(22) PATH-INDEPENDENCE

Recall ~~Let~~ $\vec{u} = X\vec{i} + Y\vec{j} + Z\vec{k}$ be a ~~continuous~~ Differentiable vector field on a Domain D of $\mathbb{R}^3$,

If $\vec{u} = \nabla F$ for $F(x,y,z)$ then

$$\int_A^B u_T\,ds = \int_A^B X\,dx + Y\,dy + Z\,dz = F(B) - F(A)$$

does not depend on the curve C (inside D) from A to B.

Thm I

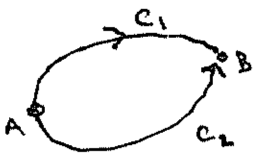
Total Differential: $dF = F_x\,dx + F_y\,dy + F_z\,dz = X\,dx + Y\,dy + Z\,dz$ is an <u>exact differential</u>

Conversely, If $\int_A^B u_T\,ds$ is path-independent on D then $\vec{u} = \nabla F$ for some $F(x,y,z)$.

<u>Thm II</u> $\int u_T\,ds$ is path-independent in D

$\iff$ $\oint_C u_T\,ds = 0$ for every closed curve C in D.

<u>Pf</u>



$\cancel{\int_A^B u_T\,ds = \int_{C_1} u_T\,ds = \int_{C_2} u_T\,ds}$

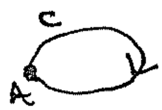
Assume $\oint_C u_T\,ds = 0$ for all C.

Let C be the closed curve obtained by following C_1 & then C_2 in reverse direction. If $\oint_C u_T\,ds = 0$

$$\int_{C_1} u_T\,ds - \int_{C_2} u_T\,ds =$$

$$\Rightarrow \int_{C_1} u_T\,ds = \int_{C_2} u_T\,ds$$

Conversely, if $\int u_T\,ds$ is path-independent, and C is closed, $\oint_C u_T\,ds = F(A) - F(A) = 0$ where $\vec{u} = \nabla F$.



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Summary:

$\int_C u_T\,ds$ path indep. in D $\Leftrightarrow$ $\oint_C u_T\,ds = 0$ for all closed curves C in D

$\Leftrightarrow$ $\vec{u} = \nabla F$

any of these $\Rightarrow$ $\text{curl}(\vec{u}) = 0$ ($\vec{u}$ is irrotational)
(since $\text{curl}(\nabla F) = 0$)

Conversely, if $\text{curl}(\vec{u}) = 0$ on D and D is simply connected, then $\vec{u} = \nabla F$ on D.

Def A Domain D in $\mathbb{R}^3$ is simply connected if Any closed curve C in D is the boundary of some surface S in D.

proof By Stokes' Thm, for any closed curve C in D

$$\oint_C u_T\,ds = \iint_S \text{curl}(\vec{u})\,ds = 0.$$

$\therefore$ $\int u_T\,ds$ is path indep. $\ni$ $\vec{u} = \nabla F$.

eg $$\int_{(1,0,1)}^{(-1,0,-1)} -\sin x\cos y\,dx - \cos x\sin y\,dy + 2z\,dz \qquad C = \begin{cases} x = \cos t \\ y = \sin t \\ z = \cos t + \sin t \\ 0 \le t \le \pi. \end{cases}$$

$F = \int -\sin x\cos y\,dx = \cos x\cos y + C(y,z)$

we need

$F_y = -\cos x\sin y + C_y = -\cos x\sin y \Rightarrow C_y = 0$

and $F_z = C_z = 2z. \qquad \Rightarrow C_z = 2z$

so $C(y,z) = z^2$.

$F(x,y,z) = \cos x\cos y + z^2$.

$$\int_{(1,0,1)}^{(-1,0,-1)} dF = F(-1,0,-1) - F(1,0,1) = \cos(-1) + 1 - (\cos(1) + 1)$$
$$= \cos(-1) - \cos(1)$$
$$= 0 \text{ (since } \cos x \text{ is an even function)}$$

"Path"- Independence for Surface Integrals.

Thm IV Let $\vec{u} = L\vec{i} + M\vec{j} + N\vec{k}$ be a diff. vect. field in a spherical Domain D. If $\operatorname{div}(\vec{u}) = 0$ in D, then there exists a vector field $\vec{v}$ on D such that $\vec{u} = \operatorname{curl}(\vec{v})$. on D.

ie, If $\vec{v} = X\vec{i} + Y\vec{j} + Z\vec{k}$,

$$\left.\begin{aligned} L &= Z_y - Y_z \\ M &= X_z - Z_x \\ N &= Y_x - X_y \end{aligned}\right\} \text{ this system of partial Diff Eq's has a solution.}$$

Formula for $\vec{v}$: $\boxed{\vec{v} = \int_0^1 t\, \vec{u}(xt, yt, zt) \times (x\vec{i} + y\vec{j} + z\vec{k})\, dt}$

Def If $\operatorname{div}(\vec{u}) = 0$, then $\vec{u}$ is solenoidal.

eg Gauss's Law

Let $\vec{F}(x,y,z) = \dfrac{x\vec{i} + y\vec{j} + z\vec{k}}{(x^2+y^2+z^2)^{3/2}}$ = Diff. vector Field on $D = \mathbb{R}^3 - \{(0,0,0)\}$.

Let S be the boundary of a bounded region R in $\mathbb{R}^3$.

Then $$\iint_S \vec{F} \cdot d\vec{\sigma} = \begin{cases} 0, & \text{if } (0,0,0) \text{ is not inside } R. \\ 4\pi, & \text{if } (0,0,0) \text{ is inside } R. \end{cases}$$

(outer normal)

Pf If (0,0,0) is not in R, we can use the Divergence Thm,

$$\iint_S \vec{F} \cdot d\vec{\sigma} = \iiint_R \operatorname{div}(\vec{F})\, dx\,dy\,dz = 0$$

HW a) check $\operatorname{div}(\vec{F}) = 0$ on $\mathbb{R}^3 - \{(0,0,0)\}$.

If (0,0,0) is in R, we can't use div. thm directly But we can use it to replace S with a sphere - Let S_r be a sphere of radius $r > 0$ at origin. where r is small enough so that S_r is contained in R.

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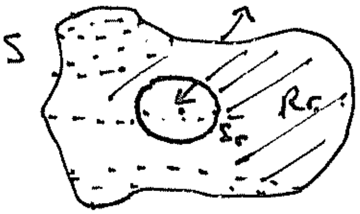
Let R_r = region outside of S_r and inside S.



Since $(0,0,0)$ is not in R_r, div. Thm

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{\sigma} - \iint_{S_r} \vec{F}\cdot d\vec{\sigma} = \iiint_{R_r} \operatorname{div}(\vec{F})\, dx\,dy\,dz$$

$$= 0.$$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{\sigma} = \iint_{S_r} \vec{F}\cdot d\vec{\sigma} \quad \text{(both outer normal)}$$

S_r has parametrization
$$\left.\begin{aligned} x &= r\cos\theta\sin\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\phi \end{aligned}\right\} \begin{array}{l} 0\le\theta\le 2\pi \\ 0\le\phi\le\pi \\ \text{inner} \\ \text{normal.} \end{array}$$

$$\iint_{S_r} \vec{F}\cdot d\vec{\sigma} = -\int_0^{\pi}\int_0^{2\pi} \frac{x}{r^3}\,dy\,dz + \frac{y}{r^3}\,dz\,dx + \frac{z}{r^3}\,dx\,dy$$

<u>Hw b)</u> $\quad = 4\pi.$

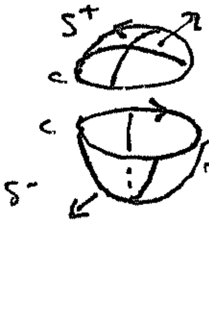
Note $\operatorname{div}(\vec{F}) = 0$ on $\mathbb{R}^3 - \{(0,0,0)\}$, but $\vec{F} \ne \operatorname{curl}(\vec{v})$ for any $\vec{v}$.

<u>Pf</u> Assume $\vec{F} = \operatorname{curl}(\vec{v})$, S = unit sphere

S^+ = top hemisphere $\quad\Big\}$ outer normal.
S^- = bottom hemisphere

$C = \partial S^+ = \partial S^-$ = unit circle $x^2+y^2=1$ in xy-plane.

Stokes' Thm $\Rightarrow \iint_{S^+} \vec{F}\cdot d\vec{\sigma} = \oint_C v_T\, ds$



$$\& \iint_{S^-} \vec{F}\cdot d\vec{\sigma} = \oint_{-C} v_T\, ds = -\oint_C v_T\, ds$$

Thus
$$\underset{\substack{\| \\ 4\pi \text{ by Gauss's Law.}}}{\iint_S \vec{F}\cdot d\vec{\sigma}} = \iint_{S^+} \vec{F}\cdot d\vec{\sigma} + \iint_{S^-} \vec{F}\cdot d\vec{\sigma} = \oint_C v_T\, ds - \oint_C v_T\, ds = 0.$$

Contradiction, so $\vec{F} \ne \operatorname{curl}(\vec{v})$.