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# Math 5C - Solutions to Midterm 2 Review Problems

1. a) Geometric: $a=-4$, $r=-1/4$

$$\boxed{a_n = -4(-1/4)^n \quad \text{for } n \geq 0}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-4(-1)^n}{4^n} = \lim_{n\to\infty} \frac{(-1)^{n+1}}{4^{n-1}} = \boxed{0}.$$

b) $\boxed{a_n = \frac{2n-1}{2n+1}, \quad n \geq 1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n-1}{2n+1} \overset{L'H}{=} \lim_{n\to\infty} \frac{2}{2} = \boxed{1}$$

c) $\boxed{a_n = \begin{cases} (-1)^{n+1}(n^2-1), & \text{if } n \geq 2 \\ 1, & \text{if } n=1 \end{cases}, \quad \text{for } n \geq 1}$

(there is a typo in the problem: $a_1$ should be 0, not 1)
then $a_n = (-1)^{n+1}(n^2-1)$ for $n \geq 1$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^{n+1}(n^2-1) = \boxed{D.N.E.}$$

since it oscillates between large negative values & large positive values.

d) $\boxed{a_n = \frac{2n+1}{2^n} \quad \text{for } n \geq 0}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n+1}{2^n} \underset{L'H}{=} \lim_{n\to\infty} \frac{2}{2^n \cdot \ln 2} = \boxed{0}$$

2. a) $\sum_{n=1}^{\infty} 2^{1-2n} = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + \cdots$

= geometric series w/ $a = 1/2$, $r = 1/4$.

$$\text{sum} = \frac{a}{1-r} = \frac{1/2}{1-1/4} = \frac{1/2}{3/4} = \boxed{\frac{2}{3}}$$

b) $\sum_{n=1}^{\infty} \left( \frac{\sqrt{2n+1}}{n} - \frac{\sqrt{2n+3}}{n+1} \right) = \left(\frac{\sqrt{3}}{1} - \frac{\sqrt{5}}{2}\right) + \left(\frac{\sqrt{5}}{2} - \frac{\sqrt{7}}{3}\right) + \left(\frac{\sqrt{7}}{3} - \frac{\sqrt{9}}{4}\right) + \cdots$

$N^{th}$ partial sum $= \sqrt{3} - \frac{\sqrt{2N+3}}{N+1}$ (telescoping series)

$$\text{Sum} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(\sqrt{3} - \frac{\sqrt{2N+3}}{N+1}\right) = \boxed{\sqrt{3}}.$$

3 a) $\sqrt{\frac{n+1}{3n^2}} \geq \sqrt{\frac{n}{3n^2}} = \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{n}}\right)$ for all $n \geq 1$.

and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{n}}\right)$ diverges since it is a p-series w/ $p = 1/2 < 1$.

$\therefore$ By comparison test, $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{3n^2}}$ $\boxed{\text{diverges.}}$

b) $a_n = \frac{1}{n \ln n}$ for $n \geq 2$ is positive and decreasing.

$\therefore$ By Integral test, since $\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{b \to \infty} \int_2^b \frac{1}{x \ln x} dx$

$= \lim_{b \to \infty} \ln|\ln x|\Big]_2^b = \lim_{b \to \infty} \ln(\ln b) - \ln(\ln 2) = \infty$ Diverges,

$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ $\boxed{\text{Diverges}}$ too.

c) $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$ is an alternating series w/ $b_n = \frac{\ln n}{n}$.

$\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{\ln n}{n} \underset{LH}{=} \lim_{n \to \infty} \frac{1/n}{1} = 0$ ✓

& $\left\{\frac{\ln n}{n}\right\}_{n \geq 2}$ is decreasing for $n \geq 3$. ✓

$f(x) = \ln x / x \Rightarrow f'(x) = \frac{x(1/x) - \ln x}{x^2} = \frac{1 - \ln x}{x^2} < 0$

when $x > e$

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$ $\boxed{\text{converges}}$ by the Alternating Series Test.

d) Ratio Test: $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty} \left|\frac{(n+1)^{n+1}/2 \cdot 4 \cdots (2n)(2n+2)}{n^n / 2 \cdot 4 \cdots 2n}\right|$

$= \lim_{n \to \infty} \left|\frac{(n+1)}{2n+2} \frac{(n+1)^n}{n^n}\right| = \frac{1}{2} \lim_{n \to \infty} (1 + 1/n)^n = \frac{1}{2} e > 1$

So it $\boxed{\text{Diverges}}$ by the Ratio test.

4 a) Ratio test $\lim_{n \to \infty} \left|\frac{(x-3)^{n+1}/4^{n+1}}{(x-3)^n / 4^n}\right| = \lim_{n \to \infty} \left|\frac{x-3}{4}\right| = \frac{|x-3|}{4} < 1$

$\Rightarrow |x-3| < 4 \Rightarrow \boxed{R = 4}$.

end points : $x = 3 - 4 = -1$ & $x = 3 + 4 = 7$.

If $x = -1$, we get $\sum_{n=0}^{\infty} \frac{(-4)^n}{4^n} = \sum_{n=0}^{\infty} (-1)^n$ $\boxed{\text{Diverges}}$

since $\lim_{n \to \infty} (-1)^n \neq 0$.

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If $x=7$, we get $\sum_{n=0}^{\infty} \frac{4^n}{4^n} = \sum_{n=0}^{\infty} 1$ $\boxed{\text{Diverges}}$

Since $\lim_{n\to\infty} 1 = 1 \neq 0$.

So $\boxed{I.C. = (-1,7)}$

b) Ratio Test: $\lim_{n\to\infty} \left| \frac{x^{2n+2}/n+2}{x^{2n}/n+1} \right| = \lim_{n\to\infty} \left| \frac{x^2(n+1)}{n+2} \right|$

$= |x^2| \lim_{n\to\infty} \left| \frac{n+1}{n+2} \right| = |x^2| < 1$

Thus we need $|x|<1$, $\therefore \boxed{R=1}$.

AT endpoints $x=\pm 1$:

If $x=-1$, we get $\sum_{n=0}^{\infty} \frac{(-1)^{2n}}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$ $\boxed{\text{Diverges}}$ b/c it's the Harmonic series

If $x=1$, we get $\sum_{n=0}^{\infty} \frac{1}{n+1}$ $\boxed{\text{Diverges}}$ (as above)

$\therefore \boxed{I.C. = (-1,1)}$

5. The series $\sum_{n=0}^{\infty} c_n (x-1)^n$ is centered at $x=1$.

Since it converges at $x=2$, $R.C. \geq |2-1| = 1$.

Since it Diverges at $x=-3$, $R.C. \leq |-3-1| = 4$.

a) at $x=1$, it $\boxed{\text{converges}}$ to $c_0$

b) at $x=0$, $\boxed{\text{we can't tell}}$ since $|0-1|=1$ may equal the R.C. (eg. I.C. could be $(0,2]$ or $[0,2]$)

c) at $x=1/2$, $|x-1| = |1/2 - 1| = 1/2 < 1 \leq R.C.$

So $x=1/2$ is in the interval of convergence, & it $\boxed{\text{converges}}$.

d) $x=3$, $\boxed{\text{we can't tell}}$ since I.C. could be $[0,2]$ or $[-1,3]$.

e) $x=5$, $\boxed{\text{we can't tell}}$ since I.C. could be $(-3,5]$ or $(-3,5)$

f) $x=6$, ~~we can't~~ $|x-1| = |6-1| = 5 > 4 \geq R.C.$

$\therefore x=6$ is not in the I.C. & it $\boxed{\text{Diverges}}$.

6. Notice $f(x) = x^2\left(\frac{1}{(1+x)^2}\right)$ and

$$\frac{1}{(1+x)^2} = -\frac{d}{dx}\left(\frac{1}{1+x}\right)$$

Since $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n$ (w/ $R=1$)

$$\frac{1}{(1+x)^2} = -\frac{d}{dx}\left(\frac{1}{1+x}\right) = -\frac{d}{dx}\left(\sum_{n=0}^{\infty}(-x)^n\right)$$

$$= -\sum_{n=0}^{\infty}\frac{d}{dx}\left[(-1)^n x^n\right]$$

$$= \sum_{n=0}^{\infty}(-1)^{n+1} n x^{n-1}$$

$$= \sum_{n=0}^{\infty}(-1)^n (n+1) x^n \quad (R=1)$$

Thus $f(x) = x^2\left(\frac{1}{(1+x)^2}\right)$

$$= x^2 \sum_{n=0}^{\infty}(-1)^n (n+1) x^n$$

$$= \sum_{n=0}^{\infty}(-1)^n (n+1) x^{n+2} = \boxed{\sum_{n=2}^{\infty}(-1)^n (n-1) x^n}$$

$R=1$, so to find I.C. we need to check convergence at the endpoints $x = \pm 1$.

at $x=-1$, we get $\sum_{n=2}^{\infty}(-1)^{2n}(n-1) = \sum_{n=2}^{\infty} n-1$ $\boxed{\text{Diverges}}$

Since $\lim_{n\to\infty}(n-1) \neq 0$.

at $x=1$, we get $\sum_{n=2}^{\infty}(-1)^n(n-1)$ = $\boxed{\text{Diverges}}$

Since $\lim_{n\to\infty}(-1)^n(n-1) \neq 0$.

$\therefore$ $\boxed{\text{I.C.} = (-1,1)}$

7. Let $f(x) = \sum_{n=1}^{\infty} n x^n$. Then $\sum_{n=1}^{\infty}\frac{n}{2^n} = \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = f\left(\frac{1}{2}\right)$

so we must find $f(x)$.

Notice $\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} n x^{n-1}$

$\therefore \frac{x}{(1-x)^2} = \left(\sum_{n=1}^{\infty} n x^{n-1}\right)x = \sum_{n=1}^{\infty} n x^n = f(x)$. $\therefore f\left(\frac{1}{2}\right) = \frac{1/2}{(1-1/2)^2} = \frac{1}{2}\Big/\frac{1}{4} = \boxed{2}$