Homework 1 Solutions

p312 #1

(a)

$x = \cos t \quad dx = -\sin t\, dt$

$y = \sin t \quad dy = \cos t\, dt \qquad 0 \le t \le 2\pi$

$z = t \quad dz = dt$

$$\int_0^{2\pi} t(-\sin t\, dt) + \cos t(\cos t\, dt) + \sin t\,(dt)$$

$$= (t\cos t - \sin t)\Big|_0^{2\pi} + \tfrac{1}{2}\int_0^{2\pi} (1+\cos 2t)\,dt$$

$$= 2\pi + \tfrac{1}{2}(2\pi) = \boxed{3\pi}$$

(b)

$x = (1-t)\,1 + t\,(2) = t+1 \qquad dx = dt$

$y = (1-t)\,0 + t\,(3) = 3t \qquad dy = 3dt \qquad 0 \le t \le 1$

$z = (1-t)\,1 + t\,(2) = t+1 \qquad dz = dt$

$$\int_0^1 (t+1)^2\,dt - (t+1)^2(3dt) + (3t)^2\,dt$$

$$= \int_0^1 -2(t^2+2t+1) + 9t^2\,dt$$

$$= -2\left(\tfrac{1}{3} + 1 + 1\right) + \tfrac{9}{3} = -2\left(\tfrac{7}{3}\right) + \tfrac{9}{3} = \boxed{-\tfrac{5}{3}}$$

(c) $x = \cos t \qquad dx = -\sin t\,dt$

$y = \cos t \qquad dy = -\sin t\,dt \qquad 0 \le t \le \pi/2$

$z = \sqrt{2}\sin t \qquad dz = \sqrt{2}\cos t\,dt$

$$\int_0^{\pi/2} (\cos^2 t)(\cos t)(\sqrt{2}\sin t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \int_0^{\pi/2} \sqrt{2}\cos^3 t \sin t \sqrt{2\sin^2 t + 2\cos^2 t}\,dt$$

$$= 2\int_0^{\pi/2} \cos^3 t \sin t\,dt$$

$$= 2\int_1^0 u^3(-du) = 2\int_0^1 u^3\,du = 2\left(\tfrac{1}{4}\right) = \boxed{\tfrac{1}{2}}$$

(d) $x = \cos t \qquad dx = -\sin t\,dt$

$y = \sin t \qquad dy = \cos t\,dt \qquad 0 \le t \le 2\pi$

$z = 2 \qquad dz = 0$

$$\int_C 2xy^2z\,dx + 2x^2yz\,dy + x^2y^2\,dz$$

$$= \int_0^{2\pi} 2\cos t \sin^2 t \cdot 2 \cdot (-\sin t)\,dt + 2\cos^2 t \cdot \sin t \cdot 2 \cdot \cos t\,dt$$

$$= \int_0^{2\pi} -4\cos t \sin t\left(\sin^2 t - \cos^2 t\right)dt$$

$$= \int_0^{2\pi} -4\left(\tfrac{1}{2}\sin 2t\right)(-\cos 2t)\,dt$$

$$= \int_0^{2\pi} 2\sin 2t \cos 2t\,dt = \int_0^{2\pi} \sin 4t\,dt = \boxed{0}$$

(e) $u = \text{curl}\, v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & z^2 & x^2 \end{vmatrix} = (-2z, -2x, -2y)$

$$\begin{array}{lll} x = 2t+1 & dx = 2\,dt & \\ y = t^2 & dy = 2t\,dt & 0 \le t \le 1 \\ z = 1+t^3 & dz = 3t^2\,dt & \end{array}$$

$$\int_c u_T\, ds = \int_c -2z\,dx - 2x\,dy - 2y\,dz$$

$$= \int_0^1 -2(1+t^3)2dt - 2(2t+1)2t\,dt - 2t^2(3t^2dt)$$

$$= \int_0^1 (-6t^4 - 4t^3 - 8t^2 - 4t - 4)\,dt$$

$$= -\frac{6}{5} - 1 - \frac{8}{3} - 2 - 4 = \frac{1}{15}(-18 - 15 - 40 - 30 - 60)$$

$$= \frac{1}{15}(-163) = \boxed{\frac{-163}{15}}$$

$\boxed{\text{p121 \#6a}}$

$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \rho^2 \sin\phi \begin{vmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{vmatrix}$$

$$= \rho^2 \sin\phi \left[\cos\phi\left(\cos\phi\cos^2\theta + \cos\phi\sin^2\theta\right) + \sin\phi\left(\sin\phi\cos^2\theta + \sin\phi\sin^2\theta\right)\right]$$

$$= \boxed{\rho^2 \sin\phi}$$

Extra Problems not in book

① $B = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1\}$

$$\text{mass} = \int_B \delta \, dV = \int_B \frac{1}{1+x^2+y^2+z^2} \, dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \frac{1}{1+\rho^2} (\rho^2 \sin\phi) \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left[\int_0^1 \frac{\rho^2 \sin\phi}{1+\rho^2} \, d\rho\right] d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left[\sin\phi \int_0^1 \left(1 - \frac{1}{1+\rho^2}\right) d\rho\right] d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left[\sin\phi \left(1 - \left[\tan^{-1}\rho\right]_0^1\right)\right] d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} (\sin\phi)\left(1 - \pi/4\right) d\phi\, d\theta$$

$$= \left(1 - \pi/4\right) \int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta$$

$$= 2\left(1 - \pi/4\right) \int_0^{2\pi} d\theta$$

$$= \boxed{4\pi\left(1 - \pi/4\right)}$$

(2) $x = \rho\cos\theta$

$y = \rho\sin\theta \rightsquigarrow dV = dx\,dy\,dz = \left|\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,z)}\right| d\rho\, d\theta\, dz = \rho\, d\rho\, d\theta\, dz$

$z = z$

$$\text{Volume} = \int_R dV = \int_0^2\int_0^{2\pi}\int_0^{4-\rho^2} \rho\, dz\, d\theta\, d\rho$$

$$= \int_0^2\int_0^{2\pi} \rho(4-\rho^2)\, d\theta\, d\rho$$

$$= 2\pi\int_0^2 \rho(4-\rho^2)\, d\rho = 2\pi\int_0^2 (4\rho - \rho^3)\, d\rho$$

$$= 2\pi\left(2\rho^2 - 1/4\,\rho^4\right)_0^2 = 2\pi(8-4) = \boxed{8\pi}$$