Homework 2 Solutions

p248 #3

(a) $P_1 = (x_u, y_u, z_u) = (-(b+a\cos v)\sin u,\ (b+a\cos v)\cos u,\ 0)$

$P_2 = (x_v, y_v, z_v) = (-a\sin v\cos u,\ -a\sin v\sin u,\ a\cos v)$

Calculate the cross-product; you get

$$P_1 \times P_2 = (b+a\cos v)(a\cos u\cos v\,\vec{i} + a\sin u\cos v\,\vec{j} + a\sin v\,\vec{k})$$

Now find the length of $P_1 \times P_2$; you should get

$$\| P_1 \times P_2 \| = a(b+a\cos v)$$

So the surface area of the torus T is

$$SA = \int_T d\sigma = \iint_0^{2\pi\ 2\pi} \| P_1 \times P_2 \|\, du\, dv = \int_0^{2\pi}\int_0^{2\pi} a(b+a\cos v)\, du\, dv$$

$$= (2\pi)^2 ab = \boxed{4\pi^2 ab}$$

(b) Same as part (a) except now you should get

$$\| P_1 \times P_2 \| = u\sqrt{4u^2+1}$$

So the surface area of the saddle S is

$$SA = \int_S d\sigma = \int_0^1 \int_0^{\pi/2} u\sqrt{4u^2+1}\, dv\, du$$

$$= \frac{\pi}{2}\int_0^1 u\sqrt{4u^2+1}\, du = \boxed{\frac{\pi}{24}(5^{3/2}-1)}$$

p313#6

(a) $\int_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$

$$= \int_S \left((u+v)\left|\frac{\partial(y,z)}{\partial(u,v)}\right| + (u-v)\left|\frac{\partial(z,x)}{\partial(u,v)}\right| + (1-2u)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \right) du\,dv$$

Find the absolute values of the determinants of the Jacobian matrices:

$$\left|\frac{\partial(y,z)}{\partial(u,v)}\right| = \text{abs}\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = 2$$

$$\left|\frac{\partial(z,x)}{\partial(u,v)}\right| = \text{abs}\begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix} = 2$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \text{abs}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2$$

So the integral becomes

$$\int_S \left[(u+v)\cdot 2 + (u-v)\cdot 2 + (1-2u)\cdot 2\right] du\,dv$$

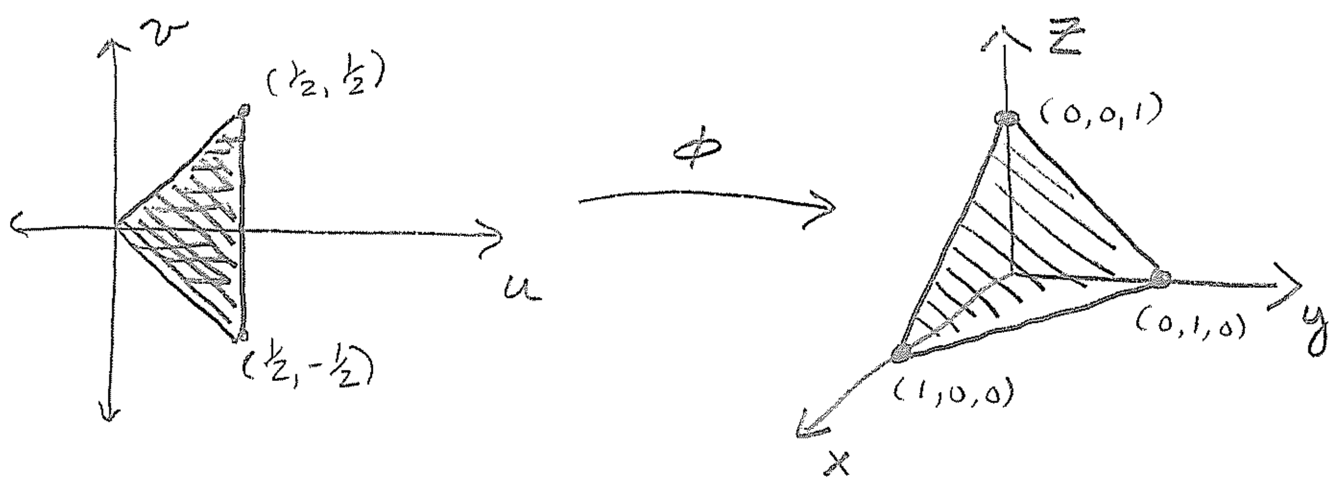
$$= \int_S 2\,du\,dv$$

Now we need to find the appropriate bounds on u and v:

When $(u,v)=(0,0)$, we have $(x,y,z)=(0,0,1)$

" $(u,v)=(\frac{1}{2},-\frac{1}{2})$, " " $(x,y,z)=(0,1,0)$

" $(u,v)=(\frac{1}{2},\frac{1}{2})$ " " $(x,y,z)=(1,0,0)$



$$\phi(u,v)=(u+v,\ u-v,\ 1-2u)$$

The surface integral is therefore

$$\int_S x\,dy\,dz+y\,dz\,dx+z\,dx\,dy = \int_0^{1/2}\int_{-v}^{v} 2\,du\,dv$$

$$= 2\int_0^{1/2} 2v\,dv = 4\left(\tfrac{1}{2}v^2\right)_0^{1/2}$$

$$= \boxed{1/2}$$

ⓑ $\int_S dy\,dz + dz\,dx + dx\,dy \qquad S = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{matrix} x^2+y^2+z^2 = 1 \\ \text{and } z \geq 0 \end{matrix} \right\}$

$$\int_S dy\,dz + dz\,dx + dx\,dy = \int_S \vec{v} \cdot \vec{n}\, d\sigma$$

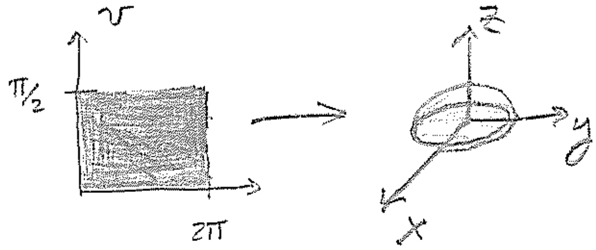
where $\vec{v} = \vec{i} + \vec{j} + \vec{k}$

$\vec{n} = x\vec{i} + y\vec{j} + z\vec{k}$

and $d\sigma = \| P_1 \times P_2 \|\, du\,dv = \text{abs} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} du\,dv$

$$= \text{abs} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos u \cos v & \cos u \sin v & -\sin u \\ -\sin u \sin v & \sin u \cos v & 0 \end{vmatrix} du\,dv$$

$$= \sin v\, du\,dv$$



So the surface integral is

$$\int_S (\vec{v} \cdot \vec{n})\, d\sigma = \int_S (x+y+z)\, d\sigma$$

$$= \int_S (\sin u \cos v + \sin u \sin v + \cos u)\, d\sigma$$

$$= \int_0^{\pi/2} \int_0^{2\pi} (\sin u \cos v \sin v + \sin u \sin^2 v + \cos u \sin v)\, dv\, du$$

$$= \int_0^{\pi/2} \left[\int_0^{2\pi} (\sin u \cos v \sin v + \sin u \sin^2 v + \cos u \sin v)\, dv \right] du$$

$$= \int_0^{\pi/2} \left[\int_0^{2\pi} \sin u \sin^2 v \, dv \right] du$$

$$= \int_0^{\pi/2} \pi \sin u \, du$$

$$= \pi \left(-\cos u \right) \Big|_0^{\pi/2}$$

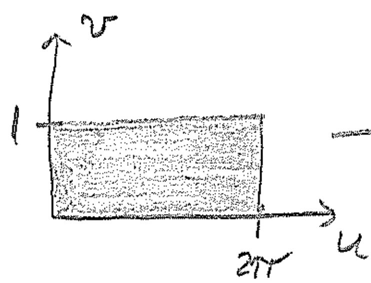
$$= \pi \left(0 - (-1) \right) = \boxed{\pi}$$

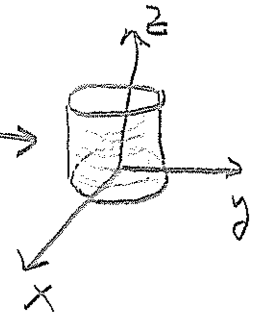
(d) $P_1 = (-\sin u, \cos u, 0)$

$P_2 = (0, 0, 1)$

$P_1 \times P_2 = (\cos u, \sin u, 0)$

$\| P_1 \times P_2 \| = 1$



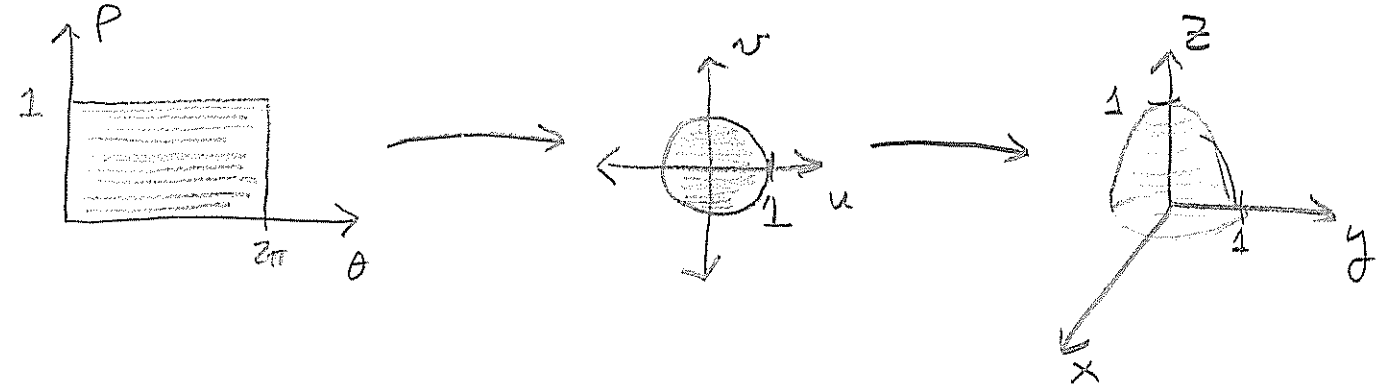


$$\int_S x^2 z \, d\sigma = \int_0^1 \int_0^{2\pi} (\cos^2 u) v \, \| P_1 \times P_2 \| \, du \, dv$$

$$= \int_0^1 \int_0^{2\pi} (\cos^2 u) v \, du \, dv = \int_0^1 \pi v \, dv = \boxed{\frac{\pi}{2}}$$

p313 #7

(a)



$$x = u \quad = \rho\cos\theta$$
$$y = v \quad = \rho\sin\theta$$
$$z = 1-u^2-v^2 \quad = 1-\rho^2$$

$$P_1 = (x_\rho, y_\rho, z_\rho) = (\cos\theta, \sin\theta, -2\rho)$$
$$P_2 = (x_\theta, y_\theta, z_\theta) = (-\rho\sin\theta, \rho\cos\theta, 0)$$

$$P_1 \times P_2 = (2\rho^2\cos\theta, 2\rho^2\sin\theta, \rho)$$

$$\vec{w} = (xy^2z, -2x^3, yz^2)$$
$$= (\rho^3\cos\theta\sin^2\theta(1-\rho^2), -2\rho^3\cos^3\theta, \rho\sin\theta(1-\rho^2)^2)$$

$$\vec{w}\cdot(P_1\times P_2) = 2\rho^5\cos^2\theta\sin^2\theta(1-\rho^2) - 4\rho^5\sin\theta\cos^3\theta$$
$$+ \rho^2\sin\theta(1-\rho^2)^2$$

So the integral becomes

$$\int_S \vec{w} \cdot \vec{n}\, d\sigma = \int_S \vec{w} \cdot (P_1 \times P_2)\, d\rho\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(2\rho^5 \cos^2\theta \sin^2\theta\,(1-\rho^2) - 4\rho^5 \sin\theta\cos^3\theta + \rho^2 \sin\theta (1-\rho^2)^2\right) d\rho\, d\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{3}\cos^2\theta\sin^2\theta - \frac{1}{4}\cos^2\theta\sin^2\theta - \frac{2}{3}\sin\theta\cos^3\theta + \sin\theta\left(\frac{3}{35}\right)\right] d\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{12}\cos^2\theta\sin^2\theta - \frac{2}{3}\sin\theta\cos\theta(\cos^2\theta) + \frac{3}{35}\sin\theta\right] d\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{48}(1+\cos 2\theta)(1-\cos 2\theta) - \frac{1}{6}\sin 2\theta\,(1+\cos 2\theta) + \frac{3}{35}\sin\theta\right] d\theta$$

$$= \frac{1}{48}\int_0^{2\pi}(1-\cos^2 2\theta)\, d\theta$$

$$= \frac{1}{48}\int_0^{2\pi}\left[1 - \frac{1}{2}(1+\cos 4\theta)\right] d\theta$$

$$= \frac{1}{48}\left(2\pi - \frac{1}{2}(2\pi)\right) = \boxed{\frac{\pi}{48}}$$

p 313 #7

(b) $\vec{w} = (1,2,3)$

$P_1 = (e^u \cos v, e^u \sin v, 0) \quad P_2 = (-e^u \sin v, e^u \cos v, \cos 2v)$

$P_1 \times P_2 = (e^u \sin v \cos 2v, -e^u \cos v \cos 2v, e^{2u})$

$$\int_S \vec{w}\cdot\vec{n}\, d\sigma = \int_S \vec{w}\cdot(P_1 \times P_2)\, du\, dv$$

$$= \int_0^{\pi/2}\int_0^1 (e^u \sin v \cos 2v - 2e^u \cos v \cos 2v + 3e^{2u})\, du\, dv$$

$$= \int_0^{\pi/2}\left[(e-1)\sin v \cos 2v - 2(e-1)\cos v \cos 2v + \tfrac{3}{2}(e^2-1)\right] dv$$

$$= \int_0^{\pi/2}\left[\frac{e-1}{4i}(e^{iv}-e^{-iv})(e^{2iv}+e^{-2iv}) - \frac{e-1}{2}(e^{iv}+e^{-iv})(e^{2iv}+e^{-2iv}) + \frac{3}{2}(e^2-1)\right] dv$$

$$= \frac{e-1}{4i}\left(\frac{1}{3i}(-i-1) - \frac{1}{i}(-i-1) - \frac{1}{i}(i-1) + \frac{1}{3i}(i-1)\right)$$

$$- \frac{e-1}{2}\left(\frac{1}{3i}(-i-1) - \frac{1}{i}(-i-1) + \frac{1}{i}(i-1) - \frac{1}{3i}(i-1)\right)$$

$$+ \frac{3\pi}{4}(e^2-1)$$

$$= \frac{e-1}{4i}\left(\frac{1}{i}\right)\left(\frac{4}{3}\right) - \frac{e-1}{2}\left(\frac{4}{3}\right) + \frac{3\pi}{4}(e^2-1)$$

$$= \frac{-(e-1)}{3} - \frac{2(e-1)}{3} + \frac{3\pi}{4}(e^2-1) = \boxed{1 - e + \frac{3\pi}{4}(e^2-1)}$$