

Math 5C, Solutions to Midterm 1 Review Problems

Winter 2007

1. Compute $\int_C yz \, dx + 2x \, dy - y \, dz$ where C is the straight line path from $(1, 2, 1)$ to $(-1, 3, 0)$.

Solution. The straight line path between the two points has parametric equations $(x, y, z) = (-2t + 1, t + 2, -t + 1)$. Thus, using the definition of line integrals, the integral becomes

$$\begin{aligned} \int_0^1 (t+2)(1-t)(-2) + 2(1-2t) - (t+2)(-1) \, dt &= \int_0^1 2t^2 - t \, dt \\ &= 2t^3/3 - t^2/2 \Big|_0^1 = 1/6. \end{aligned}$$

2. Find the surface area of the surface S , which is parametrized by

$$\phi(u, v) = \begin{cases} x(u, v) = u - v \\ y(u, v) = u + v \\ z(u, v) = uv \end{cases}$$

for all (u, v) with $u^2 + v^2 \leq 1$.

Solution. We use the formula $S.A. = \iint \sqrt{EG - F^2} \, du \, dv$. Here, $E = x_u^2 + y_u^2 + z_u^2 = 2 + v^2$, $F = x_u x_v + y_u y_v + z_u z_v = 1 - 1 + uv = uv$, and $G = x_v^2 + y_v^2 + z_v^2 = 2 + u^2$. Thus $EG - F^2 = 4 + 2(u^2 + v^2)$, and

$$\begin{aligned} S.A. &= \iint_{u^2+v^2 \leq 1} \sqrt{4 + 2(u^2 + v^2)} \, du \, dv \\ &= \int_0^{2\pi} \int_0^1 \sqrt{4 + 2r^2} \, r \, dr \, d\theta \\ &= \frac{1}{4} \int_0^{2\pi} \frac{2}{3} (4 + 2r^2)^{3/2} \Big|_0^1 \, d\theta \\ &= \frac{\pi}{3} (6^{3/2} - 4^{3/2}) = \frac{\pi}{3} (6\sqrt{6} - 8) \end{aligned}$$

3. Let S be the top half of the unit sphere (i.e., S is given by $x^2 + y^2 + z^2 = 1$ and $z \geq 0$), oriented by the outer normal. Integrate

$$\iint_S x \, dy \, dz + y \, dz \, dx + z^2 \, dx \, dy.$$

Solution. S has parametrization $x = \cos \theta \sin \phi$, $y = \sin \theta \sin \phi$ and $z = \cos \phi$ for $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi/2$. Notice that this parametrization induces the inner normal on the sphere, and so we need to multiply the integral by -1 when we write it in terms of the parameters θ and ϕ .

We now compute the Jacobians we need (these are also the components of the normal vector).

$$dy dz = \begin{vmatrix} \cos \theta \sin \phi & \sin \theta \cos \phi \\ 0 & -\sin \phi \end{vmatrix} d\theta d\phi = -\cos \theta \sin^2 \phi d\theta d\phi,$$

$$dz dx = \begin{vmatrix} 0 & -\sin \phi \\ -\sin \theta \sin \phi & \cos \theta \cos \phi \end{vmatrix} d\theta d\phi = -\sin \theta \sin^2 \phi d\theta d\phi,$$

$$dx dy = \begin{vmatrix} -\sin \theta \sin \phi & \cos \theta \cos \phi \\ \cos \theta \sin \phi & \sin \theta \cos \phi \end{vmatrix} d\theta d\phi = -\sin \phi \cos \phi d\theta d\phi,$$

Thus

$$\begin{aligned} \int \int_S x dy dz + \dots &= -\int_0^{\pi/2} \int_0^{2\pi} -\cos^2 \theta \sin^3 \phi - \sin^2 \theta \sin^3 \phi - \sin \phi \cos^3 \phi d\theta d\phi \\ &= 2\pi \int_0^{\pi/2} (\sin^2 \phi + \cos^3 \phi) \sin \phi d\phi \\ &= 2\pi \int_0^{\pi/2} (1 - \cos^2 \phi + \cos^3 \phi) \sin \phi d\phi \\ &= -2\pi(\cos \phi - \frac{1}{3} \cos^3 \phi + \frac{1}{4} \cos^4 \phi) \Big|_0^{\pi/2} = \frac{11\pi}{6}. \end{aligned}$$

4. Let S be the surface given by $z = xy^2 - 3x^2$ with upper normal \mathbf{n} , over the square with vertices $(\pm 1, \pm 1)$ in the xy -plane. If $\mathbf{w} = (z + 3x^2)\mathbf{i} + yz\mathbf{j} + y^2\mathbf{k}$, calculate

$$\int \int_S \mathbf{w} \cdot \mathbf{n} d\sigma.$$

Solution.

$$\begin{aligned} \int \int_S \mathbf{w} \cdot \mathbf{n} d\sigma &= \int_{-1}^1 \int_{-1}^1 (-w_x \frac{\partial z}{\partial x} - w_y \frac{\partial z}{\partial y} + w_z) dx dy \\ &= \int_{-1}^1 \int_{-1}^1 -(xy^2 - 3x^2 + 3x^2)(y^2 - 6x) - y(xy^2 - 3x^2)(2xy) + y^2 dx dy \\ &= \int_{-1}^1 \int_{-1}^1 6x^2 y^2 - xy^4 - 2x^2 y^4 + 6x^3 y^2 + y^2 dx dy \\ &= \int_{-1}^1 6y^2 - 4y^4/3 dy = \frac{52}{15}. \end{aligned}$$

5. Let S be the unit sphere, $x^2 + y^2 + z^2 = 1$, oriented outward, and let \mathbf{F} be the vector field $\mathbf{F}(x, y, z) = xy^2 \mathbf{i} - xz^2 \mathbf{j} + x^2z \mathbf{k}$. Use the Divergence Theorem to compute $\int \int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$.

Solution.

$$\begin{aligned}
 \int \int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma &= \int \int \int_R \operatorname{div}(\mathbf{F}) \, dx dy dz \\
 &= \int \int \int_R y^2 + x^2 \, dx dy dz \\
 &= \int_0^\pi \int_0^{2\pi} \int_0^1 (\rho^2 \sin^2 \phi) \rho^2 \sin \phi \, d\rho d\theta d\phi \\
 &= \int_0^\pi \int_0^{2\pi} \frac{1}{5} \sin^3 \phi \, d\theta d\phi \\
 &= \int_0^\pi \frac{2\pi}{5} (1 - \cos^2 \phi) \sin \phi \, d\phi \\
 &= -\frac{2\pi}{5} \left(\phi - \frac{1}{3} \cos^3 \phi \right) \Big|_0^\pi \\
 &= -\frac{2\pi}{5} (\pi + 1/3 - 0 - -1/3) = -\frac{2\pi^2}{5} - \frac{4\pi}{15}
 \end{aligned}$$

6. Let S be the cone $x^2 = y^2 + z^2$, $0 \leq x \leq 2$, oriented inward (so the normal vectors point toward the x -axis). Use Stokes' Theorem to calculate $\int \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma$, where $\mathbf{F}(x, y, z) = x^2 \mathbf{i} - z \mathbf{j} + (y^2 - z) \mathbf{k}$.

Solution. Recall $\nabla \times \mathbf{F} = \operatorname{curl}(\mathbf{F})$, so by Stokes' Theorem, the surface integral reduces to the line integral $\oint_C F_T \, ds$ where C is the boundary of S . Now S is a cone with vertex at the origin, and so its boundary is a circle lying in the plane $x = 2$. By the right-hand rule, since the normal should be roughly in the direction of the x -axis, the circle C should be traversed in the counterclockwise direction of the yz -plane. Thus C has parametric equations $(x, y, z) = (2, \cos t, \sin t)$ for $0 \leq t \leq 2\pi$, and

$$\begin{aligned}
 \int_C F_T \, ds &= \int_C x^2 \, dx - z \, dy + (y^2 - z) \, dz \\
 &= \int_0^{2\pi} 4(0) - \sin t(-\sin t) + (\cos^2 t - \sin t) \cos t \, dt \\
 &= \int_0^{2\pi} \sin^2 t + (\cos^2 t - \sin t) \cos t \, dt \\
 &= \int_0^{2\pi} \frac{1}{2}(1 - \cos(2t)) + (1 - \sin t - \sin^2 t) \cos t \, dt \\
 &= \frac{1}{2}(t - \frac{1}{2} \sin(2t)) + \sin t - \frac{1}{2} \sin^2 t - \frac{1}{3} \sin^3 t \Big|_0^{2\pi} = \pi.
 \end{aligned}$$

7. Let C be the curve given by $x = \sin t$, $y = \cos t$, $z = \cos(2t)$ for $0 \leq t \leq 2\pi$. Use Stokes' Theorem to evaluate

$$\oint_C xz \, dx + y^2 \, dy + z^2 \, dz.$$

Solution. Since $z = \cos(2t) = \cos^2 t - \sin^2 t = y^2 - x^2$, the curve C lies on the surface $z = y^2 - x^2$ (the surfaces $z = 2y^2 - 1$ or $z = 1 - 2x^2$ would also work). Since $x = \sin t$, $y = \cos t$ traces out the unit circle, the interior of C lies above the unit disk in the xy -plane. Thus C is the boundary of the surface S given by the graph of $z = y^2 - x^2$ for $x^2 + y^2 \leq 1$. Furthermore, since C is traversed in the clockwise direction, S must be given the lower normal according to the right-hand rule (thus we multiply by -1). Now, using Stokes' Theorem, we have

$$\begin{aligned} \oint_C xz \, dx + y^2 \, dy + z^2 \, dz &= \int \int_S \operatorname{curl}(xz\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}) \cdot \mathbf{n} \, d\sigma \\ &= - \int \int_S x \, dz \, dx \\ &= - \int \int_{x^2+y^2 \leq 1} -x(2y) \, dx \, dy \\ &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2xy \, dy \, dx \\ &= \int_{-1}^1 2x(1-x^2) \, dx = x^2 - x^4/2 \Big|_{-1}^1 = 0. \end{aligned}$$

8. Show that the integral

$$\int_{(-1,1,3)}^{(\pi/2,0,1)} z^2 \cos(x+y^2) \, dx + 2yz^2 \cos(x+y^2) \, dy + 2z \sin(x+y^2) \, dz$$

is independent of path and evaluate it.

Solution. To show that the integral is path independent, it suffices to find a function $F(x, y, z)$ such that the integrand equals dF . To get F , integrate the dx term with respect to x to get $F = \int z^2 \cos(x+y^2) \, dx = z^2 \sin(x+y^2) + C(y, z)$. If we now differentiate this function with respect to y and z (separately), we get the other two terms of the integrand when we let $C(y, z) = 0$. Thus the integral becomes

$$\begin{aligned} \int_{(-1,1,3)}^{(\pi/2,0,1)} d(z^2 \sin(x+y^2)) &= z^2 \sin(x+y^2) \Big|_{(-1,1,3)}^{(\pi/2,0,1)} \\ &= \sin(\pi/2) - 9 \sin 0 = 1. \end{aligned}$$

9. Let \mathbf{u} be the vector field

$$\mathbf{u}(x, y, z) = \frac{y}{x^2 + y^2} \mathbf{i} - \frac{x}{x^2 + y^2} \mathbf{j} + z^2 \mathbf{k}$$

on \mathbb{R}^3 minus the z -axis.

(a) Show that $\text{curl}(\mathbf{u}) = \mathbf{0}$ on this domain.

Solution.

$$\text{curl}(\mathbf{u}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{x^2+y^2} & \frac{-x}{x^2+y^2} & z^2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + \left(\frac{x^2 - y^2}{(x^2 + y^2)^2} - \frac{x^2 - y^2}{(x^2 + y^2)^2} \right) \mathbf{k} = \mathbf{0}.$$

(b) Show that \mathbf{u} is not the gradient vector field of any function F on this domain. (Hint: Find a closed curve C with $\int_C u_T ds \neq 0$.)

Solution. Let C be the unit circle in the xy -plane: $x = \cos t$, $y = \sin t$, $z = 0$ for $0 \leq t \leq 2\pi$. Then

$$\begin{aligned} \oint_C u_T ds &= \int_0^{2\pi} \left(\frac{\sin t}{\cos^2 t + \sin^2 t} (-\sin t) - \frac{\cos t}{\cos^2 t + \sin^2 t} (\cos t) + 0 \right) dt \\ &= \int_0^{2\pi} -(\sin^2 t + \cos^2 t) dt \\ &= -2\pi \neq 0. \end{aligned}$$

Since this integral is not zero, we know that the integral $\int u_T ds$ is not path-independent in the given domain, and hence \mathbf{u} is not a gradient vector field. (You could also prove this more directly by trying to solve for $F(x, y, z)$ with $\nabla F = \mathbf{u}$, and showing that no solutions exist.)