p.417 1a,b,g,i
p.429 1, 3a,c,d, 4

(41) ## Power Series

6.15–6.16

$$1+x+x^2+x^3+\cdots = \sum_{n=0}^{\infty} x^n \quad \text{geometric series w/ } r=x$$

$$= \begin{cases} \frac{1}{1-x}, & \text{if } |x|<1 \\ \text{Diverges}, & \text{if } |x|\geq 1. \end{cases}$$

So $f(x) = \frac{1}{1-x}$ can be approximated by partial sums of the series $\sum_{n=0}^{\infty} x^n$ for any value of x w/ $|x|<1$.

If we substitute different functions for x, we get infinite series for different functions:

eg $$\frac{1}{1-x^2} = 1+x^2+x^4+x^6+\cdots = \sum_{n=0}^{\infty} x^{2n}$$

converges for $|x^2|<1$, thus for $-1<x<1$.

eg $$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-1)^n x^n = 1-x+x^2-x^3+x^4-\cdots$$

converges for $|-x|<1$, thus for $-1<x<1$.

eg $$\frac{1}{2-x} = \frac{1}{2}\left(\frac{1}{1-\frac{x}{2}}\right) = \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \frac{1}{2}\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\cdots\right)$$

converges for $\left|\frac{x}{2}\right|<1$, thus for $-2<x<2$.

Def A power series centered at $x=a$ is an infinite series $\sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$

Goal: Given a function $f(x)$ & a value of x, $x=a$, find a power series that converges to $f(x)$ when x is close to a.
(The partial sums of such a power series will be good approximations of $f(x)$ when x is close to a.)

(42) <u>Convergence of Power Series</u>

<u>Thm</u> Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series centered at $x=a$.

Then either

(R=∞) 1) $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for all x in $\mathbb{R}$. or

(R=0) 2) $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges only for $x=a$, (& Diverges for all $x \neq a$) or

3) There is an $R>0$ s.t. $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges if $|x-a|<R$ & Diverges if $|x-a|>R$.

R = <u>radius of convergence</u>.

<u>Note</u> If $|x-a|=R$, (in case 3) convergence must be checked separately using other convergence/divergence Tests.

<u>Def</u> The <u>Interval of Convergence</u> of $\sum_{n=0}^{\infty} c_n(x-a)^n$ is the interval of all values of x that make the series converge.

Possibilities for I.C.

$I=[a,a]$ $(R=0)$, $I=(-\infty,\infty)$ $(R=\infty)$

$I=[a-R,a+R]$, $[a-R,a+R)$, $(a-R,a+R]$, $(a-R,a+R)$

Whether or not the series converges at the endpoints $a-R$ & $a+R$ is usually checked separately.

<u>eg</u> $\sum_{n=0}^{\infty} x^n$ converges for $|x|<1$: $a=0$, $R=1$, I.C. $=(-1,1)$

<u>eg</u> $\frac{1}{2-x}=\frac{1}{1-(x-1)}=\sum_{n=0}^{\infty}(x-1)^n = 1+(x-1)+(x-1)^2+(x-1)^3+\cdots$

for $|x-1|<1$. Here $a=1$, $R=1$, I.C. $=(0,2)$

<u>eg</u> $\sum_{n=1}^{\infty}\frac{x^n}{n}$: Ratio test to find I.C.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}/n+1}{x^n/n}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}x\right| = |x| \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = |x|.$$

needs to be <1 to ensure convergence so $|x|<1$.

$\boxed{R=1}$ check convergence at endpoints $x=\pm 1$:

at $x=1$, we get $\sum_{n=1}^{\infty}\frac{1^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$ Diverges.

at $x=-1$, we get $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges. by A.S.T.

$\boxed{I.C. = [-1,1)}$

eg $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ Ratio Test to find I.C.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = |x|\lim_{n\to\infty}\left|\frac{1}{n+1}\right| = 0$$

since $0<1$ for all x, $\boxed{I.C. = (-\infty,\infty),\ R=\infty}$

eg $\sum_{n=\cancel{0}}^{\infty}\frac{(x+2)^n}{n^2}$ Ratio Test to find I.C.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}/(n+1)^2}{(x+2)^n/n^2}\right| = \lim_{n\to\infty}|x+2|\cdot\left|\left(\frac{n+1}{n}\right)^2\right|$$

$$= |x+2|\lim_{n\to\infty}\left|\frac{n+1}{n}\right|^2 = |x+2| < 1. \text{ to converge.}$$

$\Rightarrow \boxed{R=1}$ Endpoints: $x+2=-1 \Rightarrow \underline{x=-3}$; $x+2=1 \Rightarrow \underline{x=-1}$

$x=-3$: $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ converges absolutely.

$x=-1$: $\sum_{n=1}^{\infty}\frac{1^n}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by p-series.

$\therefore \boxed{I.C. = [-1,3]}$

To find R in general, $\sum_{n=0}^{\infty} c_n(x-a)^n$,

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = |x-a|\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| < 1$$

$$\Rightarrow |x-a| < 1\Big/\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \boxed{\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = R}$$

USING ROOT TEST INSTEAD, $\boxed{R = \dfrac{1}{\lim_{n\to\infty} \sqrt[n]{|c_n|}}}$

TAYLOR & MACLAURIN SERIES

"**Thm**" Let $f(x)$ be a function with derivatives of all orders at $x=a$. Then the power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \qquad c_n = \frac{f^{(n)}(a)}{n!}$$

converges to $f(x)$ (when it converges, ie. when x is in I.C.)

This is the Taylor Series for $f(x)$ at $x=a$.

If $a=0$, $\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ is the Maclaurin Series for $f(x)$.

eg $f(x) = e^x$ at $x=0$.

$$\left.\begin{aligned}
f^0(x) &= e^x \Rightarrow f^0(0) = 1, & c_0 &= 1/0! = 1 \\
f'(x) &= e^x \Rightarrow f'(0) = 1, & c_1 &= 1/1! = 1 \\
f^2(x) &= e^x \Rightarrow f^2(0) = 1, & c_2 &= 1/2! \\
&\vdots \\
f^n(x) &= e^x \Rightarrow f^n(0) = 1, & c_n &= 1/n!
\end{aligned}\right\} \quad \boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}} \quad \boxed{\begin{aligned} &R = \infty \\ &I.C. = (-\infty, \infty) \end{aligned}}$$

(by prev. example)

eg $f(x) = \ln(1+x)$ at $x=0$.

$$\left.\begin{aligned}
f^0(x) &= \ln(1+x) \Rightarrow f^0(0) = 0. & c_0 &= 0 \\
f'(x) &= \frac{1}{1+x} \Rightarrow f'(0) = 1 & c_1 &= 1 \\
f^2(x) &= \frac{-1}{(1+x)^2} \Rightarrow f^2(0) = -1 & c_2 &= -1/2! \\
f^3(x) &= \frac{2}{(1+x)^3} \Rightarrow f^3(0) = 2 & c_3 &= 2/3! \\
&\vdots \\
f^n(x) &= \frac{(-1)2(-3)\cdots(-1)^{n-1}(n-1)}{(1+x)^n} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n} \Rightarrow & c_n &= \frac{(-1)^{n-1}}{n}
\end{aligned}\right\} \quad \boxed{\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}} \quad \boxed{R=1} \quad \boxed{I.C. = (-1, 1]}$$

(earlier example is similar)

- Any Power Series that converges to $f(x)$ is the Taylor series for $f(x)$.

eg $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, is a Taylor Series for $f(x) = \frac{1}{1-x}$