{2}\,du}{u}$$

$$= \lim_{b\to\infty} \tfrac{1}{2}\left(\ln(b^2+1) - \ln 2\right) = \infty$$

So $\sum \frac{n}{n^2+1}$ diverges

(6) (a) $a_n = \frac{(-1)^n}{n!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}/(n+1)!}{(-1)^n/n!}\right| = \lim_{n\to\infty} \left|\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

So $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$; thus $\sum \frac{(-1)^n}{n!}$ converges (absolutely)

⑦ (a) $a_n = \frac{(-1)^n}{\log n}$

$\sum a_n$ is an alternating series because

$a_n > 0$ for n even and $a_n < 0$ for n odd

Furthermore $\left\{ \frac{1}{\log n} \right\}$ is a decreasing sequence

i.e. $\frac{1}{\log 2} \geq \frac{1}{\log 3} \geq \frac{1}{\log 4} \geq \frac{1}{\log 5} \geq \cdots$

and $\lim_{n \to \infty} \frac{1}{\log n} = 0$. So we all of the

necessary hypotheses to apply the alternating series test. So $\sum a_n$ converges.

⑧ (a) $a_n = \frac{1}{n^n}$

$\lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} 1/n = 0 < 1$

So $\sum \frac{1}{n^n}$ converges (absolutely)

(b) $a_n = \left(\frac{n}{n+1}\right)^{n^2}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \cancel{\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n}$$

$$= \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{e} < 1$$

So by the root test $\sum a_n$ converges (absolutely).

9 (b) $$\sum_{n=1}^{\infty} \frac{1-n}{2^{n+1}} = \sum_{n=1}^{\infty} \left(\frac{n+1}{2^{n+1}} - \frac{n}{2^n}\right)$$

$$S_1 = \frac{2}{2^2} - \frac{1}{2^1}$$

$$S_2 = \left(\frac{2}{2^2} - \frac{1}{2^1}\right) + \left(\frac{3}{2^3} - \frac{2}{2^2}\right)$$

$$S_3 = \left(\frac{2}{2^2} - \frac{1}{2^1}\right) + \left(\frac{3}{2^3} - \frac{2}{2^2}\right) + \left(\frac{4}{2^4} - \frac{3}{2^3}\right)$$

$\vdots$

$$S_n = \left(\frac{2}{2^2} - \frac{1}{2^1}\right) + \cdots + \left(\frac{n+1}{2^{n+1}} - \frac{n}{2^n}\right)$$

$= \frac{n+1}{2^{n+1}} - \frac{1}{2^1}$ because almost everything subtracts out

$\lim_{n\to\infty} S_n = -\frac{1}{2}$ (by L'Hopital) Thus $\sum_{n=1}^{\infty} \frac{1-n}{2^{n+1}} = -\frac{1}{2}$.

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(a) $\left|\frac{n+4}{2n^3-1}\right| \leq \frac{2n}{2n^3-1} \leq \frac{2n}{2n^3} = \frac{1}{n^2}$ for $n \geq 4$

Since $\sum 1/n^2$ is a convergent p-series, this implies, by the comparison test, that $\sum \frac{n+4}{2n^3-1}$ converges

(b) $\left|\frac{3n-5}{n \cdot 2^n}\right| \leq \frac{3n}{n \cdot 2^n} = \frac{3}{2^n}$ for $n \geq 3$

Since $\sum 3/2^n$ is a convergent geometric series, then the comparison test tells us that $\sum \frac{3n-5}{n\,2^n}$ converges

(c) $\left|\frac{e^n}{n+1}\right| \geq \frac{1}{n+1}$

Since $\sum \frac{1}{n+1}$ diverges (by integral test) then the comparison ~~tell~~ test tells us that $\sum \frac{e^n}{n+1}$ diverges also.

(d) $\left|\dfrac{n^2}{n!+1}\right| \leq \dfrac{n^2}{n!}$

Since $\sum \frac{n^2}{n!}$ converges (by ratio test)

~~or by t~~ then the comparison test tells us that $\sum \frac{n^2}{n!+1}$ converges also.

(e) $a_n = \dfrac{n!}{3\cdot 5\cdot \cdots \cdot (2n+3)}$

Since $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{(n+1)!\,(3\cdot 5\cdot \cdots \cdot (2n+3))}{3\cdot 5\cdots(2n+5)\,n!}$

$= \lim\limits_{n\to\infty} \dfrac{n+1}{2n+5} = \frac{1}{2} < 1$

then by the ratio test $\sum a_n$ converges.

(f) $a_n = \dfrac{(-1)^n \log n}{2n+3}$

Consider the function $f(x) = \dfrac{\log x}{2x+3}$ defined for $x > 0$.

Since $f'(x) = \dfrac{(2x+3)\frac{1}{x} - 2\log x}{(2x+3)^2}$

$$= \frac{2 + 3/x - 2\log x}{(2x+3)^2} < 0$$

for all $x > 1000$, then this implies that $\left\{ \dfrac{\log n}{2n+3} \right\}$ eventually becomes a decreasing sequence, and furthermore we have

$$\lim_{n\to\infty} \frac{\log n}{2n+3} = 0 \quad (\text{by L'Hopital})$$

Then we can apply the alternating series test to conclude that $\sum a_n$ converges.