$$\sum_{n=0}^{\infty} ar^n = \lim_{N\to\infty} a\left(\frac{1-r^{N+1}}{1-r}\right) = \begin{cases} \pm\infty, & \text{if } |r|>1 \\ \text{DNE}, & \text{if } |r|=1 \end{cases}\Big\}\text{ Diverges} \quad \frac{a}{1-r},\ \text{if } |r|<1.$$

**Thm** Let $a_n = ar^n$ for $n \geq 0$ be a geometric series.
Then $\sum_{n=0}^{\infty} a_n$ converges $\iff |r|<1$
& Diverges $\iff |r| \geq 1$
If $|r|<1$, $\sum_{n=0}^{\infty} a_n = \frac{a}{1-r}$.

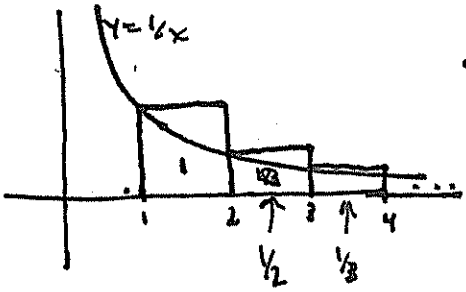
eg $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots = \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n = \frac{1/3}{1-1/3} = \frac{1}{2}.$

eg Decimals: $.315 = \frac{3}{10}+\frac{1}{100}+\frac{5}{1000} = 3\left(\frac{1}{10}\right)+1\left(\frac{1}{10^2}\right)+5\left(\frac{1}{10^3}\right)$
repeating decimals correspond to infinite sums:
$.1\overline{11} = \frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\cdots = \sum_{n=1}^{\infty}\left(\frac{1}{10}\right)^n$
= geometric w/ $a = 1/10$ & $r = 1/10$ $= \frac{a}{1-r} = \frac{1/10}{1-1/10} = \frac{1/10}{9/10} = \boxed{\frac{1}{9}}$

claim $.\overline{9} = 1$
pf $.999\ldots = \frac{9}{10}+\frac{9}{10^2}+\frac{9}{10^3}+\cdots = \sum_{n=1}^{\infty} 9\left(\frac{1}{10}\right)^n$
$= \frac{9/10}{1-1/10} = \frac{9/10}{9/10} = 1.$

Def The Harmonic Series is the series $\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$

the terms in the summation are decreasing, but the series still Diverges:
claim $S_N = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N} > \int_1^{N+1}\frac{1}{x}\,dx = \ln(N+1)$




Since $\lim_{N\to\infty}\ln(N+1) = \infty$,
$\lim_{N\to\infty} S_N = \infty$.
& thus the series Diverges.
(The partial sums increase without bound.)

(34) Convergence Tests for Series.

$\boxed{\text{(nth Term Test) } \underline{\text{Divergence Test}} \text{ If } \lim_{n\to\infty} a_n \neq 0 \text{, then } \sum_{n=0}^{\infty} a_n \text{ Diverges.}}$

(Alternatively, If $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$)

pf If $S = \lim_{n\to\infty} S_n$, the differences $S_n - S_{n-1} = a_n$ must be approaching 0. ▫

But, If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that the series $\sum_{n=0}^{\infty} a_n$ converges.

eg $a_n = \frac{1}{n}$ $\lim_{n\to\infty} a_n = 0$

But $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges.

eg $\sum_{n=1}^{\infty} \frac{3n-1}{2n}$ diverges since $\lim_{n\to\infty} \frac{3n-1}{2n} = \frac{3}{2} \neq 0$

**Integral test** Suppose $a_n = f(n)$ where $f(x)$ is

1) nonnegative
2) continuous
3) decreasing

on $[c, \infty)$

then $\sum_{n=1}^{\infty} a_n$ converges $\iff \int_c^{\infty} f(x)\,dx$ converges.

eg p-series — Let $p > 0$, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is a p-series.

$a_n = \frac{1}{n^p} = f(n)$ where $f(x) = \frac{1}{x^p}$

$f(x)$ = cont., decreasing, $> 0$, for $x > 0$

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\iff \int_1^{\infty} \frac{1}{x^p}\,dx$ converges

$\iff p \neq 1$ & $\lim_{b\to\infty} \left. \frac{x^{1-p}}{1-p} \right]_1^b$ converges.

$\iff p \neq 1$ & $\lim_{b\to\infty} \left( \frac{b^{1-p} - 1}{1-p} \right)$ converges

$\iff p \neq 1$ & $\lim_{b\to\infty} b^{1-p}$ converges

$\iff p \neq 1$ & $p \geq 1$

$\iff p > 1$.

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$\boxed{\text{p-series Test)} \quad \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges} \iff p > 1. \quad \text{diverges} \iff p \le 1.}$

**Comparison Test**

convergence: if $0 \le a_n \le b_n$ for all $n \ge 1$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Divergence: If $0 \le a_n \le b_n$ for all $n \ge 1$ and $\sum_{n=1}^{\infty} a_n$ Diverges, then $\sum_{n=1}^{\infty} b_n$ Diverges.

eg $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ $\quad a_n = \frac{1}{n^2+1} \le b_n = \frac{1}{n^2}$. for all $n \ge 1$.

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test $(p = 2 > 1)$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by comp. test.

eg $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ $\quad a_n = \frac{1}{n2^n} \le \frac{1}{2^n} = b_n$ for all $n \ge 1$

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges by Geometric series test $(|r| = \frac{1}{2} < 1)$

so $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ converges by comp. Test.

eg $\sum_{n=2}^{\infty} \frac{1}{\ln n}$, $\quad a_n = \frac{1}{\ln n} > \frac{1}{n} = b_n$ for all $n \ge 2$

$\sum_{n=2}^{\infty} \frac{1}{n}$ Diverges, since it is Harmonic.

$\therefore \sum_{n=2}^{\infty} \frac{1}{\ln n}$ Diverges by comp. Test

eg $\sum_{n=1}^{\infty} \frac{n-1}{n^2}$ $\quad a_n = \frac{n-1}{n^2} < \frac{n}{n^2} = \frac{1}{n}$ but

$\sum_{n=1}^{\infty} \frac{1}{n}$ Diverges. ~~By~~ So comp. Test can't be used w/ $b_n = \frac{1}{n}$. Instead, look for a sequence smaller than $a_n$, ~~that~~ whose sum still diverges.

$$\frac{n-1}{n^2} \geq \frac{\frac{1}{2}n}{n^2} = \frac{1}{2n} \quad \text{for } n \geq 2.$$

$$\sum_{n=2}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{n} \quad \text{Diverges (Harmonic)}$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{n-1}{n^2} \quad \text{Diverges by Comp. Test.}$$

Thus $\sum_{n=1}^{\infty} \frac{n-1}{n^2}$ Diverges as well.

<u>Note</u> Convergence/Divergence of a series only depends on terms $a_n$ for <u>large values of $n$</u>. We may remove/add any finite # of terms to the beginning of a series w/o affecting its convergence / Divergence.

<u>Alternating Series</u>

<u>Def</u> A series $\sum_{n=1}^{\infty} a_n$ is <u>alternating</u> if $a_n = \pm(-1)^n b_n$, with $b_n > 0$ for all $n \geq 1$.

~~Alt.~~ <u>eg</u> $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ $\qquad b_n = \frac{1}{n}$.

. <u>eg</u> $\sum_{n=1}^{\infty} n^2 \cos(\pi n)$ $\qquad b_n = n^2$.

<u>Alternating Series Test</u> An alternating series $\sum_{n=1}^{\infty} (-1)^n b_n$ $(b_n > 0)$ converges if $\begin{cases} \{b_n\}_{n\geq 1} \text{ is decreasing: } b_1 > b_2 > b_3 > \cdots \text{ and} \\ \lim_{n\to\infty} b_n = 0. \end{cases}$

& It diverges if $\lim_{n\to\infty} b_n \neq 0$.

<u>eg</u> $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges: $b_n = \frac{1}{n}$, $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\{b_n\}_{n\geq 1}$ is decreasing: $b_n = \frac{1}{n} > b_{n+1} = \frac{1}{n+1}$ for all $n \geq 1$.